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A CONCEPT OF WEIGHTED CONNECTIVITY ON CONNECTED GRAPHS

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Abstract. The introduction of a $\{0, 1\}$-valued game associated to a connected graph allows us to assign to each node a value of weighted connectivity according to the different solutions that for the cooperative games are obtained by means of the semivalues. The marginal contributions of each node to the coalitions differentiate an active connectivity from another reactive connectivity, according to whether the node is essential to obtain the connection or it is the obstacle for the connection between the nodes in the coalition. Diverse properties of this concept of connectivity can be derived.

Key words and phrases. Graph; Connectivity; Cooperative game; Solution; Semivalue.

Mathematics Subject Classification. Primary 91A12, 91A43; Secondary 05C40.

1 Introduction

The graph concept is a mathematical model for communication networks constituted by nodes or vertices connected by means of connection channels, that generically we name edges. Each node is directly connected to other nodes by means of edges or indirectly through paths involving different edges and intermediate nodes. The geometry of the edges, in addition to their number, determines the importance of each node in the system of the graph, so that it seems reasonable to study the contribution of each node to the set of connections. In this work we propose to associate to each connected graph a cooperative game. A desirable characteristic for the game is that it is simple. For this reason, a two-valued game is constructed. Values 1 and 0 are respectively associated with the connected and not connected coalitions according to the graph.

2 Connected graphs and games

A graph $\Gamma = (N, E)$ is a pair where N is a finite set of nodes and $E = E(\Gamma)$ is a set of non-ordered pairs of different nodes called edges and denoted by $i : j$. From now on we suppose $N = \{1, 2, ..., n\}$ and we denote by $G(N)$ the set of all graphs with these n nodes. We say that two nodes related by an edge are adjacent nodes. A path between two nodes is a series of edges that link both nodes in such a way that each edge and its following edge have a common node. A graph is a connected graph when, for every pair of nodes, there exists a path between both nodes. We denote by $CG(N)$ the set of all connected graphs on N.

A cooperative game with transferable utility is a pair (N, v), where $N = \{1, 2, ..., n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is the so-called characteristic function, which assigns to every coalition $S \subseteq N$ a real number $v(S)$, the worth of coalition S, and satisfies the condition $v(\emptyset) = 0$. We denote by G_N the set of all cooperative games on N. Given a set of players N, we identify each game (N, v) with its characteristic function v.

Definition 2.1 *We define the cooperative game $v_\Gamma \in G_N$ [1] associated to the connected graph $\Gamma \in CG(N)$ by $v_\Gamma(S) = 1$ if $S \subseteq N$ is connected by graph Γ and $|S| > 1$, and $v_\Gamma(S) = 0$ otherwise.*

Value 1 is associated to success and value 0 to failure in the connection of all members of each coalition of nodes S. Γ is a connected graph, then $v_\Gamma(N) = 1$. Value $v_\Gamma(\{i\}) = 0, \forall i \in N$, responds to the consideration of which isolated elements do not enter in communication with any other node.

A function $\Psi : G_N \to \mathbb{R}^N$ is called a solution for the cooperative games in G_N and represents a method to measure the negotiation strength of the players in the game or a way to emphasize the importance of the role that each one of them carries out. The space $\mathbb{R}^N$ is the so-called allocation space or payoff vector space.

In order to calibrate the importance of each player i in the different coalitions S, we can look at its marginal contribution, i.e., $v(S) - v(S \setminus \{i\})$. If these marginal contributions are weighted by means of equal coefficients depending only on the size of the coalitions, we arrive at the solution concept known as *semivalue*, introduced by Dubey, Neyman and Weber in 1981. These solutions $\Psi : G_N \to \mathbb{R}^N$ are characterized [3] by means of four axioms:

A1. Linearity. $\Psi[\lambda u + \mu v] = \lambda\Psi[u] + \mu\Psi[v],\ \forall u, v \in G_N,\ \forall \lambda, \mu \in \mathbb{R}$.

A2. Symmetry. $\Psi_{\pi i}[\pi v] = \Psi_i[v],\ \forall v \in G_N,\ \forall i \in N,\ \forall \pi$ permutation of N, where game πv is defined by $(\pi v)(\pi S) = v(S),\ \forall S \subseteq N$.

A3. Monotonicity. v monotonic $\Rightarrow\ \Psi_i[v] \geq 0,\ \forall i \in N$.

A4. Projection. $\Psi_i[v] = v(\{i\}),\ \forall v \in A_N$, where A_N denotes the set of additive games within G_N: games v such that $v(S \cup T) = v(S) + v(T)$ for $S \cap T = \emptyset$ and $S, T \subseteq N$.

It has been proved in [3] that there exists a one-to-one map between the semivalues on G_N and the weighting vectors (p_s), $s = 1, \ldots, n$, that verify conditions

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1 \quad \text{and} \quad p_s \geq 0 \ \text{ for } \ 1 \leq s \leq n.$$

The number of coalitions of size s that contain each player $i \in N$ is $\binom{n-1}{s-1}$. The above conditions give a probability distribution on the different coalitions that contain each player, assuming equal weight for equal size.

This way, the payoff to each player i in a game $v \in G_N$ by a semivalue ψ with weighting coefficients (p_s), $s = 1, \ldots, n$, is a weighted sum of its marginal contributions

$$\psi_i[v] = \sum_{S \subseteq N} p_s[v(S) - v(S \setminus \{i\})], \quad \text{where } s = |S|.$$

We denote the set of all semivalues defined on G_N by $Sem(G_N)$. Well known solutions for cooperative games as the Shapley value [6] and the Banzhaf value [2, 5] are semivalues.

3 Weighted connectivity

Definition 3.1 *We call weighted connectivity [1], according to semivalue $\psi \in Sem(G_N)$, of the node i in the connected graph Γ to the payoff that according to the semivalue ψ corresponds to the player i in the associated game v_Γ. We denote it by $\psi_i[v_\Gamma]$, $\forall i \in N$.*

Theorem 3.2 *The weighted connectivity $\psi_i[v_\Gamma]$ of a node i in a connected graph Γ can be decomposed in an active connectivity $\psi_i[v_\Gamma]^+$ and a reactive connectivity $\psi_i[v_\Gamma]^-$:*

$$\psi_i[v_\Gamma] = \psi_i[v_\Gamma]^+ + \psi_i[v_\Gamma]^- \quad \textit{where}$$

$$\psi_i[v_\Gamma]^+ = \sum_{s=2}^{n} \gamma_s p_s \textit{ with } \gamma_s = |\{S \subseteq N \mid i \in S, |S| = s, v_\Gamma(S) = 1 \textit{ and } v_\Gamma(S \setminus \{i\}) = 0\}| \textit{ and}$$

$$\psi_i[v_\Gamma]^- = -\sum_{s=3}^{n-1} \delta_s p_s \textit{ with } \delta_s = |\{S \subset N \mid i \in S, |S| = s, v_\Gamma(S) = 0 \textit{ and } v_\Gamma(S \setminus \{i\}) = 1\}|.$$

To prove the Theorem it suffices to consider that in cases $v_\Gamma(S) = v_\Gamma(S \setminus \{i\}) = 1$ and $v_\Gamma(S) = v_\Gamma(S \setminus \{i\}) = 0$ the marginal contributions vanish. In the first situation both coalitions are connected by Γ, whereas in the second situation the nodes are connected neither in S nor in $S \setminus \{i\}$. Case $v_\Gamma(S) = 1$, $v_\Gamma(S \setminus \{i\}) = 0$ supposes the loss of connection in S when we remove node i. The node i is essential in the connection of the nodes in S. Finally, $v_\Gamma(S) = 0$ with $v_\Gamma(S \setminus \{i\}) = 1$ shows the position of node i as an obstacle for the connection of the nodes in S, since coalition $S \setminus \{i\}$ is connected by Γ.

Example 3.3 *In the set of nodes $N = \{1, 2, 3, 4, 5, 6\}$, we consider the graph Γ whose set of edges is $E(\Gamma) = \{1:2,\ 1:4,\ 2:3,\ 2:6,\ 3:4,\ 3:5,\ 4:5,\ 5:6\}$.*

For node 1, coalitions S that compute in active connectivity are $\{1, 2\}$, $\{1, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 4, 5\}$ and $\{1, 2, 4, 6\}$, whereas for the reactive one they are $\{1, 3, 5\}$, $\{1, 5, 6\}$ and $\{1, 3, 5, 6\}$. Then: $\psi_1[v_\Gamma]^+ = 2p_2 + p_3 + 2p_4$, $\psi_1[v_\Gamma]^- = -2p_3 - p_4$ and $\psi_1[v_\Gamma] = 2p_2 - p_3 + p_4$.

Similar computations allow us to determine the weighted connectivity for the remaining nodes so that

$$\psi[v_\Gamma] = (2p_2 - p_3 + p_4,\ 3p_2 + 2p_3 + 5p_4 + 2p_5,\ 3p_2 + 2p_3 + 3p_4,\ 3p_2 + p_3 + 3p_4 + p_5,\ 3p_2 + p_3 + 3p_4 + p_5,\ 2p_2 - p_3 + p_4).$$

The symmetrical role of nodes 1 and 6, and 4 and 5, respectively, gives rise to equal allocations by every semivalue. Now, by using diverse semivalues, we can offer relative measures of the importance of the nodes in the connection scheme of the graph. If we choose the Shapley value Sh on games with six players (weighting coefficients $p_1 = p_6 = 1/6$, $p_2 = p_5 = 1/30$ and $p_3 = p_4 = 1/60$):

$$Sh[v_\Gamma] = (0.0667,\ 0.2833,\ 0.1833,\ 0.2000,\ 0.2000,\ 0.0667).$$

According to the Banzhaf value β (weighting coefficients $p_s = 1/32$, $s = 1, \dots, 6$), we have:

$$\beta[v_\Gamma] = (0.0625,\ 0.3750,\ 0.2500,\ 0.2500,\ 0.2500,\ 0.0625).$$

Finally, if we denote by ψ^* the semivalue with coefficients $p_1 = 32/243$, $p_2 = 16/243$, $p_3 = 8/243$, $p_4 = 4/243$, $p_5 = 2/243$ and $p_6 = 1/243$, the weighted connectivity of the nodes in graph Γ is

$$\psi^*[v_\Gamma] = (0.1152,\ 0.3621,\ 0.3128,\ 0.2881,\ 0.2881,\ 0.1152).$$

The weighting coefficients of semivalue ψ^* are in geometric progression with ratio 1/2: semivalue ψ^* belongs to the family of *binomial* semivalues, as they were introduced in [4].

4 General properties

Several properties of this concept of weighted connectivity can be derived. The following theorems introduce some of them, which can be easily proved from the definitions.

Given a node i in a graph Γ, the *degree* of i, denoted as $deg(i)$, is the cardinality of the set of adjacent nodes to i. The coefficient p_2 of any semivalue $\psi \in Sem(G_N)$ weights the marginal contributions to two-person coalitions. In the associated game to graph Γ, the two-person coalitions are formed by the own node i and all its adjacent nodes. Thus, the weight p_2 appears in the weighted connectivity of node i so many times as $deg(i)$.

We say that two graphs Γ_1 and Γ_2 are *isomorphic* iff there exists a one-to-one map between the respective sets of nodes, $f : N_1 \to N_2$, that preserves adjacencies, i.e., $i : j \in E(\Gamma_1) \Leftrightarrow f(i) : f(j) \in E(\Gamma_2)$. The map f is an *isomorphism* between the graphs Γ_1 and Γ_2.

Theorem 4.1 *If Γ_1 and Γ_2 are isomorphic graphs by map f, then*

$$\psi_{f(i)}(v_{\Gamma_2}) = \psi_i(v_{\Gamma_1}),\ \forall i \in N_1,\ \forall \psi \in Sem(G_N).$$

In particular, if two nodes play a symmetrical role in a given graph Γ, we can define a one-to-one map between the own nodes, so that both nodes obtain the same weighted connectivity by every semivalue.

Theorem 4.2 *Let i be a node in a graph $\Gamma \in CG(N)$. If $deg(i) = n-1, n-2$, then the reactive connectivity of vertex i vanish.*

Theorem 4.3 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, we consider the set $CG(N)$ of all connected graphs with nodes in N. Then, for every node $i \in N$,*

$$\max_{\Gamma \in CG(N)} \psi_i[v_\Gamma] = 1 - p_1.$$

We say that a node i in a graph with $deg(i) = 1$ is an *antenna*. The unique node adjacent to an antenna is the *base* of the antenna.

The *star graph* with $n-1$ points S_{n-1} is formed by $n-1$ antennas and a unique node that is the base of all antennas or star center. We denote the set of nodes of S_{n-1} by $N = \{1\} \cup \{2, ..., n\}$, where the distinguished node 1 is its center.

Corollary 4.4 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, the maximum value of weighted connectivity of the nodes in graphs of $CG(N)$ is attained in the center node of S_{n-1}.*

Theorem 4.5 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, we consider the set $CG(N)$ of all connected graphs with nodes in N. Then, for every node $i \in N$,*

$$\min_{\Gamma \in CG(N)} \psi_i[v_\Gamma] = p_2 - \sum_{s=3}^{n-1} \binom{n-2}{s-1} p_s.$$

In the set of nodes N, we call *complete graph K_N* to the graph that is formed by the $\binom{n}{2}$ possible edges.

Corollary 4.6 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, the minimum value of weighted connectivity of the nodes in graphs of $CG(N)$ is attained in an antenna node i with base in any node of $K_{N \setminus \{i\}}$.*

Corollary 4.7 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, and a graph $\Gamma \in CG(N)$, the weighted connectivity of an antenna node i with base in a node b is a value independent from the number of edges between the base b and the remaining nodes.*

5 Connectivity of graph families

In this section we consider the weighted connectivity of some families of graphs whose edges have a disposition with geometric regularity. The graph families whose connectivity we want to evaluate have an element on each set of nodes with different cardinality: complete graphs K_N, star graphs S_{n-1} with $n-1$ points or *cycle graphs* formed by a unique closed way or cycle going through the n nodes.

Theorem 5.1 *For every semivalue* $\psi \in Sem(G_N)$, $|N| \geq 3$, *with weighting coefficients* (p_s), $s = 1, \ldots, n$, *the weighted connectivity has the following values.*

(a) Complete graph: $\psi_i[v_{K_N}] = (n-1)p_2$, $\forall i \in N$.

(b) Star graph:

$$\psi_1[v_{S_{n-1}}] = 1 - p_1, \quad 1 \text{ star center;}$$
$$\psi_i[v_{S_{n-1}}] = p_2, \quad i = 2, ..., n \text{ star points.}$$

(c) Cycle graph:

$$\psi_i[v_{C_4}] = 2p_2 + p_3, \quad i = 1, 2, 3, 4;$$

$$\psi_i[v_{C_n}] = 2p_2 + \sum_{s=3}^{n-2}(2s - n - 1)p_s + (n-3)p_{n-1}, \ \forall i \in N, \ n \geq 5.$$

6 Conclusion

Semivalues form a wide family of solutions for cooperative games whose allocations to the players are frequently used in Game Theory. The introduction of a cooperative game associated to each connected graph has turned out to be a useful tool to compute weighted connectivity of the nodes by means of the family of semivalues. Several properties of this concept of connectivity have been obtained.

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ACTION OF JOIN SPACES OF CONTINUOUS FUNCTIONS ON THE UNDERLYING HYPERGROUPS OF 2-DIMENSIONAL LINEAR SPACES OF FUNCTIONS

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Abstract. Using one-to-one correspondence between the class of linear ordinary second-order homogeneous differential equations and their two-dimensional linear functional solution spaces there is constructed the multiautomaton consisting in certain action of a join space of continuous functions on the underlying hypergroup of 2-dimensional linear spaces of functions.

Key words and phrases. Join space, hypergroup, continuous function, solution space of second-order linear differential equations, multiautomaton.

Mathematics Subject Classification. Primary 60A05, 08A72; Secondary 28E10.

In this contribution we construct actions of join spaces of continuous functions on the hypergroup serving as a phase of 2-dimensional linear spaces of functions which form solution spaces of second order linear ordinary homogeneous differential equations with continuous coefficients. This constructed structure is in fact a discrete dynamical system with a phase hypergroup — a join space — of continuous functions and a phase set formed by the above mentioned hypergroup of 2-dimensional linear spaces of functions.

Join spaces are playing an important role in theories of various mathematical structures and their applications. The concept of a join space has been introduced by Walter Prenowitz and used by him and afterwards together with James Jantoisciak to reconstruct several branches of geometry — [27, 28, 29]. The other fields of applications of the mentioned structure are lattices, graphs, ordered sets and automata. Noncommutative join spaces form an interesting subclass of the class of transposition hypergroups which satisfies a postulated property of transposition — [27, 28], More precisely, if H is a set, $\mathcal{P}(H)$ is the family of all subsets of H then a mapping

$*\colon H \times H \to \mathcal{P}(H)$ is called a hyperoperation or join operation in H and the pair $(H, *)$ is said to be a hypergroupoid. The join operation is extended to subsets of H in a natural way, so that for $\emptyset \neq A \subset H$, $\emptyset \neq B \subset H$ the hyperproduct $A * B$ is given by $A * B = \bigcup\{a * b; a \in A, b \in B\}$. The relational notation $A \approx B$ (read A meets B) is used to assert that the sets A and B have nonempty intersection.

In H two hypercompositions right extension "/" and left extension "\" each being an inverse to $*$ are defined by $a/b = \{x; a \in x * b\}$ and $b\backslash a = \{x, a \in b * x\}$. Hence $x \approx a/b$ if and only if $a \approx x * b$ and $x \approx b\backslash a$ if and only if $\approx b * x$.

Now a hypergroupoid $(H, *)$ is acalled a transposition hypergroup or a (noncommutative) join space if it one satisfies three axioms:

1. $a * (b * c) = (a * b) * c$ for all $a, b, c \in H$ (Associativity),
2. $a * H = H = H * a$ for all $a \in H$ (Reproduction),
3. $b\backslash a \approx c/d$ implies $a * d \approx b * c$ for all $a, b, c, d \in H$ (Transposition).

Recall that an associative hypergroupoid (i.e. a semihypergroup) satisfying the reproduction axiom is called a hypergroup. Further a multiautomaton is a triple (S, H, δ), where H is a hypergroupoid, S is a nonempty set and $\delta\colon S \times H \to S$ is a mapping such that

$$\delta\big(\delta(s, a, b\big) \in \delta(s, a \cdot b) \qquad \text{(GMAC)}$$

for any triple $(s, a, b) \in S \times H \times H$, where $\delta(s, a \cdot b) = \{\delta(s, x); x \in a \cdot b\}$. The mapping δ is called the transition function or the next-state function or the action of the input hypergroupoid or the phase hypergroupoid H on the state or phase set S. The acronz (GMAC) means the Generalized Mixed Associativity Condition.

In what follows $\mathbb{N} = \{1, 2 \ldots\}$ is a set of all positive integers (natural numbers) and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

By a *quasi-ordered semigroup* we mean a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a semigroup and binary relation $\leq$ is a quasi-ordering (i.e is reflexive and transitive) on the set G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \bullet z \leq y \bullet z$ and $z \bullet x \leq z \bullet y$ hold. By an *ordered* (*semi*) *group* we mean (as usual) a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a (semi)group and $\leq$ is a reflexive, antisymmetrical and transitive binary relation on G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \bullet z \leq z$ and $z \bullet x \leq z \bullet y$ hold. Further, $[a)_{\leq} = \{x \in G; a \leq x\}$ is a principal and generated by $a \in G$. By an *inclusion homomorphism* we mean a mapping $f\colon (G, \bullet_G) \to (H, \bullet_H)$ such that $f(a \bullet_H b) \subset f(a) \bullet_G f(b)$ for all pairs $a, b \in G$. If equalities hold instead of inclusion and the mapping f is bijective then f is an isomorphism and we write $(G, \bullet_G) \simeq (H, \bullet_H)$.

The following lemma, which is crucial for our considerations, is proved in [16] for the first time in [12], pp 146–7.

Lemma 1 *Let a triple $(G, \cdot, \leq)$ be a quasi-ordered semigroup. Define a hyperoperation*

$$*\colon G \times G \to \mathcal{P}^*(G) \quad by \quad a * b = [a, b)_{\leq} = \{x \in G; a \cdot b \leq x\}$$

for all pairs of elements $a, b \in G$.

1. *Then $(G, *)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*

2. *Let $(G, *)$ be the above defined semihypergroup. Then $(G, *)$ is a hypergroup iff for any pair of elements $a, b \in G$ there exists a pair of elements $c, c' \in G$ with a property $a \cdot c \leq b$, $c' \cdot a \leq b$.*

Remark 1 *Notice that if $(G, \cdot, \leq)$ is a quasi-ordered group then the condition stated under 2. is satisfied, hence the final hyperstructure is a hypergroup.*

In paper [14] there were proved the following assertions:

Proposition 1 *Let $I \subset \mathbb{R}$ be an open interval, $\mathbb{LA}_2(J) = \big\{L(p,q); p, q \in \mathbb{C}(J), p(x) > 0, x \in I\big\}$. For any pair of differential operators $L(p_1, q_1), L(p_2, q_2) \in \mathbb{LA}_2(J)$ define*

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1 p_2, p_1 q_2 + q_1)$$

and $L(p_1, q_1) \leq L(p_2, q_2)$ if $p_1(x) = p_2(x)$, $q_1(x) \leq q_2(x)$ for any $x \in I$. Then $(\mathbb{LA}_2(J), \cdot, \leq)$ is a noncommutative ordered group with the unit element $L(\chi_1, \chi_0)$.

Now we apply the simple construction of a hypergroup from Lemma 1 into this considered concrete case of differential operators:

For arbitrary pair of operators $L(p_1, q_1), L(p_2, q_2) \in \mathbb{LA}_2(J)$ we put

$$\begin{aligned} L(p_1 q_1) * L(p_2, q_2) &= \big\{L(p,q) \in \mathbb{LA}_2(J); L(p_1, q_1) \cdot L(p_2, q_2) \leq L(p,q)\big\} \\ &= \big\{L(p,q) \in \mathbb{LA}_2(J); L(p_1 p_2, p_1 q_2 + q_1) \leq L(p,q)\big\} \\ &= \big\{L(p_1 p_2, q); \in \mathbb{C}(J), p_1 q_2 + q_1 \leq q\big\}. \end{aligned}$$

Then we obtain from Proposition 2.1 with respect to Lemma 1.1 immediately:

Proposition 2 *Let $I \in \mathbb{R}$ be an open interval and let $* : \mathbb{LA}_2(J) \times \mathbb{LA}_2(J) \to \mathcal{P}^*\big(\mathbb{LA}_2)\big)$ be the above defined binary hyperoperation. Then the hypergroupoid $\big(\mathbb{LA}_2(J), *\big)$ is an noncommutative hypergroup.*

Theorem 1 *Let $I \subset \mathbb{R}$ be an open interval, $\mathbb{LA}_2(J) = \big\{L(p,q); [p,q] \in \mathbb{C}_+(J) \times \mathbb{C}(J)\big\}$ be the set of ordinary linear differential operators of second order — i.e. $L(p,q)(y) = y'' + p(x)y' + q(x)y = 0$, $y \in \mathbb{C}^2(J)$. If $L(p_1, q_1) * L(p_2, q_2) = \big\{L(p,q) \in \mathbb{LA}_2(J); p_1 p_2 = p, p_1 q_2 + q_1 \leq q\big\}$ for any pair $L(p_1 q_1), L(p_2, q_2) \in \mathbb{LA}_2(J)$ then $\big(\mathbb{LA}_2(J), *\big)$ is noncommutative transposition hypergroup, i.e. a noncommutative join space.*

Now, denote $\mathbb{R}_0^+ = \{r \in \mathbb{R}, 0 \leq r\}$ and consider $\mathbb{C}^k(J)$, where $J \subset \mathbb{R}^n$ is an open set, as the linear space on the field $(\mathbb{R}, +, \cdot)$. We will construct a hypergroup with the carrier set $\mathbb{C}^k(J)$, $J \subseteq \mathbb{R}$. The generalization to the case $\mathbb{C}^k(J)$ of all real functions of one variable $F\colon J \to \mathbb{R}$. Here, for a pair of functions f, $g \in \mathbb{C}^k(J)$ we define $f \leq g$ whenever $f(x) \leq g(x)$ for any $x \in J$.

Define a binary hyperoperation $\bullet\colon \mathbb{C}^k(J) \times \mathbb{C}^k(J) \to \mathcal{P}^*\big(\mathbb{C}^k(J)\big)$ by

$$F \bullet g = \bigcup_{[a,b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+} [af + bg]_{\leq} = \bigcup_{[a,b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+} \big\{h \in \mathbb{C}^k(J); af(x) + bg(x) \leq h(x), x \in J\big\}.$$

Evidently, $\big(\mathbb{C}^k(J), \bullet\big)$ is a commutative hypergroupoid. Moreover, without any effort we verify the reproduction axiom.

Indeed, let $f \in \mathbb{C}^k(J)$ be an arbitrary function. Evidently $f \bullet \mathbb{C}^k(J) \subseteq \mathbb{C}^k(J)$. We show that the opposite inclusion is valid.

Let $h \in \mathbb{C}^k(J)$ be an arbitrary function and $a \in \mathbb{R}$ be an arbitrary positive number.

Define $g(x) = \frac{1}{a}h(x) - f(x)$, $x \in J$. Then $g \in \mathbb{C}^k(J)$, $h(x) = a\big(f(x) + g(x)\big)$, thus

$$h = af + ag \in [af + ag)_{\leq} \bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+} [af + bg)_{\leq} = f \bullet g \subseteq \bigcup_{u\in\mathbb{C}^k(J)} f \bullet u = f \bullet \mathbb{C}^k(J)\,,$$

hence $\mathbb{C}^k(J) \subseteq f \bullet \mathbb{C}^k(J)$.

Consequently the equality

$$f \bullet \mathbb{C}^k(J) = \mathbb{C}^k(J)$$

holds. Thus the hypergroupoid $\big(\mathbb{C}^k(J), \bullet\big)$ is a quasihypergroup.

In [20] there were obtained the following theorem:

Theorem 2 *Let $\mathbb{C}^k(J)$ be the ring of functions $f\colon J \to \mathbb{R}$ of the class $\mathbb{C}^k$, $k \in \{0, 1, 2, \dots, \infty\}$. Consider $(\mathbb{C}^k(J), +, \leq))$ as a linear ordered space (with the pointwise ordering of functions, which is a vector ordering) on the field $(\mathbb{R}, +, \cdot)$. Define a binary hyperoperation $\bullet\colon \mathbb{C}^k(J) \times \mathbb{C}^k(J) \to \mathcal{P}^*\big(\mathbb{C}^k(J)\big)$ by*

$$f \bullet g = \bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+} [af + bg)_{\leq} = \bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+} \big\{h \in \mathbb{C}^k(J); af(x) + bg(x) \leq h(x), x \in J\big\}\,.$$

Then the hypergroupoid $(\mathbb{C}^k(J), \bullet)$ is a join space, i.e. a commutative join hypergroup.

Now we are going to construct actions of join spaces $(\mathbb{C}^k(J), \bullet)$ on the transposition hypergroup $(\mathbb{LA}_2(J), *)$. We define a mapping

$$\delta\colon \mathbb{LA}_2(J) \times \mathbb{C}^k(J) \to \mathbb{LA}_2(J)$$

by $\delta(L(p, q), f) = L(p, q) \cdot L(1, f) = L(p, pf + q)$, for any operator $L(p, q) \in \mathbb{LA}_2(J)$ and any function $f \in \mathbb{C}^k(J)$. Then for any pair of functions $f, g \in \mathbb{C}^k(J)$ we need that

$$\delta(\delta(L(p, q), f), g) \in \delta(L(p, q), f \bullet g)\,.$$

We have

$$\delta(L(p, q), f) = L(p, q) \cdot L(1, f) = L(p, pf + q)$$
$$\delta(L(p, pf + q), g) = L(p, pf + q) \cdot L(1, g) = L(p, pg + pf + q)\,.$$

Further

$$f \bullet g = \bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+} [af + bg)_{\leq} = \big\{h; \exists [a, b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \text{ and } h \geq af + bg\big\}\,,$$

$$\delta\Big(L(p, q), \bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+} [af + bg)_{\leq}\Big) = \bigcup_{h\in f\bullet g} L(p, q) \cdot L(1, h) = \bigcup_{h\in f\bullet g} L(p, ph + q)\,.$$

As $f+g \in f \bullet g$ we obtained that $L(p, p(f+g)+q) = L(p, pf+pg+q) \in \delta(L(p,q), f \bullet g)$, thus the mapping δ is a transition map satisfying the Generalized Mixed Associativity Condition. Hence the triple $(\mathbb{LA}_2(J), (\mathbb{C}^k(J), \bullet), \delta)$ is an action of the join space $(\mathbb{C}^k(J), \bullet)$.

Let $k \in \{0, 1, \ldots, \infty\}$ and denote $\mathbb{LA}_2(J)_k = \big\{L(p,q); p, q \in \mathbb{C}^k(J)\big\}$ and similarly $\mathbb{S}(J) = \big\{T(1,p,q); p \in \mathbb{C}^k_+(J), q \in \mathbb{C}^k(J)\big\}$. We define a mapping

$$\delta_{CL} \colon \mathbb{LA}_2(J)_k \times \mathbb{C}^k(J) \to \mathbb{LA}_2(J)_k$$

by $\delta_{CL}\big(L(p,q), f\big) = L(p,q) \cdot L(1,f) = L(p, pf+q)$, for any operator $L(p,q) \in \mathbb{LA}_2(J)_k$ and any function $f \in \mathbb{C}^k(J)$. Then for any pair of functions $f, g \in \mathbb{LA}_2(J)_k$ we have

$$\begin{aligned}
\delta_{Cl}\big(\delta_{CL}(L(p,q), f), g\big) &= \delta_{CL}\big(L(p, pf+q), g\big) \\
&= L(p, pf+q) \cdot L(1,g) = L\big(p, p(f+g)+q\big) \\
&\in \big\{L(p, p(f+g)+q)\big\} \cup \big\{L(p, ph+q); f+g < h\big\} \\
&= \big\{L(p, ph+q); f+g \le h\big\} \\
&= \big\{L(p,q) \cdot L(1,h); h \in (f+g)\big\} \\
&= \big\{\delta_{CL}(L(p,q), h); h \in (f+g)\big\} = \delta_{CL}\big(L(p,q), f+h\big)\,,
\end{aligned}$$

thus the mapping δ_{CL} is a transition map (a next state function) satisfying the Generalized Mixed Associativity Condition (GMAC), hence the triad $\big(\mathbb{LA}_2(J)_k, (\mathbb{C}^k(J), +), \delta_{CL}\big)$ is a multiautomaton — an action of the join space $\big(\mathbb{C}^k(J), +\big)$.

Let $\varphi_1, \varphi_2 \in \mathbb{C}^2(J)$ be linearly independent functions, i.e. such that their Wronski determinant

$$W[\varphi_1, \varphi_2] = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} \neq 0 \quad \text{for any } x \in J.$$

Denote by $V(\varphi_1, \varphi_2)$ the two-dimensional linear space formed by all functions

$$y(x) = c_1\varphi_1(x) + c_2\varphi_2(x)\,, \qquad c_1, c_2 \in \mathbb{R}\,,$$

i.e. $V(\varphi_1, \varphi_2) = \{c_1\varphi_1 + c_2\varphi_2; \varphi_1, \varphi_2 \in \mathbb{C}^2(J), c_1, c_2 \in \mathbb{R}\}$. Denote by $\mathbb{VA}_2^+(J)$ the system of two-dimensional linear solution spaces of second order linear differential equations $L(p,q)(y) = 0$, $L(p,q) \in \mathbb{LA}_2^+(J)$, where $L(p,q,)(y) = y'' + p(x)y' + q(x)y$, $x \in J$. Denote by $D[\varphi_1, \varphi_2] = \begin{vmatrix} \varphi_1'' & \varphi_2'' \\ \varphi_1 & \varphi_2 \end{vmatrix}$. In the monography [36] there is contained a very important assertion that there is one-to-one correspondence between all equations $L(p,q)(y) = 0$ and their solution spaces $V(\varphi_1, \varphi_2)$. Notice that the differential eqaution corresponding to the space $V(\varphi_1, \varphi_2)$ where $(\varphi_1, \varphi_2) \in \mathbb{C}^2(J) \times \mathbb{C}^2(J)$, linearly independent) has the form

$$y'' + \frac{D[\varphi_1, \varphi_2]}{W[\varphi_1, \varphi_2]}\, y' + \frac{W[\varphi_1', \varphi_2']}{W[\varphi_1, \varphi_2]}\, y = 0\,.$$

For any solution space $V(\varphi_1, \varphi_2) \in \mathbb{VA}_2^+(J)$ we choose arbitrary but fixed base (e.g. $\{\varphi_1, \varphi_2\}$) which will be termed as representating fundamental solution system of the corresponding linear homogeneous second-order differential equation. In paper [4] there were obtained the following result:

Theorem 3 *Let $J \subseteq \mathbb{R}$ be an open interval. Let us define a binary hyperoperation*

$$\bullet\colon \mathbb{VA}_2^+(J) \times \mathbb{VA}_2^+(J) \to \mathcal{P}^*\big(\mathbb{VA}_2^+(J)\big)$$

in this way: Let $V(\varphi_1, \varphi_2)$, $V(\psi_1, \psi_2 \in \mathbb{VA}_2^+(J)$ be a pair of two-dimensional function spaces such that their bases $\{\varphi_1, \varphi_2\}$, $\{\psi_1, \psi_2\}$ form fundamental solution systems of second order linear homogeneous differential equations

$$y''(x) + \frac{D[\varphi_1, \varphi_2]}{W[\varphi_1, \varphi_2]}\, y'(x) + \frac{W[\varphi_1', \varphi_2']}{W[\varphi_1, \varphi_2]}\, y(x) = 0\,,$$
$$y''(x) + \frac{D[\psi_1, \psi_2]}{W[\psi_1, \psi_2]}\, y'(x) + \frac{W[\psi_1', \psi_2']}{W[\psi_1, \psi_2]}\, y(x) = 0\,.$$

Let $\mathcal{F}(\varphi_1, \varphi_2, \psi_1, \psi_2)$ be the set of all representing fundamental solution-system for differential equations

$$y''(x) + \frac{D[\varphi_1, \varphi_2]\, D[\psi_1, \psi_2]}{W[\varphi_1, \varphi_2]\, W[\psi_1, \psi_2]}\, y'(x) + v(x)\, y(x) = 0\,,$$

where

$$\frac{D[\varphi_1, \varphi_2]\, W[\psi_1', \psi_2']}{W[\varphi_1, \varphi_2]\, W[\psi_1, \psi_2]} + \frac{W[\varphi_1', \varphi_2']}{W[\varphi_1, \varphi_2]} \le v(x)\,,$$

$v \in \mathbb{C}(J)$. Then defining

$$V(\varphi_1, \varphi_2) \bullet V(\psi_1, \psi_2) = \big\{V(\omega_1, \omega_2) \in \mathbb{VA}_2^+(J), \{\omega_1, \omega_2\} \in \mathcal{F}(\varphi_1, \varphi_2, \psi_1, \psi_2)\big\}$$

we have that the hypergroupoid $(\mathbb{VA}_2^+(J), \bullet)$ is a noncommutative transposition hypergroup, i.e. a non-commutative join space.

We adopt some terms of the classical algebraic automata theory onto the concept of a multiautomaton. They are overtaken from the monography of Gécseg, Peák 1972 [8] and S. Bavel [2]:

A multiautomaton $\mathcal{M} = (S, H, \delta_{\mathcal{M}})$ is said to be

– *abelian* (or *commutative*) if $\delta_{\mathcal{M}}(s, x \cdot y) = \delta(s, y \cdot x)$ for any triad $[s, x, y] \in S \times H \times H$,

– *cyclic* if there is a state $s \in S$ such that for any state $t \in S$ there exists an element $a \in H$ with the property $\delta_{\mathcal{M}}(s, a) = t$. In the above, s is called a generator of $\mathcal{M}$. Moreover, if the set of all generators of $\mathcal{M}$ is exactly S then the multiautomaton $\mathcal{M}$ is said to be *strongly connected*. A subautomaton $\{\mathcal{T}\} = (T, H, \delta_{\mathcal{T}})$ — here $\delta_{\mathcal{M}}(t, x) \in T$ for any pair $[t, x] \in T \times H$, where $T \subseteq S$ and $\delta_{\mathcal{T}}$ is the restriction of $\delta_{\mathcal{M}}$ onto $T \times H$ — is called *separated* if $\delta_{\mathcal{M}}(S \setminus T, H) \cap T = \emptyset$. Then a multiautomaton is said to be connected if it has no separated submultiautomaton. If $\mathcal{M} = (S, H, \delta_{\mathcal{M}})$ is strongly connected we say that the (semi)hypergroup H *acts transitively* on the state set S.

A multiautomaton $\mathcal{M}$ is called *transitive* if for any pair $[s, t] \in S \times S$ there exists an automorphism ϱ of $\mathcal{M}$ (i.e. $\varrho\colon S \to S$ is a bijection such that $\varrho\big(\delta_{\mathcal{M}}(s, x)\big) = \delta_{\mathcal{M}}(\varrho(s), x)$ for any

$s \in S$ and any $x \in H$) with the property $t = \varrho(s)$, which means that the automorphism group of $\mathcal{M}$ acts transitively on the set S.

A transitive multiautomaton is said to be *quasiperfect* if it is strongly connected; similarly an abelian (i.e. commutative) multiautomaton is called *perfect* if it is strongly connected.

For a fixed positive continuous function $p\colon J \to \mathbb{R}$ we denote $\mathbb{LA}_2^+ p(J)$ the set of all second order ordinary linear differential operators $L(p,q) \in \mathbb{LA}_2^+(J)$. Defining $\delta_p\colon \mathbb{LA}_2^+ p(J) \times \mathbb{C}(J) \to \mathbb{LA}_2^+ p(J)$ by $\delta_p(L(p,q), f) = L(p, pf+q)$ for any pair $q, f \in \mathbb{C}(J)$ we have

$$\delta_p\big(\mathbb{LA}_2^+ p(J), \mathbb{C}(J)\big) = \big\{\delta_p(L(p,q), f)\,; L(p,q) \in \mathbb{LA}_2^+ p(J)\,, f \in \mathbb{C}(J)\big\} \subseteq \mathbb{LA}_2^+ p(J)$$

and

$$\delta\big(\mathbb{LA}_2^+(J) \setminus \mathbb{LA}_2^+ p(J), \mathbb{C}(J)\big) \cap \mathbb{LA}_2^+ p(J) = \emptyset\,,$$

hence the submultiautomaton $(\mathbb{LA}_2^+ p(J), \mathbb{C}(J), \delta_p)$ is separated. Since for any $p \in \mathbb{C}_+(J)$ the above submultiautomaton is a maximal connected submultiautomaton of $\mathcal{A}(J) = (\mathbb{LA}_2^+(J), \mathbb{C}(J), \delta)$ we say that $\mathcal{A}_p(J)(\mathbb{LA}_2^+(J), \mathbb{C}(J), \delta_p)$ is a component of $\mathcal{A}$ we define that the multiautomaton $\mathcal{A}$ is the cardinal sum of all $\mathcal{A}_p$, $p \in \mathbb{C}_+(J)$; in the written form: $\mathcal{A} = \sum\limits_{p\in\mathbb{C}_+(J)} \mathcal{A}_p$. Suppose $p \in \mathbb{C}_+(J)$ and denote (as above) $\mathcal{A}_p(J) = (\mathbb{LA}_2^+ p(J), \mathbb{C}(J), \delta_p)$, where

$$\delta_p\colon \mathbb{LA}_2^+ p(J) \times \mathbb{C}(J) \to \mathbb{LA}_2^+ p(J)$$

is defined by

$$\delta_p(L(p,q), f) = L(p,q) \cdot L(1,f) = L(p, pf+q)\,.$$

The multiautomaton $\mathcal{A}_p(J)$ is transitive, i.e. its automorphism group $\mathrm{Aut}\big(\mathcal{A}_p(J)\big)$ acts on the state set $\mathbb{LA}_2^+ p(J)$ transitively.

Now, let us consider the triad $(\mathbb{VA}_2^+(J), \mathbb{C}(J), \delta_{\mathcal{V}})$ (denoted by $\mathcal{V}(J)$), where

$$\delta_{\mathcal{V}}\colon \mathbb{VA}_2^+(J) \times \mathbb{C}(J) \to \mathbb{VA}_2^+(J)$$

is defined by

$$\delta_{\mathcal{V}}\big(V(\varphi_1,\varphi_2), f\big) = V(\psi_1,\psi_2)$$

for any space $V(\varphi_1,\varphi_2) \in \mathbb{VA}_2^+(J)$ and any function $f \in \mathbb{C}(J)$. Here $V(\psi_1,\psi_2)$ is the two-dimensional solution space of the second-order linear differential equation

$$y''(x) + \frac{D[\varphi_1,\varphi_2]}{W[\varphi_1,\varphi_2]}\,y'(x) + \frac{D[\varphi_1,\varphi_2]f(x) + W[\varphi_1',\varphi_2']}{W[\varphi_1,\varphi_2]}\,y(x) = 0\,,$$

$x \in J$. Then we obtain:

Theorem 4 *The system $\mathcal{V}(J) = (\mathbb{VA}_2^+(J), \mathbb{C}(J), \delta_{\mathcal{V}})$ is a semisimple abelian multiautomaton which is a (cardinal) sum of perfect submultiautomata of $\mathcal{V}(J)$ whose collection indexed by all positive continuous functions $f\colon J \to \mathbb{R}$ form all components of $\mathcal{V}(J)$.*

Proof. Denote by $\Phi\colon \mathbb{LA}_2^+(J) \to \mathbb{VA}_2^+(J)$ the above mentioned bijection, verified e.g. in the monography [36]. We show that this mapping is carrying an isomorphism of the multiautomaton $\mathcal{A}(J) = (\mathbb{LA}_2^+(J), \mathbb{C}(J), \delta\mathcal{A})$ onto the multiautomaton $\mathcal{V}(J) = (\mathbb{VA}_2^+(J), \mathbb{C}(J), \delta_{\mathcal{V}})$. Suppose $L(p,q) \in \mathbb{LA}_2^+(J)$ is an arbitrary differential operator and $f \in J \to \mathbb{R}$ is an arbitrary continuous function. Denote $V(\varphi_1, \varphi_2) = \Phi\big(L(p,q)\big)$. Then we have

$$\Phi\big(\delta_{\mathcal{A}}(L(p,q), f)\big) = \Phi\big(L(p,q) \bullet L(1,f)\big) = \Phi\big(L(p, pf+q)\big) = V(\psi_1, \psi_2)\,,$$

which is a two-dimensional solution space of the second -order differential equation

$$y'' + p(x)y' + \big(p(x)\,f(x) + q(x)\big)\,y = 0\,,$$

thus

$$V(\psi_1, \psi_2) = \delta_{\mathcal{V}}\big(V(\varphi_1, \varphi_2), f\big) = \delta_{\mathcal{V}}\big(\Phi(L(p,q), f)\big)\,.$$

Consequently, multiautomata $\mathcal{A}(J)$, $\mathcal{V}(J)$ are isomorphic. Now the theorem follows from [19] Theorem 3 which says that the multiautomaton $\mathcal{A}(J)$ is semisimple and it is a (cardinal) sum of perfect semisimple sub-multiautomata $\mathcal{A}_p(J) = \big(\mathbb{LA}_2^+ p(J), \mathbb{C}(J), \delta_p\big)$, $p \in \mathbb{C}(J)$ which are all components of $\mathcal{A}(J)$ (where for $\mathcal{A}_p(J)$ the function p is fixed).

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SANDWICH SEMIGROUPS OF SOLUTIONS OF CERTAIN FUNCTIONAL EQUATIONS AND HYPERSTRUCTURES DETERMINED BY SANDWICHES OF FUNCTIONS

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Abstract. Investigation of canonical forms of a certain class of functional differential equations needs treatment of two type equations $h \circ \varphi = \psi \circ h$ and $h \circ \varphi = \varphi \circ h$. For respond to solvability and describing the solution we use the Cayley graphs of functions which are contained in equations. This method is illustrated on examples. Moreover there is studied the algebraic structure of solution sets of these equations.

Key words and phrases. functional equation, monounanry algebra,

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1 Introduction

For the investigation of properties of linear functional-differential equations (oscillatory behavior) from certain classes of linear functional-differential equations is suitable use the pointwise transformation which preserves distribution of zeros of solutions of a functional-differential equation and its canonical forms. These canonical form of linear functional-differential equations are defined by F. Neuman [14]. The most general form of the transformation is $z(t) = f(t)y(h(t))$, where $f(t) \neq 0$ is continuous and diffeomorphism h has the first derivative $h'(t) \neq 0$.

If the linear homogeneous functional-differential equation of the first order

$$y'(x) + \sum_{i=0}^{m} p_i(x)y(\xi_i(x)) = 0, \tag{1}$$

is transformable onto equation canonical form:

$$z'(t) + \sum_{i=0}^{m} q_i(t) z(\zeta_i) = 0, \tag{2}$$

where $\zeta_i = t + ci$ with constants c_i. It means that there is a diffeomorphism h such that

$$\xi_i(h(t)) = h(\zeta_i(t)), \quad \text{for } i = 1, \ldots, n \tag{3}$$

Moreover for deviations $\xi_i(x)$ hold

$$\xi_i \circ \xi_j = \xi_j \circ \xi_i, \tag{4}$$

for this result see [16]. From the above follows that the investigations of neutrally commuting systems of functions belong to important questions. There are known some algebraic constructions which allow to us construction of solutions set or solution monoids of certain functional equations of one real variable from special classes such equations.

2 Preliminaries

If X is an arbitrary set and $\xi\ : X \to X$ a mapping, the pair (X, ξ) is called a mono-unary algebra. The $n-th$ iteration of ξ is defined for $n \in \mathbb{N}$ by $\xi^1 = \xi$, $\xi^{n+1} = \xi \circ \xi^n$, where $\circ$ stands for the binary operation of composition of functions; ξ^0 is the identity mapping, i.e. $\xi^0 = id_x$. Any self-map $\xi\ : X \to X$ determines the Kuratowski Whybourn equivalence $\sim_\xi$ on X (or KW-equivalence) defined by $x, y \in X$,

$$x \ \sim_\xi \ y \Leftrightarrow \xi^n(x) = \xi^m(y) \text{ for some pair } m, n \in \mathbb{N} \cup \{0\}.$$

Blocks of corresponding partition $X/\sim_\xi$ are called orbits of the mapping ξ or $\xi-$ orbits. Subalgebras $(Y, \xi|Y)$ of the mono-unary algebra (X, ξ), where Y is an orbit, are called components of the algebra (X, ξ). Components of (X, ξ) are maximal connected subalgebras of (X, ξ); here the algebra (X, ξ) is said to be connected if $x \ \sim_\xi \ y$ for any pair of its elements x, y and an algebra (Y, η) is a subalgebra of (X, ξ) if $Y \subseteq X$ and η is a restriction of the mapping ξ onto Y, i.e. $\eta = \xi|Y$. Therefore, if $\{(Y_r, \eta_r); r \in I\}$ is the system of all components of a mono-unary algebra (X, ξ), we write (with respect to a cardinal sum of mono-relational structures)

$$(X, \xi) = \sum_{r \in I} (Y_r, \eta_r).$$

The graphic form of the above expression is also called the Cayley graph of the considered function. The following two examples of connected mono-unary algebras are $(\mathbb{Z}, \nu)$, $(\mathbb{N}, \nu)$, where $\mathbb{Z}$ is set of all integer and $\mathbb{N}$ its subset of all positive integers and $\nu(n) = n + 1$.

We suppose that the function $\varphi, \psi : X \to X$ are given.We say that the function f is solution of equation

$$f \circ \varphi = \psi \circ f \tag{5}$$

on the set X if there is mapping $f : X \to X$ such that for any $x \in X$ holds the equation (5) i.e.

$$f(\varphi(x)) = \psi(f(x)).$$

It means that the solution f is homomorphism of mono-unary algebra (X, φ) on mono-unary algebra (X, ψ). Let us also remind that functions g, h are said to be conjugated if for some bijection $\varphi : R \to R$ we have $\varphi(g(x)) = h(\varphi(x))$ for any $x \in \mathbb{R}$. This means that $\varphi : (R, g) \to (R, h)$ is an isomorphism of mono-unary algebras (R, g), (R, h). We denote conjugacy of functions g, h as $g \backsimeq h$, or more precisely $(R, g) \backsimeq (R, h)$. By the iterated sequence or splinter (or rather ξ-splinter) generated by the element x_0 we mean a sequence $\xi^n(x_0)_{n \in N_0}$, where $\xi : R \to R$ is a function and $x_0 \in R$. Finally $S(R)$ denotes the group of all bijections of R onto itself. As usual, by a monoid we mean a semihypergroup with a unit.

3 Examples

Now we consider the functions which are suitable for demonstration of algebraic construction. This construction operate with Cayley graph of these function and $n - th$ iteration of these.

$$\varphi_a(x) = \frac{ax}{a + |x|}, \quad \text{where} \ \ a > 0, \ \ x \in \mathbb{R} \tag{6}$$

$$\psi_a(x) = \frac{ax}{\sqrt{x^2 + a^2}}, \quad \text{where} \ \ a > 0, \ \ x \in \mathbb{R} \tag{7}$$

$$y = x^{2k+1}, , \quad \text{where} \ \ k \in \mathbb{N}\ x \in R \tag{8}$$

$$y = a^x, \quad \text{where} \ \ a > 0, \ \ x \in \mathbb{R} \tag{9}$$

For this functions we generate these.

Function $\varphi_a(x)$ has the $n - th$ iteration

$$\varphi_a^n(x) = \frac{ax}{a + n\,|x|}$$

and orbit structure of this function has form

$$(\mathbb{R}, \varphi_a) = \{0\} \quad + \sum_{\alpha \in (0,1)} (X_\alpha, \xi_\alpha),$$

where $\xi_\alpha = \varphi_a | X_\alpha$, $\alpha \in (0, 1)$ a $(X_\alpha, \xi_\alpha) \cong (\mathbb{N}, \nu)$.

Similarly the function $\psi_a(x)$ has the $n - th$ iteration

$$\psi_a^n(x) = \frac{ax}{\sqrt{a^2 + nx^2}}$$

and orbit structure of this function has the isomorphic form with the function $\varphi_a(x)$ i.e.

$$(\mathbb{R}, \psi_a) = \{0\} \quad + \sum_{\alpha \in (0,1)} (X_\alpha, \xi_\alpha),$$

where $\xi_\alpha = \varphi_a|X_\alpha$, $\alpha \in (0,1)$ a $(X_\alpha, \xi_\alpha) \cong (\mathbb{N}, \nu)$.

We note that these formulae for the functions $\varphi_a^n(x)$, $\psi_a^n(x)$ hold for all integer n, if for $n = -1$ denotes inverse function and for $-n$, where n is positive denotes of the $n-th$ iteration of inverse function (these functions are defined only on subset of $\mathbb{R}$) . For $n = 0$ the formulae give the identity.

The function $q_k(x) = x^{2k+1}$ has $n-th$ iteration $q_k^n(x) = x^{n(2k+1)}$ and orbit structure of this function has form

$$(\mathbb{R}, q_k) = \{-1\} \quad + \{0\} \quad + \{1\} \quad + \sum_{\alpha \in (0,1)} (X_\alpha, \zeta_\alpha),$$

where $\zeta_\alpha = p_a|X_\alpha$, $\alpha \in (0,1)$ a $(X_\alpha, \zeta_\alpha) \cong (\mathbb{Z}, \nu)$ for every element $(0,1)$.

Finally for the function $p_a(x) = a^x$ we have $n-th$ iteration

$$p_a^n(x) = \underbrace{\ln a \exp(\ldots \ln a \exp(\ln a \exp(x)) \ldots)}_{n\text{times}}$$

and orbit structure of this function has form

$$(\mathbb{R}, p_a) = \sum_{\alpha \in (0,1)} (X_\alpha, \xi_\alpha),$$

where $\xi_\alpha = p_a|X_\alpha$, $\alpha \in (0,1)$ a $(X_\alpha, \xi_\alpha) \cong (\mathbb{N}, \nu)$ for every element $\alpha \in$ (0,1).

4 Solving of equations

Now we use the orbit structure of the functions $\psi_a(x)$ and $\varphi_b(x)$ for specification of the set all solutions of equation

$$f \circ \varphi_b = \psi_a \circ f, \tag{10}$$

which is possible to rewrite in the form

$$f\left(\frac{bx}{b+|x|}\right) = \frac{af(x)}{\sqrt{a^2 + (f(x))^2}}$$

For $x_0 \in G^0_{\varphi_b} = (-\infty, -b\rangle \cup \langle b, \infty)$ there holds $\varphi_b^{-1}(x_0) = \emptyset$. We use the orbit structure the functions φ_b and ψ_a for the construction of all solution f of the equation (10). For every function $\gamma : \quad G^0_{\varphi_b} \to \mathbb{R}$ with property $\gamma(0) = 0$ we construct function f_γ such that:

For $x = 0$ we put $f_\gamma(0) = 0$. Let $x \in \mathbb{R} - \{0\}$, then there is just nonnegative integer n and number $x_0 \in G^0_{\varphi_b}$ with property

$$x = \varphi_b^n(x_0) = \frac{bx_0}{b + n|x_0|}.$$

Then we put

$$f_\gamma(x) = \psi_a^n \gamma(x_0) = \frac{a\gamma(x_0)}{\sqrt{a^2 + n(\gamma(x_0))^2}}.$$

Using this construction we define for $x \in \mathbb{R}$ the function $f_\gamma : \quad \mathbb{R} \to \mathbb{R}$, which satisfies:

$$f_\gamma(\varphi_b(x)) = f_\gamma(\varphi_b^{n+1}(x_0)) = \frac{a\gamma(x_0)}{\sqrt{a^2 + (n+1)(\gamma(x_0))^2}} = \psi_a\left(\frac{a\gamma(x_0)}{\sqrt{a^2 + n(\gamma(x_0))^2}}\right) = \psi_a(f_\gamma(x))$$

for every $x \in \mathbb{R}$. Consequently the function f_γ is solution of the the functional equation (10). In this case is possible easy state some relations between the properties of the function γ and the solution f_γ.

At the first the solution f_γ is isomorphism if and only if the function $\gamma : G^0_{\varphi_b} \to G^0_{\psi_a}$ is bijection. Then inverse transformation $g = f^{-1}$ is the solution of equation

$$\varphi_b \circ g = g \circ \psi_a. \tag{11}$$

For study of continuity of solution f_γ is suitable denote

$$G^n_{\varphi_b} = \varphi_b^n(G^0_{\varphi_b}) = \left(-\frac{b}{n}, -\frac{b}{n+1}\right\rangle \cup \left\langle\frac{b}{n+1}, \frac{b}{n}\right).$$

The continuity of solution in interior of intervals $G^n_{\varphi_b}$ is equivalent with the continuity on the intervals $G^0_{\varphi_b}$ i.e. with continuity of function γ. Continuity of solution in the point of the boundary of intervals is equivalent the condition $f_\gamma\left(-\frac{b}{n}\right) = \lim_{x \to -\frac{b}{n}+} f_\gamma(x)$ which is equivalent for $n \in \mathbb{N}$

$$\frac{b\gamma(-b)}{\sqrt{b^2 + (n-1)(\gamma(-b))^2}} = \lim_{x \to -\infty} \frac{b\gamma(x)}{\sqrt{b^2 + n(\gamma(x))^2}}.$$

and this is equivalent the existence the finite limit $\gamma(-\infty) = \lim_{x\to-\infty} \gamma(x)$ satisfying the relation

$$\gamma(-b) = \frac{b\gamma(-\infty)}{\sqrt{b^2 + (\gamma(-\infty))^2}}$$

At the second the solution f_γ is continuous if nad only if the function γ is continuous and there are finite limits $\gamma(\pm\infty) = \lim_{x\to\pm\infty} \gamma(x)$ satisfying the relation

$$\gamma(\pm b) = \frac{b\gamma(\pm\infty)}{\sqrt{b^2 + (\gamma(\pm\infty))^2}}$$

Using the analogous consideration we may discussed the equations which contain other pair of functions which Cayley graphs were above described.

Since for components which are isomorphic with algebra (Z, ν_z) do not exist homomorphisms onto components which are isomorphic with algebra (N, ν) and there is no homomorphism which maps the simple loop in the infinite components, the functional equations $a^{f(x)} = f(\varphi_a(x))$, $a^{f(x)} = f(\psi_a(x))$, $a^{f(x)} = f(x^{2n+1})$ has no solution but for the restriction of functions which are in the equations $a^{f(x)} = f(\varphi_a(x))$ and $a^{f(x)} = f(\psi_a(x))$ onto the set $R - \{0\}$ the set of solution of these equations has the cardinality of continuum. and equations $f(x^{2n+1} = \varphi(f)$ and $f(x^{2n+1}) = \psi(f)$ has just one trivial solution. The other equation have the set of solutions with the cardinality of continuum.

5 P-hypergroups and P-hypersemigroups

In the paper [24] the authors define so called P-hypergroups in the connection with cyclic hypergroups. This useful structure is developed in other published papers e.g. [11]. Recall the basic concept which can be generalized as a P-semihypergroups.

Let $(G, \cdot)$ be an arbitrary semigroup and $\emptyset \neq P \subseteq G$, i.e. P is an arbitrary nonempty subset of G

Define the hyperoperation as: $*^P : \ G \times G \to \mathcal{P}(G)$ such that

$$x *^P y = x \cdot P \cdot y = \{x \cdot p \cdot y; p \in P\},$$

for any pair of elements $x, y \in G$. This operation is associative: for any triad $x, y, z \in G$ there holds

$$x *^P (y *^P z) = x \cdot P \cdot y \cdot P \cdot z = (x *^P y) *^P z.$$

The structure $(G, *^P)$ is called P-semihypergroup. If this hyperoperation satisfies reproduction axiom:

$$x *^p G = G = G *^P x \quad \text{for any } x \in G$$

above defined P-semihyergroup is P-hypergroup.

In the case of centralizer functional equations of one variable it is natural the solution sets endow with an appropriate algebraic structure. In fact a centralizer of a function φ within the ring of all real functions is the monoid of all functions $f \ : \ \mathbb{R} \to \mathbb{R}$ commuting with φ. In the case of solution sets of functional equations of the form

$$f \circ \varphi_1 = \varphi_2 \circ f,$$

i.e. $f(\varphi(x)) = \varphi(f(x))$, $x \in J$, $J \subseteq \mathbb{R}$, where $\varphi_1, \varphi_2 \ \ J \to \mathbb{R}$ are given different functions we have the another natural possibily. In order to define binary structure on solution sets of the just mention equations which is based on the binary operation of composition of function we are concerning to the concept of a sandwich semigroup. These semigroups were investigated since the end of sixtieth (by Kenneth D.Magill Jr. - 1967) of the last century; let us mentioned at least papers [9, 12, 13] and recall the basic notion.

According to paper [9] for any element a of a semigroup S we may define a "sandwich" operation $\cdot$ on the set S by $x \cdot y = xay$, $x, y \in S$. Under this operation the set S is again a semigroup; it is denoted by (S, a) and called a variant of S (in [9]). A certain generalization of a variant suitable for our purposes are sandwich semigroups investigated in papers of Kenneth D. Magill, Jr. and his collaborates.

Consider the following two functional equations (10) and (11) of one real variable with two real parameters a, b.

Let $f_0 : \mathbb{R} \to \mathbb{R}$ be an arbitrary solution of the functional equation (10) which is surjective, i.e. $f_0(\mathbb{R}) = \mathbb{R}$ it has a fixed point 0, i.e. $f_0(0) = 0$. Denote by $S(\varphi_b, \psi_a)$ the solution set of the equation (11) in which we define the following binary operation:

For an arbitrary pair $g_1, g_2 \in S(\varphi_b, \psi_a)$ we put $g_1 \cdot g_2 = g_1 \circ f_0 \circ g_2$. Then we get a sandwich semigroup will be denoted by $S(\varphi_b, \psi_a, g_0) = (S(\varphi_b, \psi_a), \cdot)$. Evidently

$$\varphi_b \circ g_1 \cdot g_2 = \varphi_b \circ g_1 \circ f_0 \circ g_2 = g_1 \circ \psi_a \circ f_0 \circ g_2 = g_1 \circ f_0 \circ \varphi_b \circ g_2 = \\ g_1 \circ f_0 \circ g_2 \circ \psi_a = g_1 \cdot g_2 \circ \psi_a \;\Rightarrow\; g_1 \cdot g_2 \in S(\varphi_b, \psi_a)$$

As the composition of mapping is associative we see that obtained structure is semigroup. Analogous way for the set bijective solutions of the equation (11) $(S_B(\varphi_b, \psi_a)$ we derive subsemigroup $S_B(\varphi_b, \psi_a, g_0)$. For the same function this structure turns into monoid $((S\varphi_a, \varphi_a), \cdot)$ or semigroup $((S_B\varphi_a, \varphi_a), \cdot)$.

Now we suppose that is given any nonempty subset $F \subseteq S(\psi_a, \varphi_b)$ and we define From the above mentioned results contained in the monograph [5] there follows the following binary hyperoperation:

For any pair $g_1, g_2 \in S(\varphi_b, \psi_a)$ we put $g_1 *^F g_2 = \{g_1 \circ f \circ g_2 \mid f \in F\}$. Then we get a sandwich hypersemigroup which will be denoted by $S(\varphi_b, \psi_a, F) = (S(\varphi_b, \psi_a), *^F)$. For this hypersemigroup $S(\varphi_b, \psi_a, F)$ is possible by suitable choose $F = S_B(\psi_a, \varphi_b)$ obtain that this structure includes the set of neutral elements which coincides with $S_B(\psi_a, \varphi_b)$.

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COMMUTATIVE HYPERSTRUCTURES CONSTRUCTED ON A SET OF TRANSFORMATION OPERATORS

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Abstract. The contribution is related to the theory of transformation operators of complex functions of one variable defined on a compact set. Some "natural" operations (sum, product, real multiple) on the set of all such operators are discussed. Using these operations "natural" commutative hyperoperations on the set of transformation operators are defined and respective structures are studied.

Key words and phrases. hypergroup, hyperoperation, ordered structures, semihypergroup, transformation operator.

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1 Motivation

Motivated by the use of concepts such as Laplace and Fourier transforms in a number of applications studied at faculties of electrical engineering of universities of technology, contributions such as [5], [6], [7], [8], [9] deal with the concept of operators of certain types. Contributions [7], [8] and [9] develop the theory of transformation operators

$$T(\lambda, F, \varphi) : \mathbb{C}^\Omega \to \mathbb{C}^\Omega,$$

where $\lambda \in \mathbb{C}$, and $F, \varphi \in \mathbb{C}^\Omega$, i.e. $F, \varphi \in \{f : \Omega \to \mathbb{C}\}$, where $\emptyset \neq \Omega \subseteq \mathbb{C}$ is a compact, i.e. closed and bounded, set. Notice that $\mathbb{C}^\Omega$ is a normed algebra of all complex functions of one variable defined on Ω. The operators $T(\lambda, F, \varphi)$ are defined by

$$T(\lambda, F, \varphi)(f(z)) = \lambda F(z)f(z) + \varphi(z),$$

for any $f \in \mathbb{C}^\Omega$ and any $z \in \mathbb{C}$, which is usually shortened to the form

$$T(\lambda, F, \varphi)(f) = \lambda F f + \varphi.$$

The set of all such operators is denoted by $\mathcal{T}(\Omega)$.

So far, non-commutative operations and hyperoperations on $\mathcal{T}(\Omega)$ have been studied. In [8] the operation "." defined by (using a slightly different notation)

$$T_1(\lambda, F, \varphi).T_2(\mu, G, \psi) = T(\lambda\mu, \lambda G + F, \varphi\psi)$$

is defined and investigated. Other contributions, especially [7] and [9], study the issue of – again non-commutative – composition of transformation operators, for which the equality

$$T_1(\lambda, F, \varphi) \circ T_2(\mu, G, \psi) = T(\lambda\mu, FG, \lambda F\psi + \varphi)$$

holds. Using this concept, iterations T^n of transformation operators have been introduced in Lemma 2 of [7]. Since composition of transformation operators is a non-commutative operation, equivalent conditions for the fact that operators $T_1(\lambda, F, \varphi)$ and $T_2(\mu, G, \psi)$ commute have been formulated in Lemma 3 of [7].

Contributions [7] and [9] define certain binary hyperoperations on the centralizer of the operator T_0, i.e. on the set

$$Ct_{\mathcal{T}}(T_0) = \{T(\lambda, F, \varphi); T(\lambda, F, \varphi) \circ T_0 = T_0 \circ T(\lambda, F, \psi)\}.$$

In further contributions and articles, these hyperoperations are used in constructions of certain automata, whose input sets are the couples of $Ct_{\mathcal{T}}(T_0)$ and the hyperoperations.

This article introduces and studies certain *commutative* operations and hyperoperations on the set of transformation operators $T(\lambda, F, \varphi)$.

2 Preliminaries

Recall first some basic definitions and ideas from the hyperstructures theory. A *hypergrupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet : H \times H \to \mathcal{P}^*(H)$ is a binary hyperoperation on H. Symbol $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H. If the associativity axiom $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$, then the pair $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom: for any element $a \in H$ equalities $a \bullet H = H = H \bullet a$ hold, is satisfied, then the pair $(H, \bullet)$ is called a *hypergroup*. A hypergroup $(H, \bullet)$ is called a *transposition hypergroup* if it satisfies the following transposition axiom: For all $a, b, c, d \in H$ the relation $b\backslash a \approx c/d$ implies $a \bullet d \approx b \bullet c$, where $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$. Sets $b\backslash a = \{x \in H; a \in b \bullet x\}$ and $c/d = \{x \in H; c \in x \bullet d\}$ are called *left* and *right extensions*, or *fractions*, respectively. A commutative transposition hypergroup is called a *join space* (sometimes the terms *transposition hypergroup* and *join space* are treated as equivalents and adjective *commutative* / *non-commutative* are used). A hypergroupoid $(H, \bullet)$, where a reproduction axiom holds, is called a *quasi-hypergroup*.

If $(H, \bullet)$ is a commutative hypergroupiod, the test of associativity may be simplified as the following lemma holds (for proof cf. [4], p. 151):

Lemma 2.1 *In a commutative hypergroupoid* $(H, \bullet)$ *the following assertions are equivalent:*

1^0 *There holds* $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ *for any triad* $a, b, c \in H$.

2^0 *There holds* $(a \bullet b) \bullet c \subseteq a \bullet (b \bullet c)$ *for any triad* $a, b, c \in H$.

3^0 *There holds* $a \bullet (b \bullet c) \subseteq (a \bullet b) \bullet c$ *for any triad* $a, b, c \in H$.

A connection between hyperstructures and ordered structures can been established using the following simple but important construction from [4]. Notice that this construction has been used in a number of works by Chvalina, Chvalinová, Moučka, Hošková, Račková, Novák and others. By a *quasi-ordered (semi)group* we mean a triple $(G, \cdot, \leq)$, where $(G, \cdot)$ is a (semi)group and $\leq$ is a reflexive and transitive binary relation on G such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ hold. Further, $[a)_{\leq} = \{x \in G; a \leq x\}$ is a principal end generated by $a \in G$. The following lemma holds (for proof cf. [4], pp. 146–7):

Lemma 2.2 *Let a triple* $(G, \cdot, \leq)$ *be a quasi-ordered semigroup. Define a hyperoperation*

$$* : G \times G \to \mathcal{P}^*(G) \qquad \text{by} \qquad a * b = [a \cdot b)_{\leq} = \{x \in G; a \cdot b \leq x\}$$

for all pairs of elements $a, b \in G$.

1. *Then* $(G, *)$ *is a semihypergroup which is commutative if the semigroup* $(G, \cdot)$ *is commutative.*

2. *Let* $(G, *)$ *be the above defined semihypergroup. Then* $(G, *)$ *is a hypergroup if and only if for any pair of elements* $a, b \in G$ *there exists a pair of elements* $c, c' \in G$ *with a property* $a \cdot c \leq b, c' \cdot a \leq b$.

Remark 2.3 *Notice that if* $(G, \cdot, \leq)$ *is a quasi-ordered group then the condition stated under 2. is satisfied, hence the final hyperstructure is a hypergroup.*

In the article we attempt at creating hyperstructures with identities or scalar identities. An element of $e \in H$, where $(H, \bullet)$ is a hyperstructure, is called an *identity* if for $\forall x \in H$ there holds $x \bullet e \ni x \in e \bullet x$. If for $\forall x \in H$ there holds $x \bullet e = \{x\} = e \bullet x$, then $e \in H$ is called a *scalar identity*.

3 New results

Throughout the article notation $\mathcal{T}_{\lambda \neq 0}(\Omega)$ will mean the subset of $\mathcal{T}(\Omega)$ such that if $T(\lambda, F, \varphi) \in \mathcal{T}_{\lambda \neq 0}(\Omega)$, then $\lambda \neq 0$. If not explicitely stated otherwise, the shortened way of defining the transformation operator, i.e. $T(\lambda, F, \varphi)(f) = \lambda F f + \varphi$, will be used.

First, let us study the sum of two operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda \neq 0}(\Omega)$. For any two operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda \neq 0}(\Omega)$ and any function $f \in \mathbb{C}^{\Omega}$ we have

$$T_1(\lambda, F, \varphi)(f) + T_2(\mu, G, \psi)(f) = \lambda F f + \varphi + \mu G f + \psi = \lambda\mu(\tfrac{1}{\mu}F + \tfrac{1}{\lambda}G)f + \varphi + \psi = \\ = T(\lambda\mu, \tfrac{1}{\mu}F + \tfrac{1}{\lambda}G, \varphi + \psi)(f).$$

Therefore define a "natural sum $\oplus$" for any two operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ in the following way:

$$T_1(\lambda, F, \varphi) \oplus T_2(\mu, G, \psi) = T(\lambda\mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, \varphi + \psi).$$

Proposition 3.1 *$(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus)$ is a commutative group with neutral element $T(1, 0, 0)$.*

Proof. Suppose $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi), T_3(\kappa, H, \rho) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ are arbitrary. Then

1. commutativity is obvious as $T_1(\lambda, F, \varphi) \oplus T_2(\mu, G, \psi) = T(\lambda\mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, \varphi + \psi) = T(\mu\lambda, \frac{1}{\lambda}G + \frac{1}{\mu}F, \psi + \varphi) = T_2(\mu, G, \psi) \oplus T_1(\lambda, F, \varphi)$,
2. associativity axiom is also valid as

   a) $[T_1(\lambda, F, \varphi) \oplus T_2(\mu, G, \psi)] \oplus T_3(\kappa, H, \rho) = T(\lambda, \mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, \varphi + \psi) \oplus T_3(\kappa, H, \rho) = T(\lambda\mu\kappa, \frac{1}{\kappa}(\frac{1}{\mu}F + \frac{1}{\lambda}G) + \frac{1}{\lambda\mu}H, \varphi + \psi + \rho) = T(\lambda\mu\kappa, \frac{1}{\mu\kappa}F + \frac{1}{\lambda\kappa}G + \frac{1}{\lambda\mu}H, \varphi + \psi + \kappa)$,

   b) $T_1(\lambda, F, \varphi) \oplus [T_2(\mu, G, \psi) \oplus T_3(\kappa, H, \rho)] = T_1(\lambda, F, \varphi) \oplus T(\mu\kappa, \frac{1}{\kappa}G + \frac{1}{\mu}H, \psi + \rho) = T(\lambda\mu\kappa, \frac{1}{\mu\kappa}F + \frac{1}{\lambda}(\frac{1}{\kappa}G + \frac{1}{\mu}H), \varphi + \psi + \rho) = T(\lambda\mu\kappa, \frac{1}{\mu\kappa}F + \frac{1}{\lambda\kappa}G + \frac{1}{\lambda\mu}H, \varphi + \psi + \kappa)$.
3. for $T_1(\lambda, F, \varphi) \oplus T(1, 0, 0) = T(\lambda, \frac{1}{1}F + \frac{1}{\lambda}0, \varphi + 0) = T(1, 0, 0) \oplus T_1(\lambda, F, \varphi)$, thus the operator $T(1, 0, 0)$ is the neutral element of $(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus)$,
4. since $T_1(\lambda, F, \varphi) \oplus T(\frac{1}{\lambda}, -\lambda^2 F, -\varphi) = T(\lambda\frac{1}{\lambda}, \frac{1}{\frac{1}{\lambda}}F + \frac{1}{\lambda}(-\lambda^2 F), \varphi + (-\varphi)) = T(1, 0, 0)$ and the operation $\oplus$ is commutative, then to every operator there exists an inverse element from $\mathcal{T}_{\lambda\neq 0}(\Omega)$.

Therefore $(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus)$ is a commutative group.

Remark 3.1 *In contributions [7] and [9], in which the binary operation of composition of operators is studied, it is the element $T(1, 1, 0)$ that plays the role of the neutral element. Since for an arbitrary $f \in \mathbb{C}^\Omega$ we have $T(1, 1, 0)(f) = f$, this is an identity operator. The meaning of $T(1, 0, 0)$ is different, though. For an arbitrary $f \in \mathbb{C}^\Omega$ we have that $T(1, 0, 0)(f) \equiv 0$, i.e. the operator maps any function to a zero function.*

In a number of works dealing with ordered structures, hyperoperations and hyperstructeres are constructed using an important result proved in [4], which has been included above as Lemma 2.2. Let us now inspect whether the group $(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus)$ is qusi-ordered.

For an arbitrary pair of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ define that $T_1 \leq T_2$ if simultaneously $|\lambda| \leq |\mu|$ and $|\frac{1}{\mu}F| \leq |\frac{1}{\lambda}G|$, where the latter signs $\leq$ denote normal ordering of numbers or functions respectively. This is in fact only a slight variation of the quasi-ordering suggested in [9] for a different structure.

Proposition 3.2 *The commutative group $(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus, \leq)$ is a quasi-ordered group.*

Proof. Suppose and arbitrary triple of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi), T_3(\kappa, H, \rho) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ such that $T_1 \leq T_2$ and $T_2 \leq T_3$. Then

1. the relation $\leq$ is obviously reflexive,
2. the fact that $T_1(\lambda, F, \varphi) \leq T_2(\mu, G, \psi)$ implies $|\lambda| \leq |\mu|$ and $|\frac{1}{\mu}F| \leq |\frac{1}{\lambda}G|$ while the fact that $T_2(\mu, G, \psi) \leq T_3(\kappa, H, \rho)$ implies that $|\mu| \leq |\kappa|$ and $|\frac{1}{\kappa}G| \leq |\frac{1}{\mu}H|$. Then
   a) the fact that $|\lambda| \leq |\kappa|$ is obvious,
   b) since $|\lambda| \leq |\mu|$ is equivalent to $|\frac{1}{\mu}| \leq |\frac{1}{\lambda}|$ and since $|\frac{1}{\mu}F| \leq |\frac{1}{\lambda}G|$ holds, we get that $|F| \leq |G|$. In the same way we get that $|G| \leq |H|$. Since the relation $\leq$ on the set of functions is transitive, the relation $\leq$ defined on $\mathcal{T}_{\lambda\neq 0}(\Omega)$ is transitive as well.
3. Suppose now that $T_1 \leq T_2$ and $T(\kappa, H, \rho)$ is arbitrary. Then $T_1 \oplus T = T_1(\lambda, F, \varphi) \oplus T(\kappa, H, \rho) = T(\lambda\kappa, \frac{1}{\kappa}F + \frac{1}{\lambda}H), \varphi + \rho$ and $T_2(\mu, G, \psi) \oplus T(\kappa, H, \rho) = T(\mu\kappa, \frac{1}{\kappa}G + \frac{1}{\mu}H, \psi + \rho)$ and
   a) since $T_1 \leq T_2$, there holds $|\lambda| \leq |\mu|$, i.e. also $|\lambda\kappa| \leq |\mu\kappa|$.
   b) Further we need to verify that $|\frac{1}{\mu\kappa}(\frac{1}{\kappa}F + \frac{1}{\lambda}H)| \leq |\frac{1}{\lambda\kappa}(\frac{1}{\kappa}G + \frac{1}{\mu}H)|$. However, $|\frac{1}{\mu\kappa^2}F + \frac{1}{\lambda\mu\kappa}H| \leq |\frac{1}{\lambda\kappa^2}G + \frac{1}{\lambda\mu\kappa}H|$ is equivalent to $|\frac{1}{\mu}F| \leq |\frac{1}{\lambda}G|$, which is supposed to hold.

   Therefore $T_1 \oplus T \leq T_2 \oplus T$ and since the operation $\oplus$ is commutative, there also holds $T \oplus T_1 \leq T \oplus T_2$.

Therefore we may define a hyperoperation $\bullet$ on $\mathcal{T}_{\lambda\neq 0}(\Omega)$ in the following way: for any two operators $T_1(\lambda, F, \varphi)$, $T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ we set

$$T_1 \bullet T_2 = [T_1 \oplus T_2)_{\leq} = \{T \in \mathcal{T}_{\lambda\neq 0}(\Omega) : T_1 \oplus T_2 \leq T\}.$$

Theorem 3.2 *$(\mathcal{T}_{\lambda\neq 0}(\Omega), \bullet)$ is a join space with an identity $T(1, 0, 0)$.*

Proof. Denote $T_1(\lambda, F, \varphi)$ an arbitrary operator from $\mathcal{T}_{\lambda\neq 0}(\Omega)$. Then:

1. Since $(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus)$ is a commutative group, the fact that $(\mathcal{T}_{\lambda\neq 0}(\Omega), \bullet)$ is a commutative hypergroup follows from lemma 2.2 and the above considerations.
2. In order to show that $(\mathcal{T}_{\lambda\neq 0}(\Omega), \bullet)$ is a join space we have to further show that for an arbitrary quadruple $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi), T_3(\kappa, H, \rho), T_4(\sigma, K, \xi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ the fact that $T_2 \backslash T_1 \cap T_3 / T_4 \neq \emptyset$ implies that $T_1 \bullet T_4 \cap T_2 \bullet T_3 \neq \emptyset$. However

$$T_1 \bullet T_4 = \{R \in (\mathcal{T}_{\lambda\neq 0}(\Omega) : T_1 \oplus T_4 \leq R\} = \{R \in (\mathcal{T}_{\lambda\neq 0}(\Omega) : T(\lambda\sigma, \tfrac{1}{\sigma}F + \tfrac{1}{\lambda}K, \varphi + \xi) \leq R\} =$$
$$= \{R(\beta, C, \tau) \in \mathcal{T}_{\lambda\neq 0}(\Omega) : |\lambda\sigma| \leq |\beta|, |\tfrac{1}{\beta}(\tfrac{1}{\sigma}F + \tfrac{1}{\lambda}K)| \leq |\tfrac{1}{\lambda\sigma}C|\}$$
$$T_2 \bullet T_3 = \{S \in \mathcal{T}_{\lambda\neq 0}(\Omega) : T_2 \oplus T_3 \leq S\} = \{S \in \mathcal{T}_{\lambda\neq 0}(\Omega) : T(\mu\kappa, \tfrac{1}{\kappa}G + \tfrac{1}{\mu}H, \psi + \rho) \leq S\} =$$
$$= \{S(\delta, D, \epsilon) \in \mathcal{T}_{\lambda\neq 0}(\Omega) : |\mu\kappa| \leq |\delta|, |\tfrac{1}{\delta}(\tfrac{1}{\kappa}G + \tfrac{1}{\mu}H)| \leq |\tfrac{1}{\mu\kappa}D|\}$$

and if we now regard an operator $U(\max\{|\lambda\sigma|,|\mu\kappa|\},|\frac{1}{\sigma}F+\frac{1}{\lambda}K|+|\frac{1}{\kappa}G+\frac{1}{\mu}H|,\omega)$, where ω is an arbitrary function from $\mathbb{C}^{\Omega}$, then obviously $U\in\mathcal{T}_{\lambda\neq 0}(\Omega)$ and it can be easily verified that $U\in T_1\bullet T_4\cap T_2\bullet T_3$. Therefore the transposition axiom holds trivially.

3. For an arbitrary $T_1(\lambda,F,\varphi)\in\mathcal{T}_{\lambda\neq 0}(\Omega)$ there holds

$$T_1(\lambda,F,\varphi)\bullet T(1,0,0)=T(1,0,0)\bullet T_1(\lambda,F,\varphi)=[T\oplus T_1)_{\leq}=[T_1)_{\leq},$$

because $T(1,0,0)$ is a neutral element of the group $(\mathcal{T}_{\lambda\neq 0}(\Omega),\oplus)$. Since for an arbitrary $T_1(\lambda,F,\varphi)\in\mathcal{T}_{\lambda\neq 0}(\Omega)$ there holds $T_1\in[T_1)_{\leq}$, operator $T(1,0,0)$ is an identity of the hypergroup $(\mathcal{T}_{\lambda\neq 0}(\Omega),\bullet)$.

Remark 3.3 *However, $T(1,0,0)$ is not a* scalar *identity because $[T_1)_{\leq}\neq T_1$.*

Further, we may study the product of two operators. Regard an arbitrary pair of operators $T_1(\lambda,F,\varphi),T_2(\mu,G,\psi)\in\mathcal{T}_{\lambda\neq 0}(\Omega)$ and an arbitrary function $f\in\mathbb{C}(\Omega)$. We get that

$$T_1(\lambda,F,\varphi)(f)\cdot T_2(\mu,G,\psi)(f)=(\lambda fF+\varphi)\cdot(\mu Gf+\psi)=\lambda\mu GFf^2+\lambda F\psi f+\mu G\varphi f+\varphi\psi,$$

which – if we ignore the square of f – may be rewritten to

$$[\lambda\mu(FG+\frac{1}{\mu}F\psi+\frac{1}{\lambda}G\varphi)]f+\varphi\psi=T(\lambda\mu,FG+\frac{1}{\mu}F\psi+\frac{1}{\lambda}G\varphi,\varphi\psi).$$

We may therefore define a new operation $\odot$ on the set of $\mathcal{T}_{\lambda\neq 0}(\Omega)$ by

$$T_1(\lambda,F,\varphi)\odot T_2(\mu,G,\psi)=T(\lambda\mu,FG+\frac{1}{\mu}F\psi+\frac{1}{\lambda}G\varphi,\varphi\psi)$$

for any two operators T_1,T_2.

Proposition 3.3 *$(\mathcal{T}_{\lambda\neq 0}(\Omega),\odot)$ is a commutative semigroup with neutral element $T(1,0,1)$.*

Proof. Let $T_1(\lambda,F,\varphi),T_2(\mu,G,\psi),T_3(\kappa,H,\rho)$ be arbitrary operators from $\mathcal{T}_{\lambda\neq 0}(\Omega)$ and $\odot$ the above defined operation. Then

1. commutativity of the operation $\odot$ is obvious,

2. associativity may be proved as follows:

   a) $[T_1(\lambda,F,\varphi)\odot T_2(\mu,G,\psi)]\odot T_3(\kappa,H,\rho)=T(\lambda\mu,FG+\frac{1}{\mu}F\psi+\frac{1}{\lambda}G\varphi,\varphi\psi)\odot T(\kappa,H,\rho)=$ $T(\lambda\mu\kappa,H(FG+\frac{1}{\mu}F\psi+\frac{1}{\lambda}G\varphi)+\frac{1}{\kappa}(FG+\frac{1}{\mu}F\psi+\frac{1}{\lambda}G\varphi)\rho+\frac{1}{\lambda\mu}H\varphi\psi,\varphi\psi\rho)$

   b) $T_1(\lambda,F,\varphi)\odot[T_2(\mu,G,\psi)\odot T_3(\kappa,H,\rho)]=T_1(\lambda,F,\varphi)\odot T(\mu\kappa,GH+\frac{1}{\kappa}G\rho+\frac{1}{\mu}H\psi,\psi\rho)=$ $T(\lambda\mu\kappa,F(GH+\frac{1}{\kappa}G\rho+\frac{1}{\mu}H\psi)+\frac{1}{\mu\kappa}F\psi\rho+\frac{1}{\lambda}(GH+\frac{1}{\kappa}G\rho+\frac{1}{\mu}H\psi)\varphi,\varphi\psi\rho)$,

   which – as can be seen after we expand their second components – are two identical operators.

3. the fact that $T(1,0,1)$ is a neutral element of the operation $\odot$ can be verified easily by simple evaluating $T_1(\lambda, F, \varphi) \odot T(1,0,1)$. Since the operation is commutative the other equality need not be verified.

Remark 3.4 *The meaning of the operator* $T(1,0,1)$ *is for any function* $f \in \mathbb{C}^\Omega$ *the following:* $T(1,0,1)(f) \equiv 1$. *The semigroup* $(\mathcal{T}_{\lambda\neq 0}(\Omega), \odot)$ *is not a group because the inverse operators would have their last component* $\psi = \frac{1}{\varphi}$, *the existence of which may not be guaranteed due to the issue of zero points.*

As in the case of the operation $\oplus$ the hyperoperation will be constructed with the help of an ordering. Define therefore that $T_1(\lambda, F, \varphi) \leq_3 T_2(\mu, G, \psi)$ if $|\varphi| \leq |\psi|$, i.e. we say that one operator is smaller that the other if it shifts each function less.

Proposition 3.4 *The commutative semigroup* $(\mathcal{T}_{\lambda\neq 0}(\Omega), \odot, \leq_3)$ *is a quasi-ordered semigroup.*

Proof. Reflexivity and transitivity of the relation $\leq_3$ is obvious. Further, regard an arbitrary pair of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ such that $T_1 \leq_3 T_2$ and an arbitrary operator $T \in \mathcal{T}_{\lambda\neq 0}(\Omega)$. Then $T_1 \odot T = T(\lambda\mu, FH + \frac{1}{\kappa}F\rho + \frac{1}{\lambda}H\varphi, \varphi\rho)$ and $T_2 \odot T = T(\mu\kappa, GH + \frac{1}{\kappa}G\rho + \frac{1}{\mu}H\psi, \psi\rho)$, i.e. $T_1 \odot T \leq_3 T_2 \odot T$, because since $|\varphi| \leq |\psi|$ then also $|\varphi\rho| \leq |\psi\rho|$. Since the operation $\odot$ is commutative, the other inequality need not be verified.

Remark 3.5 *We could also define that* $T_1(\lambda, F, \varphi) \leq_{13} T_2(\mu, G, \psi)$ *if* $|\lambda| \leq |\mu|$ *and* $|\varphi| \leq |\psi|$. *Then it can be easily verified that* $(\mathcal{T}_{\lambda\neq 0}(\Omega), \odot, \leq_{13})$ *is a quasi-ordered semigroup too.*

Define now another hyperoperation $\circledast$ by the following: for an arbitrary pair of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ set

$$T_1 \circledast T_2 = [T_1 \odot T_2)_{\leq} = \{T \in \mathcal{T}_{\lambda\neq 0}(\Omega) : T_1 \odot T_2 \leq_3 T\}.$$

Theorem 3.6 $(\mathcal{T}_{\lambda\neq 0}(\Omega), \circledast)$ *is a commutative semihypergroup with an identity* $T(1,0,1)$.

Proof. The proof is similar to the proof of Theorem 3.2. It follows from Lemma 2.2 and repeats part 3 of the proof of Theorem 3.2. Furthermore, a remark analogous to Remark 3.3 may be formulated.

Now we examine the real multiple of an operator. Suppose arbitrary $T(\lambda, F, \varphi) \in (\mathcal{T}_{\lambda\neq 0}(\Omega)$ and $m \in \mathbb{R}$, $m \neq 0$. Then

$$m[T(\lambda, F, \varphi)(f)] = m[\lambda F f + \varphi] = m\lambda F f + m\varphi = T(m^k\lambda, \frac{1}{m^{k-1}}F, m\varphi)$$

for an arbitrary $k \in \mathbb{R}$. Since all operators of such a form are identical, we may set

$$m \cdot T(\lambda, F, \varphi) = T(m\lambda, F, m\varphi)$$

as a "natural real multiple" of an operator. The following simple result is almost obvious:

Lemma 3.7 *For an arbitrary* $m \in \mathbb{R}$ *and an arbitrary pair of operators* $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ *there holds*

$$m \cdot (T_1 \oplus T_2) = m \cdot T_1 \oplus m \cdot T_2$$

Proof. Suppose an arbitrary $m \in \mathbb{R}$ and an arbitrary pair of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$. Then:

1. $m \cdot (T_1 \oplus T_2) = m \cdot T(\lambda\mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, \varphi + \psi) = T(m\lambda\mu, \frac{1}{\mu}F + \lambda\frac{1}{\lambda}G, m(\varphi + \psi))$

2. $m \cdot T_1 \oplus m \cdot T_2 = T(m\lambda, F, m\varphi) \oplus T(m\mu, G, m\psi) = T(m^2\lambda\mu, \frac{1}{m\mu}F + \frac{1}{m\lambda}G, m\varphi + m\psi) = T(m^2\lambda\mu, \frac{1}{m}(\frac{1}{\mu}F + \frac{1}{\lambda}G), m(\varphi + \psi)) = T(m\lambda\mu, \frac{1}{\mu}F + \lambda\frac{1}{\lambda}G, m(\varphi + \psi))$.

Let us now define hyperoperation $\circledast$ using the above multiplication $\cdot$ in the following way: for any two operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ we set

$$T_1 \circledast T_2 = \{m \cdot (T_1 \oplus T_2), m \in \mathbb{N}\} = \{T(m\lambda\mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, m(\varphi + \psi)), m \in \mathbb{N}\}$$

Theorem 3.8 $(\mathcal{T}_{\lambda\neq 0}(\Omega), \circledast)$ *is a commutative quasi-hypergroup with an identity* $T(1, 0, 0)$.

Proof. The commutativity of the operation is obvious. Therefore suppose an arbitrary operator $T(\lambda, F, \varphi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$. Then

1. we get that

   $$T \circledast \mathcal{T}_{\lambda\neq 0}(\Omega) = \{T(m\lambda\mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, m(\varphi + \psi)), m \in \mathbb{N}, \mu \in \mathbb{C}, \mu \neq 0, G, \psi \in \mathbb{C}^{\Omega} \text{ arbitrary}\},$$

   therefore both inclusions $T \circledast \mathcal{T}_{\lambda\neq 0}(\Omega) \subseteq \mathcal{T}_{\lambda\neq 0}(\Omega)$ and $\mathcal{T}_{\lambda\neq 0}(\Omega) \subseteq T \circledast \mathcal{T}_{\lambda\neq 0}(\Omega)$ hold, which means that the reproduction axiom holds,

2. $T_1(\lambda, F, \varphi) \circledast T(1, 0, 0) = \{T(m\lambda, F, m\varphi), m \in \mathbb{N}\}$, i.e. $T \in T_1 \circledast T$. Since the operation $\circledast$ is commutative, we have also that $T \in T \circledast T_1$, i.e. $T(1, 0, 0)$ is an identity.

If we wanted to construct a commutative quasi-hypergroup with a *scalar* identity, we could use the operation $\cdot$ in the following way. For an arbitrary pair of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ define number t as follows:

$$t = \begin{cases} \min\{|\lambda|, |\mu|\} & \text{if } |\lambda| > 1 \wedge |\mu| > 1 \\ 1 & \text{if } |\lambda| \leq 1 \vee |\mu| \leq 1 \end{cases}$$

and define a new hyperoperation $*$ on $\mathcal{T}_{\lambda\neq 0}(\Omega)$ by

$$T_1 * T_2 = \{m \cdot (T_1 \oplus T_2), m \leq t, m \in \mathbb{N}\} = \{T(m\lambda\mu, \frac{1}{\mu}F + \frac{1}{\lambda}G, m(\varphi + \psi)), m \leq t, m \in \mathbb{N}\}$$

for any two operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi)$.

Theorem 3.9 $(\mathcal{T}_{\lambda\neq 0}(\Omega), *)$ *is a commutative quasi-hypergroup with scalar identity* $T(1,0,0)$.

Proof. Commutativity of the hyperoperation is obvious. The test of validity of the reproduction axiom is analogous to the test performed in the proof of Theorem 3.8. Further suppose $T_1(\lambda, F, \varphi)$ is an arbitrary operator an m an arbitrary non-zero real number and regard the result of $T_1 * T(1,0,0)$. The above defined t equals 1. Therefore $T_1 * T = \{m \cdot (T_1 \oplus T), m \leq 1, m \in \mathbb{N}\} = T_1$ because $T(1,0,0)$ is a neutral element of the commutative group $(\mathcal{T}_{\lambda\neq 0}(\Omega), \oplus)$.

The operation $\cdot$ may be used in defining yet another "natural" commutative hyperoperation. Define for an arbitrary pair of operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$

$$T_1(\lambda, F, \varphi) \blacktriangle T_2(\mu, G, \psi) = \{\lambda \cdot T_2\} \cup \{\mu \cdot T_1\} = \{T(\lambda\mu, G, \lambda\psi), T(\mu\lambda, F, \mu\varphi)\}.$$

Theorem 3.10 $(\mathcal{T}_{\lambda\neq 0}(\Omega), \blacktriangle)$ *is a commutative hypergroup.*

Proof. The fact that $\blacktriangle$ is a commutative hyperoperation is obvious.

1. For the test of associativity regard arbitrary operators $T_1(\lambda, F, \varphi), T_2(\mu, G, \psi), T_3(\kappa, H, \rho)$. Then:

   a) $[T_1 \blacktriangle T_2] \blacktriangle T_3 = S \blacktriangle T_3$, where $S \in \{T(\lambda\mu, G, \lambda\psi), T(\mu\lambda, F, \mu\varphi)\}$, i.e. $[T_1 \blacktriangle T_2] \blacktriangle T_3 = \{T(\lambda\mu\kappa, H, \lambda\mu\rho), T(\kappa\lambda\mu, G, \kappa\lambda\psi), T(\kappa\mu\lambda, F, \kappa\mu\varphi)\}$

   b) $T_1 \blacktriangle [T_2 \blacktriangle T_3] = T_1 \blacktriangle S$, where $S \in \{T(\mu\kappa, H, \mu\rho), T(\kappa\mu, G, \kappa\psi)\}$, i.e. $T_1 \blacktriangle [T_2 \blacktriangle T_3] = \{T(\lambda\mu\kappa, H, \lambda\mu\rho), T(\mu\kappa\lambda, F, \kappa\mu\varphi), T(\lambda\kappa\mu, G, \lambda\kappa\psi)\}$,

   i.e. the hyperoperation $\blacktriangle$ is associative.

2. In order to prove that the reproduction axiom holds, regard an arbitrary operator $T(\lambda, F, \varphi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ and study $T \blacktriangle \mathcal{T}_{\lambda\neq 0}(\Omega)$. We get that

$$T(\lambda, F, \varphi) \blacktriangle \mathcal{T}_{\lambda\neq 0}(\Omega) = \bigcup_{S \in \mathcal{T}_{\lambda\neq 0}(\Omega)} T(\lambda, F, \varphi) \blacktriangle S(\mu, G, \psi) =$$
$$= \{T(\lambda\mu, G, \lambda\psi), T(\mu\lambda, F, \mu\varphi), \mu \in \mathbb{C}, G, \psi \in \mathbb{C}^\Omega\}.$$

   Therefore both inclusions $T \blacktriangle \mathcal{T}_{\lambda\neq 0}(\Omega) \subseteq \mathcal{T}_{\lambda\neq 0}(\Omega)$ and $\mathcal{T}_{\lambda\neq 0}(\Omega) \subseteq T \blacktriangle \mathcal{T}_{\lambda\neq 0}(\Omega)$ are satisfied.

Remark 3.11 *If we regarded the set* $\mathcal{T}_{\lambda=1}(\Omega)$ *of all operators* $T(\lambda, F, \varphi) \in \mathcal{T}_{\lambda\neq 0}(\Omega)$ *such that* $\lambda = 1$, *then* $(\mathcal{T}_{\lambda=1}(\Omega), \blacktriangle)$ *would be a minimal extensive hypergroup.*

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SOME PROPERTIES OF DIRECTED GRAPHS CORRESPONDING TO THE COMPANION MATRIX

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Abstract. The distance matrix $D(G) = (d_{ij})$ of a weighted digraph G is arranged from the weights d_{ij} of the shortest path from a vertex v_i to a vertex v_j of G. The sum of all terms of the distance matrix $D(G)$ is known as the Wiener index $W(G)$ of the given digraph G. It is one of the topological indices of a graph which correlate with many physico-chemical properties of related organic compounds. In this paper we deal with digraphs corresponding to a special case of the companion matrix. The Wiener index, the determinant of the distance matrix and a recursion formula for the distance polynomial for this class of digraphs are obtained.

Key words. Digraph, companion matrix, distance matrix, Wiener index, distance polynomial

Mathematics Subject Classification: Primary 05C12, 05C20, Secondary 11C20, 11C08

1. Introduction

In the linear algebra and its application the companion matrix belongs to the most frequently used types of matrices. The companion matrix can be defined as the $n \times n$ square matrix

$$C_n = C_n(u_1, u_2, \dots, u_n) = \begin{pmatrix} u_1 & u_2 & u_3 & \dots & u_{n-1} & u_n \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots\dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $u_1, u_2, \dots, u_n$ may be regarded as indeterminates [3]. Its characteristic equation has the form $\lambda^n - u_1\lambda^{n-1} - u_2\lambda^{n-2} - \dots - u_n = 0.$

We can introduce a weighted digraph G which has the companion matrix as its adjacency matrix. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G then the arc (v_1, v_j), $j = 1, 2, \ldots, n$, has the weight u_j and the arc(v_j, v_{j-1}) has the weight 1.

Definition 1.
Let G be a strongly connected weighted digraph with n vertices and without loops. Then the distance matrix of G is defined as the $n \times n$ matrix $D(G) = D = (d_{ij})$, where

$$\begin{aligned} d_{ij} &= \text{ the distance from } v_i \text{ to } v_j, \\ &= 0, \text{ if } v_i = v_j. \end{aligned}$$

The distance from v_i to v_j is the weight of the shortest path from v_i to v_j The sum $\sum d_{ij}$ of all terms of the distance matrix is called the Wiener index of a graph G .

Definition 2.
Let D be the distance matrix of a digraph G. The distance polynomial of G is defined as $P(G; x) = \det(x I - D)$, where I is the unit matrix of the size $n \times n$.

It is rather complicated to find the determinant of the distance matrix, eigenvalues of this matrix or the distance polynomial for any digraph. We obtained these quantities for a cycle.

Theorem 1. ([4], Theorem 6).
For a cycle S_n on $n \geq 2$ vertices the following statements hold:

$$\det D(S_n) = (-1)^{n-1} \binom{n}{2} n^{n-2},$$

$$P(S_n; x) = \left(x - \binom{n}{2} \right) \prod_{j=2}^{n} \left(x - \frac{\varepsilon_j}{(1 - \varepsilon_j)^2} (1 - n\varepsilon_j^{n-1} + (n-1)\varepsilon_j^n \right),$$

where ε_j, $j = 1, \ldots, n$, are the n-th roots of unity. The matrix $D(S_n)$ has the eigenvalues

$$x_1 = \binom{n}{2}, \ x_j = \frac{\varepsilon_j}{(1-\varepsilon_j)^2} \left(1 - n\varepsilon_j^{n-1} + (n-1)\varepsilon_j^n\right).$$

for $2 \leq j \leq n$.

2. The main results

Throughout this paper we are concerned with the digraph G_n (Fig. 1) corresponding to the companion matrix $C_n(0, 1, 1, \ldots, 1)$.

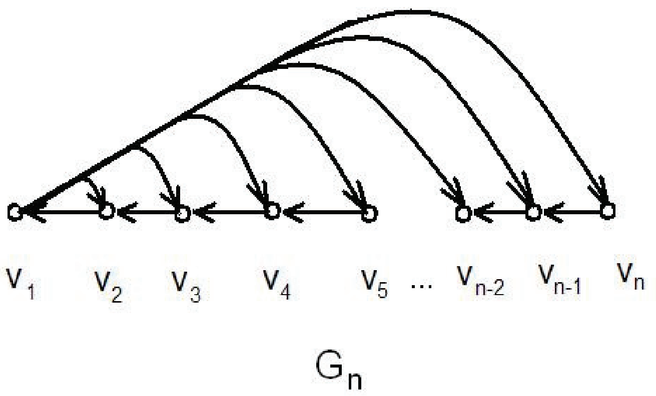

Fig. 1.

It is easy to see that the distance matrix D_n of such digraph G_n has the form

$$D_n = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 1 & 0 & 3 & \cdots & 3 & 3 \\ 3 & 2 & 1 & 0 & \cdots & 4 & 4 \\ & & \cdots\cdots\cdots & & & & \\ n-2 & n-3 & n-4 & n-5 & \cdots & 0 & n-1 \\ n-1 & n-2 & n-3 & n-4 & \cdots & 1 & 0 \end{pmatrix}.$$

Theorem 2.
The Wiener index of the above mentioned digraph G_n is given by the relation

$$W(G_n) = 2\binom{n+1}{3}$$

for any integer $n \geq 2$.

Proof.
It follows from the definition of the Wiener index and the expression of the distance matrix. Then we have

$$W(G_n) = 2\sum_{k=1}^{n-1}\sum_{m=0}^{k} m = 2\sum_{k=1}^{n-1}\binom{k+1}{2} = 2\binom{n+1}{3}$$

using simple combinatorial identities.

We will prove the further theorems by using the following well–known property of determinants.

Proposition.
Let A_1, A_2, B be the $n \times n$ square matrices such that

$$A_1 = \begin{pmatrix} a_{11}\dots a_{1n} \\ \dots\dots \\ a_{i1}\dots a_{in} \\ \dots\dots \\ a_{n1}\dots a_{nn} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11}\dots a_{1n} \\ \dots\dots \\ b_{i1}\dots b_{in} \\ \dots\dots \\ a_{n1}\dots a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \dots\dots & \\ a_{i1}+b_{i1} & \dots & a_{in}+b_{in} \\ & \dots\dots & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Then $\det B = \det A_1 + \det A_2$ and the statement holds for an arbitrary i-th row (or column), where $i = 1, 2, \dots, n$.

Theorem 3.
For the determinant of the distance matrix of a graph G_n the following recurrence holds $\det D_n = (-1)^{n+1}(n-1)! - n \det D_{n-1}$, where $n \geq 2$, and $\det D_1 = 0$.

Proof.
Consider a matrix A_n obtained from the matrix D_n by changing only the last row subtracting one from each term of this row. With respect to Proposition we have successively

$$\det A_n = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 2 & 2 & \dots & 2 & 2 \\ 2 & 1 & 0 & 3 & \dots & 3 & 3 \\ 3 & 2 & 1 & 0 & \dots & 4 & 4 \\ & & & \dots\dots & & & \\ n-2 & n-3 & n-4 & n-5 & \dots & 0 & n-1 \\ n-2 & n-3 & n-4 & n-5 & \dots & 0 & -1 \end{vmatrix} = \det D_n + \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 2 & 2 & \dots & 2 & 2 \\ 2 & 1 & 0 & 3 & \dots & 3 & 3 \\ 3 & 2 & 1 & 0 & \dots & 4 & 4 \\ & & & \dots\dots & & & \\ n-2 & n-3 & n-4 & n-5 & \dots & 0 & n-1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 \end{vmatrix} =$$

$$= \det D_n + \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 2 & 2 & \dots & 2 & 2 \\ 2 & 1 & 0 & 3 & \dots & 3 & 3 \\ 3 & 2 & 1 & 0 & \dots & 4 & 4 \\ & & & \dots\dots & & & \\ n-2 & n-3 & n-4 & n-5 & \dots & 0 & n-1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \end{vmatrix} = \det D_n + (-1)^n \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 2 & 2 & \dots & 2 & 2 \\ 1 & 0 & 3 & \dots & 3 & 3 \\ & & \dots\dots & & & \\ n-3 & n-4 & n-5 & \dots & 0 & n-1 \end{vmatrix} =$$

$$= \det D_n + (-1)^n \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 3 & \dots & 3 & 3 \\ & & \dots\dots & & & \\ n-3 & n-4 & n-5 & \dots & 0 & n-1 \end{vmatrix} = \det D_n + 2(-1)^n \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 3 & \dots & 3 & 3 \\ & & \dots\dots & & \\ n-4 & n-5 & \dots & 0 & n-1 \end{vmatrix} =$$

$$= \det D_n + 2(-1)^n \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ -3 & 0 & \dots & 0 & 0 \\ & & \dots\dots & & \\ n-4 & n-5 & \dots & 0 & n-1 \end{vmatrix} = \det D_n + 3\cdot 2(-1)^n \begin{vmatrix} 1 & \dots & 1 & 1 \\ & \dots\dots & & \\ n-5 & \dots & 0 & n-1 \end{vmatrix} = \dots =$$

$$= \det D_n + (n-2)\cdot \dots \cdot 3\cdot 2(-1)^n \begin{vmatrix} 1 & 1 \\ 0 & n-1 \end{vmatrix} = \det D_n + (-1)^n (n-1)!$$

We can also calculate the determinant of the matrix A_n by subtracting the $(n-1)$-st row from the n-th row. Then

$$\det A_n = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 2 & 2 & \dots & 2 & 2 \\ 2 & 1 & 0 & 3 & \dots & 3 & 3 \\ 3 & 2 & 1 & 0 & \dots & 4 & 4 \\ & & & \dots\dots & & & \\ n-2 & n-3 & n-4 & n-5 & \dots & 0 & n-1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -n \end{vmatrix} = -n \det D_{n-1}.$$

Comparing the both expressions of det A_n we obtain $\det D_n + (-1)^n (n-1)! = -n \det D_{n-1}$ and the required recurrence follows.

Consider the integer sequence $\{a_n\}$ of generalized Stirling numbers which can be defined by the recurrence $a_n = n a_{n-1} + (n-1)!$ with $a_1 = 0$. This sequence is A 001705 in Neil Sloane´s On-line Encyclopedia of Integer Sequences [6]. The closed formula for the generalized Stirling numbers has the form $a_{n+1} = n! \sum_{k=0}^{n-1} \frac{k+1}{n-k}$.

Corollary.
For any positive integer n the relation $\det D_n = (-1)^{n+1} a_n$ holds, where a_n is the n-th generalized Stirling number.

Proof.
We proceed by induction on n. It is obvious that $\det D_1 = a_1 = 0$. Let us suppose the validity of the relation for an arbitrary integer n. Then

$$\det D_{n+1} = (-1)^n n! - (n+1) \det D_n = (-1)^n n! - (n+1)(-1)^{n+1} a_n = (-1)^n \left[n! + (n+1) a_n \right] = (-1)^n a_{n+1}$$

and the proof is over.

Now, we will derive a recursive formula for the distance polynomial $P(G_n; x)$.

Lemma.
Let $\{B_n\}$ be a sequence of matrices, where $B_1=(-1)$ and

$$B_n = \begin{pmatrix} x & -1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & x & -2 & -2 & \ldots & -2 & -2 \\ -2 & -1 & x & -3 & \ldots & -3 & -3 \\ -3 & -2 & -1 & x & \ldots & -4 & -4 \\ & & & \ldots\ldots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & x & -(n-1) \\ -(n-1) & -(n-2) & -(n-3) & -(n-4) & \ldots & -1 & -n \end{pmatrix}$$

for $n \geq 2$. Then the recurrence $\det B_n = x \det B_{n-1} - (x+1)(x+2)\ldots(x+n-1)$, with $\det B_1 = -1$, holds for the corresponding sequence of the determinants

Proof.
We introduce a sequence of matrices $\{H_n\}$, where $H_1 = (0)$ and for $n \geq 2$

$$H_n = \begin{pmatrix} x & -1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & x & -2 & -2 & \ldots & -2 & -2 \\ -2 & -1 & x & -3 & \ldots & -3 & -3 \\ -3 & -2 & -1 & x & \ldots & -4 & -4 \\ & & & \ldots\ldots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & x & -(n-1) \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & 0 & -(n-1) \end{pmatrix}.$$

It is easy to see with respect to Proposition that

$$\det H_n = \det B_n + \begin{vmatrix} x & -1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & x & -2 & -2 & \ldots & -2 & -2 \\ -2 & -1 & x & -3 & \ldots & -3 & -3 \\ -3 & -2 & -1 & x & \ldots & -4 & -4 \\ & & & \ldots\ldots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & x & -(n-1) \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 \end{vmatrix}.$$

Adding the n-th row to the first row in the last determinant we have

$$\det H_n = \det B_n + \begin{vmatrix} x+1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & x & -2 & -2 & \dots & -2 & -2 \\ -2 & -1 & x & -3 & \dots & -3 & -3 \\ -3 & -2 & -1 & x & \dots & -4 & -4 \\ & & & \dots\dots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \dots & x & -(n-1) \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix} =$$

$$= \det B_n + (x+1) \begin{vmatrix} x & -2 & -2 & \dots & -2 & -2 \\ -1 & x & -3 & \dots & -3 & -3 \\ -2 & -1 & x & \dots & -4 & -4 \\ & & \dots\dots & & & \\ -(n-3) & -(n-4) & -(n-5) & \dots & x & -(n-1) \\ 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix} .$$

By adding twice the last row to the first row, by expanding the determinant with respect to the first row and by repeating of this procedure we get $\det H_n = \det B_n + (x+1)(x+2)...(x+n-1)$.

Further we can also express det H_n after subtracting the (n-1) – st row from the n-th row in the following form

$$\det H_n = \begin{vmatrix} x & -1 & -1 & -1 & \dots & -1 & -1 \\ -1 & x & -2 & -2 & \dots & -2 & -2 \\ -2 & -1 & x & -3 & \dots & -3 & -3 \\ -3 & -2 & -1 & x & \dots & -4 & -4 \\ & & & \dots\dots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \dots & x & -(n-1) \\ 0 & 0 & 0 & 0 & \dots & -x & 0 \end{vmatrix} = x \det B_{n-1} .$$

Comparing the both expressions of det H_n we obtain the proved recurrence immediately.

We can calculate several first terms of the sequence {det B_n} from Lemma. Then $\det B_1 = -1$, $\det B_2 = -(2x+1)$, $\det B_3 = -(3x^2 + 4x + 2)$,
$\det B_4 = -(4x^3 + 10x^2 + 13x + 6)$, $\det B_5 = -(5x^4 + 20x^3 + 48x^2 + 56x + 24)$.

Now, we will focus in deriving a recurrence for the distance polynomial of the graphs G_n.

Theorem 4.
A recursive formula for the distance polynomial $P(G_n\ ; x)$ has the form

$$P(G_n;x) = (x+n)P(G_{n-1};x) + x\det B_{n-1} - (x+1)(x+2)\ldots(x+n-1)$$

for any integer $n \geq 2$ and $P(G_1;x) = x$; where the matrices B_n are as in Lemma.

Proof.
It is made in a similar way as for Theorem 3. Denote for $n \geq 2$ a matrix

$$K_n = \begin{pmatrix} x & -1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & x & -2 & -2 & \ldots & -2 & -2 \\ -2 & -1 & x & -3 & \ldots & -3 & -3 \\ -3 & -2 & -1 & x & \ldots & -4 & -4 \\ & & & \ldots\ldots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & x & -(n-1) \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & 0 & x+1 \end{pmatrix}.$$

Then we can write

$$\det K_n = P(G_n;x) + \begin{vmatrix} x & -1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & x & -2 & -2 & \ldots & -2 & -2 \\ -2 & -1 & x & -3 & \ldots & -3 & -3 \\ -3 & -2 & -1 & x & \ldots & -4 & -4 \\ & & & \ldots\ldots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & x & -(n-1) \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 \end{vmatrix} =$$

$$= P(G_n;x) + \begin{vmatrix} x+1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ -1 & x & -2 & -2 & \ldots & -2 & -2 \\ -2 & -1 & x & -3 & \ldots & -3 & -3 \\ -3 & -2 & -1 & x & \ldots & -4 & -4 \\ & & & \ldots\ldots & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \ldots & x & -(n-1) \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 \end{vmatrix} =$$

$$= P(G_n;x) + (x+1)\begin{vmatrix} x & -2 & -2 & \dots & -2 & -2 \\ -1 & x & -3 & \dots & -3 & -3 \\ -2 & -1 & x & \dots & -4 & -4 \\ & & \dots\dots & & & \\ -(n-3) & -(n-4) & -(n-5) & \dots & x & -(n-1) \\ 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix}.$$

Repeating this procedure we obtain $\det K_n = P(G_n;x) - (x+1)(x+2)\dots(x+n-1)$.

We can also express $\det K_n$ in a different way after subtracting the (n-1)-st row from the n-th row

$$\det K_n = \begin{vmatrix} x & -1 & -1 & -1 & \dots & -1 & -1 \\ -1 & x & -2 & -2 & \dots & -2 & -2 \\ -2 & -1 & x & -3 & \dots & -4 & -4 \\ -3 & -2 & -1 & x & \dots & -4 & -4 \\ & & \dots & & & & \\ -(n-2) & -(n-3) & -(n-4) & -(n-5) & \dots & x & -(n-1) \\ 0 & 0 & 0 & 0 & \dots & -x & x+n \end{vmatrix} = (x+n)P(G_{n-1};x) + x \det B_{n-1}.$$

Comparing the both expressions for $\det K_n$ leads to the required recurrence.

By application the proved statements in this paper we easy get the expressions of the used quantities of the graphs G_n for $n \le 6$ (Table 1).

Table 1. The Wiener index $W(G_n)$, the determinant of the distance matrix det $D(G_n)$, the distance polynomial $P(G_n; x)$

n	$W(G_n)$	det $D(G_n)$	$P(G_n; x)$
1	0	0	x
2	2	–1	$x^2–1$
3	8	5	$x^3 – 5x – 5$
4	20	–26	$x^4 + x^3 – 9x^2 –27x –20$
5	40	154	$x^5 + 2x^4 – 14x^3 – 85x^2 – 161x – 100$
6	70	–1044	$x^6 + 3x^5 – 22x^4 – 217x^3 – 727x^2 – 1090x – 600$

3. Concluding remark

We concentrated in this paper only on some quantities for graphs relating to a special case of companion matrices. It would by interesting to investigate properties of diagraphs with the companion matrix as their adjacency matrix for another values of the variables u_1, u_2, …, u_n. Some further topological indices for such digraphs would be possible to apply (see e.g. [5]).

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FUZZY APPROACH TO QUANTITATIVE INTERPRETATION OF MMPI-2

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Abstract. MMPI-2 is a psychological test searching for pathological personality features. Based on results of the test, a patient is described by a codetype, which characterizes the patient's personality. There are two steps to the quantitative interpretation of the test. In the first one, a codetype is determined using the original approach of the MMPI test. In the second step, introduced in the MMPI-2, the codetype is verified with help of detailed prototypic profiles, which are unique for each specific codetype. In this paper we design a mathematical model for codetype determination and evaluation of overlap between the measured data and the prototypic profiles. The model is composed of two fuzzy expert systems. The first is used to formally express the linguistic description of the method employed to determine the codetype. The second model presents a simplified description of prototypic profiles and allows to find effectively the degree of agreement between the profiles and data obtained from the patient. The proposed mathematical model is realized in the MATLAB Fuzzy Logic toolbox.

Key words and phrases. MMPI-2, Psychometry, Fuzzy expert systems, Base of rules, Linguistic variable.

Mathematics Subject Classification. Primary 03E75, 91E45.

1 Introduction

MMPI-2 (Minnesota Multiphasic Personality Inventory) is one of the most frequently used tests for characterization of personality features and psychic disorders. The first version of the test, MMPI, was developed by psychologist S. R. Hathaway and psychiatrist J. C. McKinley [5] of the Minnesota University. Their goal was to develop an instrument to describe patient's personality more effectively than what was allowed by the psychiatric interview with the patient, [1]. At the same time it was desirable to replace a great number of tests, focusing on single features, by a single test capable of full characterization. The fruit of their labor was an extensive testing method with applications far beyond the clinical practice. Today, a revised version of the test, MMPI-2 [2], is used. MMPI-2 is an important screening method for detecting pathological personality features, which is used in clinical practice, as well as in entrance interviews for universities, military, police, or leading positions [6].

Use of the MMPI-2 is very demanding. The examiner needs to posses knowledge of theory and use of psychological tests; he/she should have a Master degree in personal psychology and psychopathology, [6]. Furthermore, correct interpretation of the test requires experience with the MMPI-2 and a special training. For this reason, a software with transparent results providing solid basis for the clinic deliberation is clearly needed.

An important part of the testing process is quantitative interpretation, [4]. Answers to questionnaire questions are used to saturate a large number of scales. Their rough point values are then transformed into linear T-scores. Based on values of these, a codetype of the patient is determined.

The basis for the MMPI-2 interpretation is determination of codetype, if possible. Each codetype is defined by T-scores of ten clinical scales (1-Hypochondriasis, 2-Depression, 3-Hysteria, 4-Psychopathic deviate, 5-Masculinity-Feminity, 6-Paranoia, 7-Psychasthenia, 8-Schizophrenia, 9-Hypomania, 0-Social introversion). Value of each T-score comes from the interval $[0, 120]$. Values higher than 65 are considered significantly elevated. According to number and type of increased clinical scales we define 55 different codetypes. Codetypes with one significantly elevated clinical scale are designated “Spike” (ten possible types), while two significantly elevated scales represent a “Two Point” (45 possible types). For a codetype to be well defined, there has to be at least five point difference between the T-scores of the highest scales and remaining T-scores. If this is not satisfied, there is a possibility of triad, for example, and it is not possible to use codetypes.

After finding the codetype, the agreement between patient’s data and the respective prototypic profile is checked. In this testing, T-scores of all scales need to be considered. Each of 55 prototypic profiles is defined by specific values of all scales. To have a perfect match between the patient and a given prototypic profile, T-scores of patient’s scales must not differ from T-scores of the profile by more than ten points.

For finding T-scores and determination of codetype the MMPI-2 software was developed [6]. This software finds the codytpe only from the two highest T-scores and rest of the data is not involved in the process. This leads to loss of information and it is wasteful of the full MMPI-2 potential. Furthermore, the software does not strictly adhere to the five-point-difference condition and therefore may return an erroneous result.

The aim of this work is to create a mathematical model which can be used to find several codetypes most likely describing the patient. Some of them are found by the fuzzified MMPI methodology, while the others are obtained by a direct comparison of patient’s scales with prototypic profiles. Furthermore, we aim to realize this model in form of a software.

The Czech version of the MMPI-2 does not work with all of the scales. It uses and saves values of only 79 of them. The mathematical model will consider this simplified version of the MMPI-2.

2 Used mathematical tools

The codetype determination requiring full satisfaction of all 79 conditions of a prototypic profile is problematic. Classification based on such a crisp mathematical model may not work, because only rarely a patient satisfies fully a prototypic profile. It will be shown that in a situation like this, as well as in many areas of social sciences and psychology, it is effective to use the so called fuzzy approach.

Fuzzy set theory is a relatively young branch of mathematics, [3], [7]. Its basic notion is a fuzzy set. Classical set theory is based on binary logic: for each element we can say whether it belongs to a given set (value 1) or not (value 0). But in the real life this decision can not be often made easily, because the property defining the set may be of vague nature. In fuzzy set theory not only an element can have a property fully or not at all, but it can also have the property only partially. The measure of the property possession is quantified by a number from the interval $[0, 1]$.

Fuzzy set theory gives us a tool to model the vagueness phenomenon. It allows us to describe mathematically linguistic values and linguistically defined rules. This is the reason why in this special case, where we look for a mathematical model of a linguistically described methodology, description by fuzzy sets is very helpful.

Let U be a nonempty set. A fuzzy set A on U is defined by the mapping $A : U \to [0, 1]$. For each $x \in U$ the value $A(x)$ is called a membership degree of the element x in the fuzzy set A, $A(\cdot)$ is a membership function of the fuzzy set A.

A height of a fuzzy set A on U is a real number $\mathrm{hgt}(A) = \sup_{x\in U}\{A(x)\}$. An intersection of fuzzy sets A, B on U is a fuzzy set $A \bigcap B$ on U with a membership function $(A \bigcap B)(x) = \min\{A(x), B(x)\}$ for any $x \in U$.

A fuzzy number A is a fuzzy set on $\mathbb{R}$ which fulfills the following conditions: the kernel of the fuzzy set A, $\mathrm{Ker}A = \{x \in \mathbb{R}|A(x) = 1\}$, is a non-empty set, the α-cuts of the fuzzy set A, $A_\alpha = \{x \in \mathbb{R}|A(x) \geq \alpha\}$, are closed intervals for all $\alpha \in (0, 1]$, the support of A, $\mathrm{Supp}A = \{x \in \mathbb{R}|A(x) > 0\}$, is bounded.

The family of all fuzzy numbers on $\mathbb{R}$ is denoted by $F_N(\mathbb{R})$. If $\mathrm{Supp}A \subseteq [a, b]$ then A is referred to as a fuzzy number on the interval $[a, b]$. The family of all fuzzy numbers on the interval $[a, b]$ is denoted by $F_N([a, b])$. A linear fuzzy number on the interval $[a, b]$ that is determined by four points $(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0)$, $a \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq b$, is a fuzzy number A with the membership function depending on parameters x_1, x_2, x_3, x_4, as follows

$$\forall x \in [a, b] : A(x, x_1, x_2, x_3, x_4) = \begin{cases} 0, & \text{for } x < x_1; \\ \frac{x-x_1}{x_2-x_1}, & \text{for } x_1 \leq x < x_2; \\ 1, & \text{for } x_2 \leq x \leq x_3; \\ \frac{x_4-x}{x_4-x_3}, & \text{for } x_3 < x \leq x_4; \\ 0, & \text{for } x_4 < x. \end{cases}$$

This linear fuzzy number A will be denoted by $A \sim (x_1, x_2, x_3, x_4)$.

A linguistic variable is a quintuple $(X, T(X), U, G, M)$, where X is a name of the variable, $T(X)$ is a set of its linguistic values (linguistic terms), U is an universe, which the mathematical meanings of the linguistic terms are modelled on, G is a syntactical rule for generating the linguistic terms, and M is a semantic rule, which to every linguistic term $\mathcal{A}$ assigns its meaning $M(\mathcal{A})$ as a fuzzy set on U. If the set of linguistic terms is given explicitly, then the linguistic variable is denoted by $(X, T(X), U, M)$.

Let $(X_j, T(X_j), U_j, M_j),\ j = 1, 2, \ldots, m$, and $(Y, T(Y), V, N)$ be linguistic variables. Let $\mathcal{A}_{ij} \in T(X_j)$ and $M(\mathcal{A}_{ij}) \in F_N(U_j),\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. Let $\mathcal{B}_i \in T(Y)$ and $M(\mathcal{B}_i) \in F_N(V),\ i = 1, 2, \ldots, n$. Then the following scheme F

If X_1 is $\mathcal{A}_{11}$ and $\ldots$ and X_m is $\mathcal{A}_{1m}$, then Y is $\mathcal{B}_1$
If X_1 is $\mathcal{A}_{21}$ and $\ldots$ and X_m is $\mathcal{A}_{2m}$, then Y is $\mathcal{B}_2$
....
If X_1 is $\mathcal{A}_{n1}$ and $\ldots$ and X_m is $\mathcal{A}_{nm}$, then Y is $\mathcal{B}_n$

is called a linguistically defined function (base of rules).

The process of calculating linguistic values of an output variable for the given linguistic values of input variables by means of such a rule base is called an approximate reasoning. There are several methods of approximate reasoning. The most popular and the most widely used one is the Mamdani algorithm.

Let F be the base of rules defined above and let us assume the observed values to be

$$X_1 \text{ is } \mathcal{A}'_1 \text{ and } X_2 \text{ is } \mathcal{A}'_2 \text{ and } \ldots \text{ and } X_m \text{ is } \mathcal{A}'_m,$$

then by entering the observed values into the base of rules F, according to the Mamdani algorithm, we obtain the output value $Y = \mathcal{B}'$, where the $\mathcal{B}'$ is the linguistic approximation ([3]) of a fuzzy set B^M. The membership function of the fuzzy set B^M is defined for all $y \in V$ as follows $B^M(y) = \max\{B_1^M(y), \ldots, B_n^M(y)\}$, where $B_i^M(y) = \min\{h_i, B_i(y)\}$, $h_i = \min\{\text{hgt}(A_{i1} \bigcap A'_1),\ \ldots,\ \text{hgt}(A_{im} \bigcap A'_m)\}$, for $i = 1, \ldots, n$.

3 Designed mathematical model

The quantitative interpretation of the MMPI-2 is performed in two steps. First, based on values of clinical scales, a patient's codetype is determined. This is followed by the verification, where the relevant prototypic profile is compared with the patient's data.

The proposed mathematical model respects this structure of MMPI-2. In the first step, the model finds the three clinical scales with the highest T-scores, and with help of the linguistically described function decides on a codetype. In the second step, the model works with values of all 79 scales and calculates the overlap between the linear T-scores of the patient and the prototypic profile of the codetype found in the previous step. Simultaneously the model searches for other prototypic profiles, which agree well with patient's data.

3.1 Codetype determination

Two conditions are important for correct determination of the codetype. First, T-scores of significantly elevated scales must be higher than 65. Second, values of the highest scales must be at least five points higher than values of all remaining scales. In practice, it is often difficult to strictly fulfill this conditions. It has shown to be more effective to use the fuzzy approach and define these conditions linguistically. Furthermore, use of the fuzzy set theory was instrumental in finding more variants of the codetype, which can be presented to the evaluator.

Prior to further processing, the scales need to be ordered from the highest T-score to the lowest. Based on the above mentioned requirements, we then define linguistic variables as:

1. $\langle$The First Scale Elevation,
   {Insignificant, Significant}, $[0, 120], M_1\rangle$,
2. $\langle$The Second Scale Elevation,
   {Insignificant, Significant}, $[0, 120], M_1\rangle$,
3. $\langle$The Third Scale Elevation,
   {Insignificant, Significant}, $[0, 120], M_1\rangle$,
4. $\langle$The Difference between the First Two Scales,
   {Small, Big Enough}, $[0, 120], M_2\rangle$,
5. $\langle$The Difference between the 2^{nd} and the 3^{rd} Scale,
   {Small, Big Enough}, $[0, 120], M_2\rangle$,
6. $\langle$Codetype Shape,
   {Spike, Two Point, Potential Triad, Within-Normal-Limits}, $\{1, 2, 3, 4\}, N\rangle$,

where
M_1(Insignificant) $= IE \sim (0, 0, 63, 65)$, M_1(Significant) $= SE \sim (63, 65, 120, 120)$, M_2(Small) $= SM \sim (0, 0, 0, 5)$, M_2(Big Enough) $= BE \sim (0, 5, 120, 120)$, N(Spike) $= S \sim (1, 1, 1, 1)$, N(Two Point) $= 2P \sim (2, 2, 2, 2)$, N(Potential Triad) $= PT \sim (3, 3, 3, 3)$, N(Within-Normal-Limits) $= WNL \sim (4, 4, 4, 4)$. Some of defined variables are illustrated in Fig. 1 and 2.

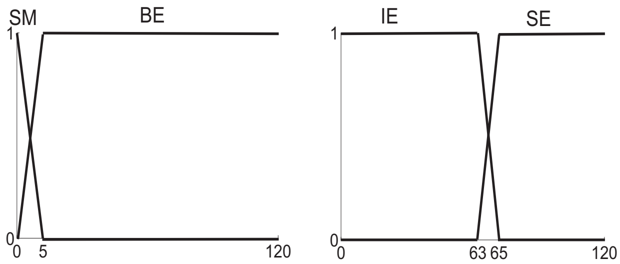

Figure 1: Input linguistic variables
Left: *The Difference between the First Two Scales* and its two linguistic values *Small* and *Big Enough* modelled by fuzzy numbers SM and BE.
Right: *The First Scale Elevation* and its two linguistic values *Insignificant* and *Significant* modelled by fuzzy numbers IE and SE.

With help of these six linguistic variables and four rules we construct a base of rules F:

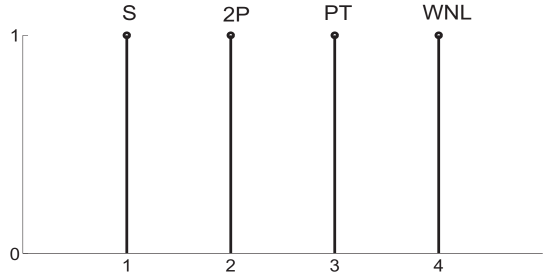

Figure 2: Output linguistic variable *Codetype Shape* and its four linguistic values *Spike, Two Point, Potential Triad* and *Within-normal-limits* modelled by fuzzy numbers *S,2P,PT* and *WNL*.

rule 1 If The First Scale Elevation is Significant and The Second Scale Elevation is Insignificant and The Difference between the First Two Scales is Big Enough, then the Codetype Shape is a Spike.

rule 2 If The First Scale Elevation is Significant and The Second Scale Elevation is Significant and The Difference between the 2^{nd} and the 3^{rd} Scale is Big Enough, then the Codetype Shape is Two Point.

rule 3 If The First Scale Elevation is Significant and The Second Scale Elevation is Significant and The Third Scale Elevation is Significant and The Difference between the 2^{nd} and the 3^{rd} Scale is Small, then the Codetype Shape is Potential Triad.

rule 4 If The First Scale Elevation is Insignificant, then the Codetype Shape is Within-Normal-Limits.

The base of rules F has five input linguistic variables - the three highest T-scores of clinical scales and the two differences between them - and one output linguistic variable, which determines the shape of the codetype.

Together with the Mamdani approximate reasoning algorithm, the linguistic function F forms an expert system for determination of the codetype shape. With values of clinical scales as an input, the model produces a fuzzy set B^M that helps the evaluator to determine possible codetype shapes. The membership degree of an element of the set $\{1, 2, 3, 4\}$ in fuzzy set B^M, representing a particular codetype shape, equals to the degree of satisfaction of the respective rule. See, for example, Fig. 3. To determine the complete codetype of the patient, we need to combine the information about the codetype shape with knowledge of the initial ordering of clinical scales. For example, if the codetype shape is Spike and the designation of the highest scale is 8-Schizophrenia, then the codetype is Spike 8.

3.2 Verification by comparison to prototypic profiles

In the second part of the mathematical model we calculate the degree of agreement between the found prototypic profiles and the patient's data. In addition, the model searches for other

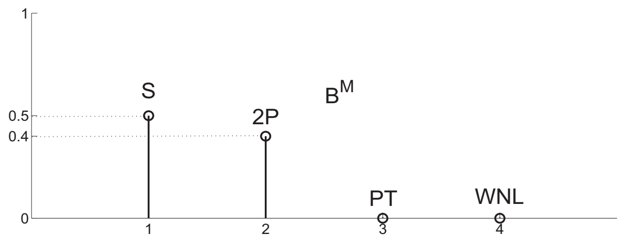

Figure 3: The fuzzy set B^M as obtained by entering input values [67 64 62 3 2] into the first expert system. The degrees of satisfaction express the possibility that the corresponding codetype shape is a Spike (possibility 50%) or a Two Point (possibility 40%).

prototypic profiles with a good overlap. Each profile is described by a vector of 79 integers representing values of the 79 scales with the T-scores ranging from 0 to 120. For a patient's profile to match a prototypic profile, the patient's T-scores must be within 10 point distance from the prototypic values. It is readily apparent that probability of any patient's profile fully matching a prototypic one is very low.

With help of tools of the fuzzy set theory we have simplified the description of profiles. We have defined 80 linguistic variables, of which 79 are input and one is output. The input variables correspond to 79 scales, such as *APS Scale Elevation*, for example. The output variable determines the prototypic profile.

All the input linguistic variables were defined on the interval $[0, 120]$ and a four-element set of terms {Low, Standard, High, Extremely High} was assigned to each of them. Each term was modelled by a fuzzy number: $\widetilde{M}(\text{Low}) = L \sim (0, 0, 42, 47)$, $\widetilde{M}(\text{Standard}) = ST \sim (42, 47, 62, 67)$, $\widetilde{M}(\text{High}) = H \sim (62, 67, 87, 92)$, $\widetilde{M}(\text{Extremely High}) = EH \sim (87, 92, 120, 120)$. In this way, we have replaced the original interval $[0, 120]$ by one fuzzy scale same for all input variables. The fuzzy scale is illustrated by the Fig. 4.

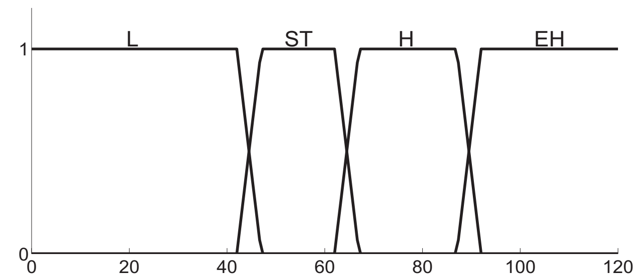

Figure 4: Input linguistic variable *APS Scale Elevation* and its four values *Low, Standard, High* and *Extremely High* modelled by fuzzy numbers *L, ST, H, EH.*

At the same time we have defined one output linguistic variable $\langle$Prototypic profile, T, $\{1, 2, \ldots, 55\}, \widetilde{N}\rangle$. The set T contains names of all 55 prototypic profiles, i.e. $T =$ {Spike 1, 1221, ..., Spike 0}. Their meanings are defined by function $\widetilde{N}$: $\widetilde{N}$(Spike 1) $= S1 \sim$

$(1,1,1,1)$, $\widetilde{N}(1221) = 1221 \sim (2,2,2,2)$, ..., $\widetilde{N}(\text{Spike } 0) = S0 \sim (55,55,55,55)$. That is, meaning of each prototypic profile is modelled by one integer between 1 and 55.

With these linguistic variables we have performed a linguistic description of the prototypic profiles. We have thus created a new base of rules G with 79 input linguistic variables and one output variable. The base of rules is formed by 55 rules; each of them represents a linguistic description of a prototypic profile, such as:

Rule 1 If "?" Scale Elevation is Low and "L" scale Elevation is Standard and ... and "APS" Scale Elevation is Standard, then Prototypic Profile is Spike 1.

For use with the linguistically defined function G we had to devise a new algorithm different from the Mamdani approximate reasoning. This was because during initial tests of the model the overlap between the patient's and the prototypic profile was often determined as zero, even if these profiles were very similar. It appears that the operation minimum, employed by the Mamdani approximate reasoning, is too severe for number of variables this high. The operation of average has shown itself to be the most suitable operation for finding the appropriate prototypic profile:

$$h_i = \frac{1}{79}\sum_{j=1}^{79} \mathrm{hgt}(A_{ij} \bigcap A'_j), \ i = 1, 2, \ldots, 55. \tag{1}$$

The degree of rule satisfaction here represents the average satisfaction of all the 79 conditions, in contrast to the Mamdani approach, where the degree of rule satisfaction stands for satisfaction of the least satisfied rule. This is illustrated in Figs. 5 and 6.

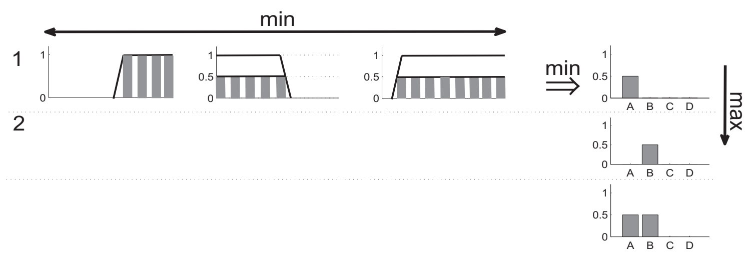

Figure 5: The Mamdani approximate reasoning mechanism.

Entering values of the 79 observed scales into the linguistically described function G and applying our modified algorithm for the approximate reasoning we arrive at a fuzzy set B^{MU} on $\{1, 2, \ldots, 55\}$. This fuzzy set represent the overlap of prototypic profiles with the profile of the patient. It is illustrated in Fig. 7.

4 MATLAB implementation of the mathematical model

The proposed mathematical model we have realized in MATLAB. With help of the Fuzzy Logic Toolbox we have created the two bases of rules and set the relevant approximate reasoning algorithms, all of which was described in detail in previous sections.

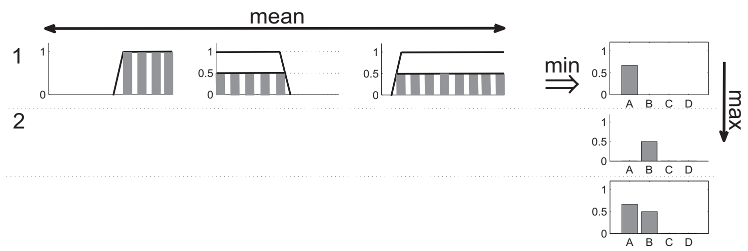

Figure 6: The modified Mamdani approximate reasoning mechanism.

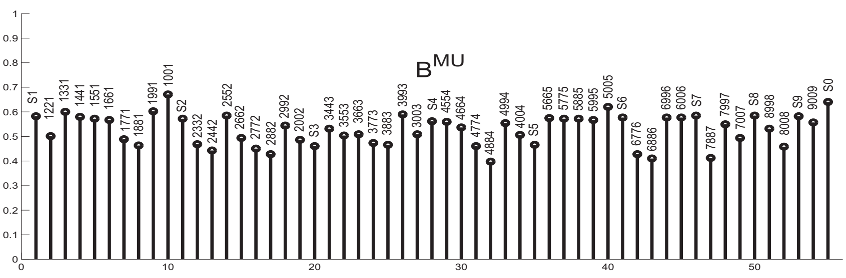

Figure 7: The fuzzy set B^{MU} as obtained by entering 79 input values into the second expert system. The degrees of satisfaction represent the overlap between the prototypic profiles and the pacient's profile.

The model was then tested on real data from clinical practice. An example of output can be seen in Fig. 8. The output of the utility is in the form of three figures and linguistic description of the situation. The first figure presents values of clinical scales as obtained from the patient - the patient's profile. The second figure presents possible codetypes, together with their respective degrees of satisfaction. The third figure shows all prototypic profiles and their overlap with the patient's profile. The evaluator can therefore decide, whether the found codetypes are in good agreement with all available patient's data. The linguistic output presents possible codetypes and three prototypic profiles with the best agreement. In addition it comments on a possibility of a triad or scales within normal limits.

In Fig. 8 we demonstrate performance of the implementation. According to clinical scales values, codetype 6-9 was determined. The result is in agreement with the original software. However, during the prototypic profile analysis, the codetype 6-9 didn't show sufficient agreement. The three most faithful profiles were those of codetypes 6-8/8-6, 7-8/8-7, and 2-8/8-2, with 6-8/8-6 showing the best overlap. This suggests that for further deliberation, codetypes 6-8/8-6 should be considered in addition to 6-9.

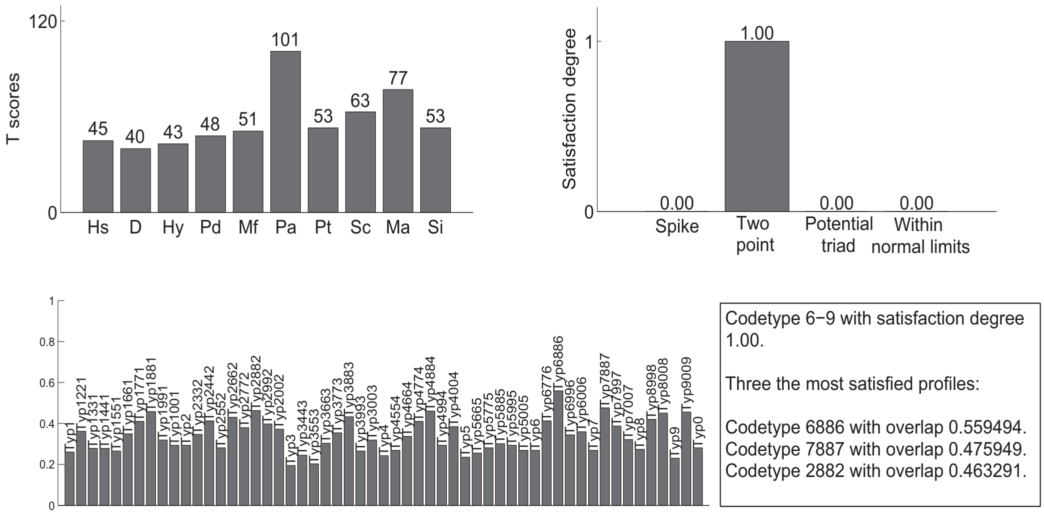

Figure 8: Three figures and linguistic description as returned by the MATLAB implementation of the model.

5 Conclusion

We have designed a fuzzy model which reliably processes results of the MMPI-2 test, which is the third test most often used in psychology and psychiatry.

For determination of a MMPI codetype we have employed a fuzzy expert system to formally express linguistically described method of data analysis.

Another fuzzy expert system has been used to verify the overlap between the prototypic profiles of the found codetypes and the patient's data. In both these situations, the fuzzy approach has shown itself to be suitable and reliable.

The created fuzzy model was implemented in the MATLAB Fuzzy Logic toolbox. The created utility, employing the fuzzy approach, can analyze the data in similar way as a human psychiatrist or psychologist, and thus avoid shortcomings of the existing software "MMPI-2" we mentioned. We have successfully tested the software on data from clinical practice.

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GENERATED FUZZY IMPLICATORS AND THEIR PROPERTIES

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Abstract. Conjunctors in MV-logic with truth values range $[0,1]$ are monotone extensions of the classical conjunction. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function, such that $f(1) = 0$, then we can define conjunctor $C : [0,1]^2 \to [0,1]$ by

$$C(x,y) = f^{(-1)}(f(x) + f(y)),$$

where the pseudo-inverse $f^{(-1)}$ is given by $f^{(-1)}(x) = \sup\{t \in [0,1]; f(t) > x\}$, f is called an additive generator of C. Dual operator to the conjunctor C called the disjunctor. Commonly used disjunctors in MV-logic are the triangular conorms S. A function $I : [0,1]^2 \to [0,1]$ is said to an implicator if and only if $I(1,0) = 0$ and $I(0,0) = I(0,1) = I(1,1) = 1$ and I is non-increasing in its first component and non-decreasing in its second component. The unary operator $n : [0,1] \to [0,1]$ is called negator if for any $a, b \in [0,1]$ holds

$$a \leq b \Rightarrow n(b) \leq n(a),$$

$$n(0) = 1, n(1) = 0.$$

Starting with the triangular conorm S and standard negation $N_s(x) = 1 - x$, we can introduce the implication operator in $[0,1]-$ valued logic as follows: $I(x,y) = S(N(x),y)$. Another way of extending the classical binary implication operator to the unit interval $[0,1]$ uses the *residuation* I with respect to the left-continuous conjunctor C

$$I(x,y) = \sup\{z \in [0,1]; C(x,z) \leq y\}.$$

There exists several constructions of implicators. We will investigate some genereted implicators and their properties.

Key words and phrases. fuzzy implicators, pseudoinverse, generators.

Mathematics Subject Classification. Primary 03B52, 03E72; Secondary 39B99.

1 Preliminaries

We briefly recall definitions of the most important connectives in MV-logic.

Definition 1.1 *An unary operator* $n : [0,1] \to [0,1]$ *is called a negator if, for any* $a, b \in [0,1]$,

- $a < b \Rightarrow n(b) \leq n(a)$, *and*
- $n(0) = 1, n(1) = 0.$

The negator n is called a *strong negator* if and only if the mapping n is one to one. Evidently, a strong negator is continuous and its inverse n^{-1} is a strong negator too. The negator n is called *involutive negator* if and only if $n(n(a)) = a$ for all $a \in [0,1]$. It can be easily proved that an involutive negator is strong and $n^{-1}(a) = n(a)$.

Example 1.2 *The following are some examples of fuzzy negators:*

- $N_s(a) = 1 - a$ *involutive negator, standard negation,*
- $n(a) = 1 - a^2$ *strong, non-involutive negator,*
- $n(a) = \sqrt{1 - a^2}$ *involutive negator,*
- $N_{G_1}(0) = 1, n(a) = 0$ *if* $a > 0$ *not-continuous, smallest, Gödel negator,*
- $N_{G_2}(1) = 0, n(a) = 1$ *if* $a < 1$ *not-continuous, greatest, dual Gödel negator.*

Definition 1.3 *A non-decreasing mapping* $C : [0,1]^2 \to [0,1]$ *is called a conjunctor if, for any* $a, b \in [0,1]$, *it holds*

- $C(a, b) = 0$ *whenever* $a = 0$ *or* $b = 0$, *and*
- $C(1, 1) = 1.$

Commonly used conjunctors in MV-logic are the triangular norms.

Definition 1.4 *A triangular norm (t-norm for short) is a binary operation on the unit interval* $[0,1]$, *i.e., a function* $T : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$, *the following four axioms are satisfied:*
(T1) Commutativity $T(x, y) = T(y, x)$,
(T2) Associativity $T(x, T(y, z)) = T(T(x, y), z)$,
(T3) Monotonicity $T(x, y) \leq T(x, z)$ *whenever* $y \leq z$,
(T4) Boundary Condition $T(x, 1) = x$.

Example 1.5 *The following are the four basic t-norms:*

- *Minimum* T_M *given by*

$$T_M(x, y) = \min(x, y).$$

- *Product T_P given by*

$$T_P(x, y) = x \cdot y.$$

- *Lukasiewicz t-norm T_L given by*

$$T_L(x, y) = \max(0, x + y - 1).$$

- *Drastic product T_D given by*

$$T_D(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.6 *Note that the dual operator to the conjunctor C, defined by a non-decreasing mapping $D : [0, 1]^2 \to [0, 1]$ such that $D(a, b) = 1$ whenever $a = 1$ or $b = 1$ and $D(0, 0) = 0$, is called the disjunctor D. Commonly used disjunctors in MV-logic are the triangular conorms. Triangular conorms (also called $S-$norms) are dual to $t-$norms under the order reversing operation which assigns $1 - x$ to x on $[0, 1]$.*

In the literature, especially at the beginnings, we can find several different definitions of fuzzy implications. In this article we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [5].

Definition 1.7 *A function $I : [0, 1]^2 \to [0, 1]$ is called a fuzzy implicator if it satisfies the following conditons:*

(I1) I is decreasing in its first variable,

(I2) I is increasing in its second variable, and

(I3) $I(1, 0) = 0$, $I(0, 0) = I(1, 1) = 1$.

Now, we recall definitions of some important properties of implicators which we will investigate.

Definition 1.8 *A fuzzy implicator $I : [0, 1]^2 \to [0, 1]$ is said to satisfy:*

(NP) the left neutrality property, or is said to be left neutral, if

$$I(1, y) = y; \quad y \in [0, 1],$$

(EP) the exchange principle if

$$I(x, I(y, z)) = I(y, I(x, z)) \text{ for all } x, y, z \in [0, 1],$$

(IP) the identity principle if

$$I(x, x) = 1; \quad x \in [0, 1],$$

(OP) the ordering property if

$$x \leq y \iff I(x, y) = 1; \quad x, y \in [0, 1],$$

(CP) the contrapositive symmetry with respect to a given negator n if

$$I(x, y) = I(n(y), n(x)); \quad x, y \in [0, 1].$$

Definition 1.9 *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. The function N_I defined by $N_I(x) = I(x, 0)$ for all $x \in [0,1]$, is called the natural negation of I.*

Definition 1.10 *A function $I : [0,1]^2 \to [0,1]$ is called an $(S, N)-$implication if there exist a t-conorm S and fuzzy negation N such that*

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1].$$

If N is a strong negation, then I is called a strong implication.

The following characterization of $(S, N)-$implications is from [1].

Theorem 1.11 *For a function $I : [0,1]^2 \to [0,1]$, the following statements are equivalent:*

- *I is an (S, N)-implication generated from some t-conorm and some continuous (strict, strong) fuzzy negation N.*
- *I satisfies (I2), (EP) and N_I is a continuous (strict, strong, respectively) fuzzy negation.*

Another way of extending the classical binary implication operator to the unit interval $[0, 1]$ uses the *residuation* I with respect to a left-continuous triangular norm T

$$I(x, y) = \max\{z \in [0, 1]; T(x, z) \leq y\}.$$

Theorem 1.12 *For a function $I : [0,1]^2 \to [0,1]$, the following statements are equivalent:*

- *I is an R-implication based on some left-continuous t-norm T.*
- *I satisfies (I2), (OP), (EP), and $I(x, .)$ is a right-continuous for any $x \in [0, 1]$.*

Our constructions of implicators will make use of extending the classical inverse of function. One way of extending is described in next definitions.

Definition 1.13 *Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function. The function $\varphi^{(-1)}$ which is defined by*

$$\varphi^{(-1)}(x) = \sup\{z \in [0, 1]; \varphi(z) < x\},$$

is called the pseudo-inverse of the function φ, with the convention $\sup \emptyset = 0$.

Definition 1.14 *Let $f : [0,1] \to [0,1]$ be a non-increasing function. The function $f^{(-1)}$ which is defined by*

$$f^{(-1)}(x) = \sup\{z \in \ [0,1]; f(z) > \ x\},$$

is called the pseudo-inverse of the function f, with the convention $\sup\emptyset = 0$.

One of the main contributions of our paper are, in fact, corrollaries of the following technical result.

Proposition 1 *Let c be a positive real number, then for pseudo-inverse of positive multiple of any left-continuous function f we get*

$$(c \cdot f(x))^{(-1)} = f^{(-1)}\left(\frac{x}{c}\right).$$

Proof. Let f be a non-decreasing function. Then

$$f^{(-1)}(x) = \sup\{z \in \ [0,1]; f(z) < \ x\},$$

and

$$(c \cdot f)^{(-1)}(x) = \sup\left\{z \in \ [0,1]; c \cdot f(z) < \ x\right\} = \sup\left\{z \in \ [0,1]; f(z) < \ \frac{x}{c}\right\} = f^{(-1)}\left(\frac{x}{c}\right).$$

The proof for the case of non-increasing function is analogous.

2 New generated implicators

It is possible to generate implicators in a similar way as t-norms. Two of theese posibilities are in the next examples.

Example 2.1 *Let $f_1, f_2, f_3 : [0,1] \to [0,1]$ be functions defined as follows:*

$$f_1(x) = \begin{cases} 1 - x & \textit{if } x \leq 0.5, \\ 0.5 - 0.5x & \textit{otherwise}, \end{cases}$$

$$f_2(x) = \frac{1}{x} - 1,$$

$$f_3(x) = -\ln(x).$$

Therefore, we have for $f_1^{(-1)}(x), f_2^{(-1)}(x), f_3^{(-1)}(x)$:

$$f_1^{(-1)}(x) = \begin{cases} 1 - 2x & \textit{if } x \leq 0.25, \\ 0.5 & \textit{if } 0.25 < x \leq 0.5, \\ 1 - x & \textit{otherwise}, \end{cases}$$

$$f_2^{(-1)}(x) = \min\left\{\frac{1}{1+x}, 1\right\},$$

$$f_3^{(-1)}(x) = \min\{e^{-x}, 1\}.$$

Now, we will investigate operator $I_f(x,y) : [0,1]^2 \to [0,1]$ *which is given for decreasing function* $f(x)$ *by*

$$I_f(x,y) = f^{(-1)}(f(y^+) - f(x)).$$

For our functions $f_1(x), f_2(x), f_3(x)$ *we get*

$$I_{f_1}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - 2x + 2y & \text{if } x \leq 0.5, y < 0.5, x - y \leq 0.25, \\ 0.5 & \text{if } x \leq 0.5, y < 0.5, x - y > 0.25, \\ 0.5 & \text{if } x > 0.5, y < 0.5, x \leq 2y, \\ 0.5 + y - 0.5x & \text{if } x > 0.5, y < 0.5, x > 2y, \\ 1 - x + y & \text{if } x > 0.5, y \geq 0.5, \end{cases}$$

$$I_{f_2}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1}{\frac{1}{y} - \frac{1}{x} + 1} & \text{otherwise}, \end{cases}$$

$$I_{f_3}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{y}{x} & \text{otherwise}. \end{cases}$$

Note that I_{f_3} *is well-known Gödel implicator. We can see that previous mappings are implicators:*

$$I_{f_1}(1,1) = I_{f_2}(1,1) = I_{f_3}(1,1) = 1, \quad I_{f_1}(0,1) = I_{f_2}(0,1) = I_{f_3}(0,1) = 1,$$

$$I_{f_1}(0,0) = I_{f_2}(0,0) = I_{f_3}(0,0) = 1, \quad I_{f_1}(1,0) = I_{f_2}(1,0) = I_{f_3}(1,0) = 0,$$

I_{f_1}, I_{f_2} *and* I_{f_3} *are increasing in first and decreasing in second variable.*

Let's turn our attention to the next example:

Example 2.2 *Let* $g_1, g_2 : [0,1] \to [0,1]$ *be given by*

$$g_1(x) = \begin{cases} x & \text{if } x \leq 0.5, \\ 0.5 + 0.5x & \text{otherwise}, \end{cases}$$

$$g_2(x) = -\ln(1-x).$$

For functions $g_1^{(-1}(x)$ *and* $g_2^{(-1)}(x)$ *we get*

$$g_1^{(-1)}(x) = \begin{cases} x & \text{if } x \leq 0,5, \\ 0,5 & \text{if } 0,5 < x \leq 0,75, \\ 2x - 1 & \text{if } 0,75 < x \leq 1, \\ 1 & \text{if } 1 < x, \end{cases}$$

and

$$g_2^{(-1)}(x) = 1 - e^{-x} \text{ for } x \in [0, \infty].$$

Now we will investigate operator $I^g : [0,1]^2 \to [0,1]$, *which is given for increasing function* $g(x)$ *by*

$$I^g(x,y) = g^{-1}(g(1-x) + g(y)).$$

For functions g_1 *and* g_2 *we get*

$$I^{g_1}(x,y) = \begin{cases} 1 - x + y & \text{if } x \geq 0.5, y \leq 0.5, x - y \geq 0.5, \\ 0.5 & \text{if } x \geq 0.5, y \leq 0.5, 0.25 \leq x - y < 0.5, \\ 1 - 2x + 2y & \text{if } x \geq 0.5, y \leq 0.5, x - y < 0.25, \\ \min(1 - x + 2y, 1) & \text{if } x < 0.5, y \leq 0.5, \\ \min(2 - 2x + y, 1) & \text{if } x \geq 0.5, y > 0.5, \\ 1 & \text{if } x < 0.5, y > 0.5, \end{cases}$$

$$I^{g_2}(x,y) = 1 - e^{\ln(x(1-y))} = 1 - x + xy.$$

Note that I^{g_2} *is Kleene-Dienes implicator. As in previous example, also* I^{g_1} *and* I^{g_2} *are implicators.*

Our investigations we can use and form a theorem generalizing previous examples.

Theorem 2.3 *Let* $f : [0,1] \to [0,\infty]$ *be a strictly decreasing function such that* $f(1) = 0$ *and* $g : [0,1] \to [0,\infty]$ *be a strictly increasing function such that* $g(0) = 0$. *Then functions* I_f *and* I^g *defined as follows*

$$I_f(x,y) = f^{(-1)}(f(y^+) - f(x)),$$
$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y)),$$

are implicators.

The last example in this section shows that implicator I^g can be generalize.

Example 2.4 *We deal with operator which is given by*

$$I_n^g(x,y) = g^{-1}(g(n(x)) + g(y)).$$

We will use functions $g_1(x)$, $g_2(x)$ *and* $n(x) = 1 - x^2$. *And we get*

$$I_n^{g_1}(x,y) = \begin{cases} 1 - x^2 + y & \text{if } x \geq \frac{1}{\sqrt{2}}, y \leq 0.5, x^2 - y \geq 0.5, \\ 0.5 & \text{if } x \geq \frac{1}{\sqrt{2}}, y \leq 0.5, 0.25 \leq x^2 - y < 0.5, \\ 1 - 2x^2 + 2y & \text{if } x \geq \frac{1}{\sqrt{2}}, y \leq 0.5, x^2 - y < 0.25, \\ \min(1 - x^2 + 2y, 1) & \text{if } x < \frac{1}{\sqrt{2}}, y \leq 0.5, \\ \min(2 - 2x^2 + y, 1) & \text{if } x \geq \frac{1}{\sqrt{2}}, y > 0.5, \\ 1 & \text{if } x < \frac{1}{\sqrt{2}}, y > 0.5, \end{cases}$$

$$I_n^{g_2}(x,y) = 1 - x^2 + x^2 y.$$

Obviously, $I_n^{g_1}$ *and* $I_n^{g_2}$ *are implicators. Note that the mappings* I^{g_i} *from previous example are only special case of* I_n^g, *when* $n(x) = 1 - x$.

Now we are able to generalize the previous theorem:

Theorem 2.5 *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$ and n be a fuzzy negator. Then the function I_n^g:*

$$I_n^g(x,y) = g^{(-1)}(g(n(x)) + g(y)),$$

is an implicator.

3 Basic properties of generated implicators

First, we will investigate the properties of I_f implicators:

Proposition 2 *Let $f : [0,1] \to [0,\infty]$ be a left-continuous, strictly decreasing function such that $f(1) = 0$. Then I_f satisfies ordering and neutrality properties. Moreover, if f is continuous strictly decreasing function such that $f(1) = 0$, then implicator I_f satisfies exchange principle.*

It is well known that generators of continuous Archimedean t-norms are unique up to a positive multiplicative constant, and this is also true for the f generators of I_f implicators. The next theorem is a corrollary of Proposition 1.

Theorem 3.1 *The f generator of an I_f implicator is uniquely determined up to a positive multiplicative constant.*

Second, we turn our attention to the implicators I^g and their properties.

Proposition 3 *Let $g : [0,1] \to [0,\infty]$ be a left-continuous, strictly increasing function such that $g(0) = 0$. Then I^g satisfies neutrality property and it is a contrapositive implicator. Moreover, if g is a continuous, strictly increasing function such that $g(0) = 0$, then the implicator I^g satisfies the exchange principle.*

Theorem 3.2 *The generator g of an I^g implicator is uniquely determined up to a positive multiplicative constant.*

Remark 3.3 *Note, that if $f(x) = g(1-x)$, then the implicators I_f and I^g are identical.*

For implicators I_n^g and their properties we get next propositions.

Proposition 4 *Let n be a negator, and g be a left-continuous, strictly increasing function such that $g(0) = 0$. Then the function $I_n^g : [0,1]^2 \to [0,1]$ which is defined by*

$$I_n^g(x,y) = g^{(-1)}(g(n(x)) + g(y)),$$

satisfies the neutrality property.

Proposition 5 *Let n be an involutive negator, g be a left-continuous, strictly increasing function such that $g(0) = 0$. Then implicator I_n^g is a contrapositive implicator with respect to the negator n.*

Proposition 6 *Let g be a continuous, strictly increasing function such that $g(0) = 0$. Then implicator I_n^g satisfies the exchange principle.*

Theorem 3.4 *The g generator of an I_n^g implicator is uniquely determined up to a positive multiplicative constant.*

4 Other properties of our generated implicators

In this section we will investigate properties of natural negations, which are based on our generated implicators.

Example 4.1 *Let $f_1(x), f_2(x), f_3(x)$ be functions which were mentioned in Example 2.1. Then the natural negators N_I are given by:*

$$N_{I_{f_1}}(x) = \begin{cases} 1 - 2x & \text{if } x \leq 0.25, \\ 0.5 & \text{if } 0.25 < x \leq 0.5, \\ 0.5 - 0.5x & \text{otherwise}, \end{cases}$$

$$N_{I_{f_2}}(x) = N_{I_{f_3}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 4.2 *Note, that N_{f_2} and N_{f_3} coincide with Gödel negator. More, it is easy to proof, that if $f : [0, 1] \to [0, \infty]$ is a strictly decreasing function such that $f(0) = \infty$ and f is right-continuous in $x = 0$, then $N_I(x)$ given by $I_f(x, y)$ is Gödel negation $N_{G_1}(x)$.*

Proposition 7 *Let $f : [0, 1] \to [0, \infty]$ be a left-continuous, strictly decreasing function, such that $f(1) = 0$. Then I_f is a R-implicator based on a conjunctor C, which is given by the additive generator f.*

Remark 4.3 *Note that $I^g = R_{C^*}$, where C^* is the conjunctor generated by the additive generator f^* and $f^*(x) = g(1 - x)$.*

Let us turn our attention to increasing functions g_1 and g_2 from Example 2.2:

Example 4.4 *If we deal with functions g_1 and g_2 then natural negators are given by:*

$$N_{I^{g_1}} = N_{I^{g_2}} = 1 - x.$$

Also for $n = 1 - x^2, g_1, g_2$ and implicators I_n^g we get:

$$N_{I_n^{g_1}} = N_{I_n^{g_2}} = 1 - x^2.$$

Since g is left-continuous and strictly increasing, we have $\forall\, x \in [0,1];\ g^{(-1)}(g(x)) = x.$ Therefore

$$N_I(x) = g^{(-1)}(g(n(x)) + g(0)) = g^{(-1)}(g(n(x))) = n(x).$$

This property and our investigation in the previous example lead us to the next statement.

Proposition 8 *Let n be a negator, and g be a left-continuous, strictly increasing function such that $g(0) = 0$. Then for I_n^g holds:*

$$\forall x \in\ [0,1];\ N_{I_n^g}(x) = n(x).$$

Theorem 4.5 *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing, continuous function, $g(0) = 0$, and n be a continuous (strict, strong) negation. Then $I_n^g(x,y) = g^{(-1)}(g(n(x))+g(y))$ is (S,N)-implication.*

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PSEUDOMETRICS AND FUZZY RELATIONS

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Abstract. The main goal of this article is to discuss the problem of a connection between pseudometrics and two types of binary fuzzy relations on **R**, namely so called coherent nearnesses and similarity relations, mainly the question whether they are "inverse" one another.

Key words and phrases. Pseudometric, Coherent nearness, Similarity relation, Fuzzy quantity.

1 Introduction

In this paper we are going to study some metric and other properties of two types of real binary fuzzy equivalence relations. Fuzzy equivalence relations play a significant role in fuzzy set theory, analogical to the role of crisp equivalence relations in classical theory.

They model in a sense proximity, indistinguishability, or similarity of elements from an arbitrary universe X. We will deal with two kinds of fuzzy equivalence relations. First of them was introduced in 1971 by Lotfi Zadeh, a founder of the fuzzy set theory and it is called a **similarity relation**. The other type, a fuzzy equivalence called nearness, was defined and studied in several articles ([1],[2],[4],[5]). We will pay attention to a special type of a nearness, called a **coherent nearness**.

Although both these fuzzy relations originally are defined on an arbitrary universe X, we will study them only on the set of all real numbers $\mathbb{R}$. We want to discuss two problems.

Both follow from the fact, that while real binary fuzzy equivalences measure in some sense propinquity or closeness of two real numbers, metrics or pseudometrics on $\mathbb{R}$ measure their distance. From this point of view fuzzy equivalences on one side and pseudometrics on the other side are complementary, one can say "inverse" each to other. We are interested whether, if d is a pseodometric on $\mathbb{R}$ with values from the interval $\langle 0, 1\rangle$, the binary fuzzy relation $1 - d$ is always a coherent nearness, or even a similarity.

And conversely, if we have a coherent nearness N (or a similarity S), whether the binary fuzzy relation $1 - N$ (or $1 - S$) is always a pseudometric.

First let us recall some basic notions and concepts.

2 Preliminaries

Definition 1.
A binary fuzzy relation S on a universe X is called a **similarity relation** on X if and only if it is reflexive, symmetric and transitive w.r.t.minimum, it means, if and only if for any $x, y, z \in X$:

(S1) $S(x, x) = 1$

(S2) $S(x, y) = S(y, x)$

(S3) $\min(S(x, y), S(y, z)) \leq S(x, z)$.

In this article we will restrict ourselves to the case of real similarity relations, if the universe $X = \mathbb{R}$.

From Theorem 5.1. and Proposition 5.4. in [3], it follows

Proposition 1.
A fuzzy relation S on the universe $\mathbb{R}$ is a similarity relation if and only if there exists a fuzzy set $h \in \mathcal{F}(\mathbb{R})$ such that

$$S(x, y) = \begin{cases} 1, & \textit{if} \quad x = y, \\ \min(h(x), h(y)), & \textit{if} \quad x \neq y \end{cases}$$

Another approach to the fuzzification of proximity is based on the concept of a nearness (see, for example, [1], [2], [4], [5]). In [1] a type of nearness, called a **coherent nearness**, modeling equivalence or proximity of two elements in a universe X, was introduced as follows.

Definition 2.
Let X be a universe. A binary fuzzy relation N on X is called a coherent nearness on X, if:

(N1) $N(x, x) = 1$, for each $x \in X$

(N2) $N(x, y) = N(y, x)$, for each $x, y \in X$

(N3) For each $\epsilon > 0$ there exists $\delta < 1$ such that

$$N(x, y) > \delta \Longrightarrow |N(x, z) - N(y, z)| < \epsilon, \text{ for each } x, y, z \in X.$$

For more detailed terminology of fuzzy notions see e.g. [6].

In what follows, for abbreviation, let **similarity** stand for a similarity relation on $\mathbb{R}$ and **nearness** for a coherent nearness on the universe $\mathbb{R}$ as well.

3 Nearnesses and pseudometrics

Relationship between similarities and nearnesses was studied in [2]. There is proved (Proposition 3) that each similarity satisfies (N3), hence it is a nearness, but opposite is not true (Example 1).

Now let d be a pseudometric on $\mathbb{R}$. We restrict ourselves to the pseudometrics with values from the interval $\langle 0,1\rangle$. Under this condition a pseudometric d is a real binary fuzzy relation, which is reflexive, symmetrical and satisfies the triangular inequality. In the remainder of this paper we assume d to be such a pseudometric.

Regarding the mentioned complementarity of pseudometrics and fuzzy equivalences, we can expect that a binary fuzzy relation R, defined on $\mathbb{R}$ by:

$$R(x,y) = 1 - d(x,y)$$

is a nearness. Let us prove it.

Proposition 2.
Suppose d is a pseudometric on $\mathbb{R}$. Then binary fuzzy relation R, defined by:

$$R(x,y) = 1 - d(x,y)$$

satisfies the property (N3), therefore, it is a nearness.

Proof. We want to prove that for each $\epsilon > 0$ there exists $\delta < 1$ such that

$$R(x,y) > \delta \Longrightarrow |R(x,z) - R(y,z)| < \epsilon, \text{ for each } x,y,z \in \mathbb{R}.$$

Let $\varepsilon > 0$. We can suppose, that $\varepsilon < 1$. Put $\delta = 1 - \varepsilon$. Let x, y be two real numbers, such that $R(x,y) > \delta$, let z be any real number. Then

$$|R(x,z) - R(y,z)| = |1 - d(x,z) - (1 - d(y,z))| = |d(x,z) - d(y,z)|.$$

But from the triangular inequality it follows that

$$|d(x,z) - d(y,z)| \leq d(x,y) = 1 - R(x,y) < 1 - \delta = \varepsilon.$$

Opposite is not true. It means, if $N(x,y)$ is a nearness, then binary fuzzy relation

$$R(x,y) = 1 - N(x,y)$$

can, but need not be a pseudometric.

Proposition 3.
Let $f : \langle 0,\infty) \to \langle 0,1\rangle$ be a non-decreasing subadditive function. Then the binary fuzzy relation $d(x,y) = f(|x-y|)$ is a pseudometric on $\mathbb{R}$.

Proof. Reflexivity and symmetricity of d is trivial. What about the triangular inequality, for each $x, y, z \in \mathbb{R}$

$$d(x,y) = f(|x-y|) \leq f(|x-z| + |y-z|) \leq f(|x-z|) + f(|y-z|) = d(x,z) + d(y,z)$$

Example 1.

Let for each $x, y \in \mathbb{R}$ be

$$N(x,y) = \frac{1}{1+|x-y|}$$

As follows from [1], N is a nearness. Define

$$R(x,y) = 1 - N(x,y) = \frac{|x-y|}{1+|x-y|}$$

Since the function $f : y = \frac{x}{1+x}$ satisfies assumptions of Proposition 2, it follows that $R(x,y) = f(|x-y|)$ is a pseudometric. Figure 1 shows graphs of this couple of real binary fuzzy relations on the interval $\langle -4, 4\rangle \times \langle -4, 4\rangle$.

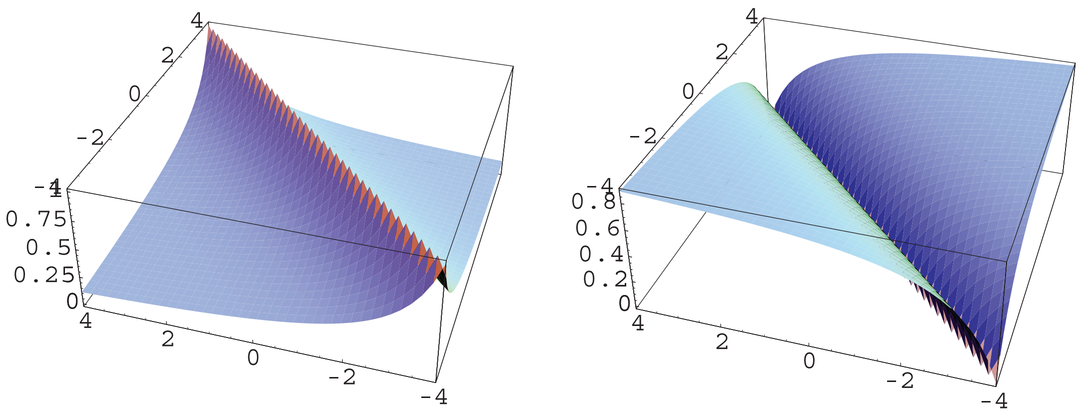

Figure 1.

In the next example we present a nearness, such that its "inverse" is not a pseudometric.

Example 2.

Let N be a nearness, defined as follows:

$$N(x,y) = \begin{cases} 1, & \text{if } x = y, \\ \frac{3}{4}, & \text{if } 0 < |x-y| \leq 1, \\ \frac{1}{4}, & \text{if } |x-y| > 1 \end{cases}$$

It is simple to see, that N satisfies the property (N3), therefore, it is coherent. Now define a binary fuzzy relation

$$R(x,y) = 1 - N(x,y)$$

It is obvious, that R is reflexive and symmetrical, but it does not satisfy the triangular inequality:

$$R(-1,0) + R(0,1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < \frac{3}{4} = R(-1,1)$$

In spite of the fact, that the "inverse" of a nearness need not be a pseudometric, it always satisfies a property, similar to the property (N3), a little weaker than the triangular inequality.

Proposition 4.

Suppose N is a nearness on $\mathbb{R}$. Then reflexive and symmetrical binary fuzzy relation R, defined by:

$$R(x,y) = 1 - N(x,y)$$

satisfies the following property: For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$R(x,y) < \delta \Longrightarrow |R(x,z) - R(y,z)| < \epsilon, \text{ for each } x,y,z \in X.$$

Proof. Trivial proof is left to the reader.

4 Similarities and pseudometrics

Since each similarity is a nearness, but opposite is not valid ([2]), we may ask whether analogy of Proposition 2 is still true for similarities. Hence, if d is a pseudometric, whether the fuzzy relation $R(x,y) = 1 - d(x,y)$ is always a similarity. Consider the following

Example 3.

Define $d(x,y) = \min(|x-y|, 1)$ for each $x,y,z \in \mathbb{R}$ and

$$R(x,y) = 1 - d(x,y) = 1 - \min(|x-y|, 1)$$

Since d is a pseudometric, according Proposition 2 R is a nearness. But, because it does not satisfy the property (S3), it is not a similarity. For example:

$$\min\left(R\left(1,\frac{3}{2}\right), R\left(2,\frac{3}{2}\right)\right) = \min\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2} > 0 = R(x,y)$$

On the other hand, in Example 2 it is presented such a nearness N, that the fuzzy relation $R(x,y) = 1 - N(x,y)$ does not satisfy the triangular inequality, therefore, it is no pseudometric. After the following proposition, it cannot happen, if the nearness is replaced by a similarity.

Proposition 5.

Suppose S is a similarity on $\mathbb{R}$. Then reflexive and symmetrical binary fuzzy relation R, defined by:

$$R(x,y) = 1 - S(x,y)$$

is a pseudometric.

Proof. Reflexivity and symmetricity are straightforward. Let us prove the triangular inequality. Let x,y,z be any real numbers. Then

$$S(x,y) \geq \min(S(x,z), S(y,z)) \geq \min(S(x,z), S(y,z)) + \max(S(x,z), S(y,z)) - 1 =$$

$$= S(x,z) + S(y,z) - 1 \Rightarrow -S(x+y) \leq 1 - S(x,z) - S(y,z) \Rightarrow 1 - S(x,y) \leq$$
$$\leq 1 - S(x,z) + 1 - S(y,z) \Rightarrow R(x,y) \leq R(x,z) + R(y,z)$$

Corollary.

Suppose h is a fuzzy quantity. Then reflexive and symmetrical binary fuzzy relation R, defined by

$$R(x,y) = \begin{cases} 0, & \textit{if} \quad x = y, \\ 1 - \min(h(x), h(y)), & \textit{if} \quad x \neq y \end{cases}$$

is a pseudometric on $\mathbb{R}$.

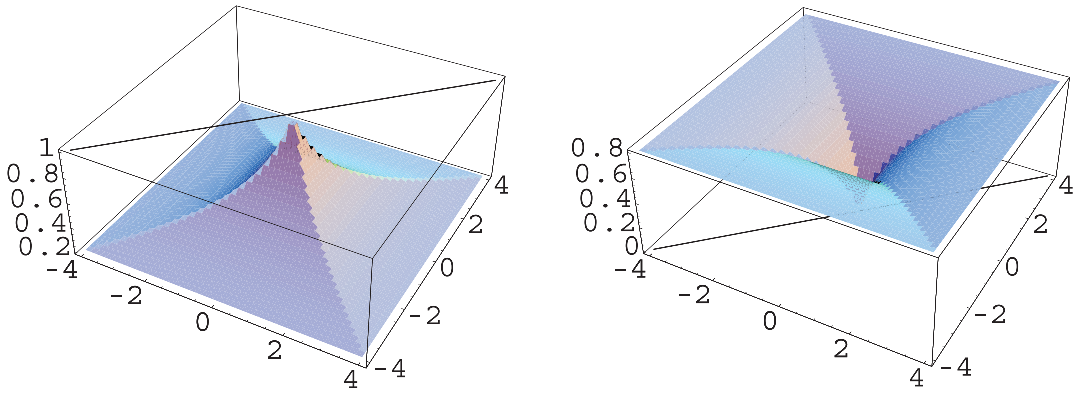

Figure 2.

Figure 2. above shows graphs of two mutually "inverse" binary fuzzy relations

$$S(x,y) = \begin{cases} 1, & \text{if} \quad x = y, \\ \min(h(x), h(y)), & \text{if} \quad x \neq y \end{cases} \qquad \text{and} \qquad d(x,y) = 1 - \min(h(x), h(y))$$

for $h(x) = \frac{1}{1+|x|}$ on the interval $\langle -4, 4 \rangle \times \langle -4, 4 \rangle$.

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SOFTWARE SYSTEM FOR TIME SERIES PREDICTION BASED ON F-TRANSFORM AND LINGUISTIC RULES

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Abstract. The article describes software tool for prediction of time series based on various methods derived from F-transformation and linguistic rules. The tool has been created on Institute for Research and Applications of Fuzzy Modeling, University of Ostrava. It is developing into a really powerful application giving good results in some cases.

Key words. Fuzzy logic, time series prediction, F-transform, linguistic rules.

Mathematics Subject Classification: Primary 37M10; Secondary 94D05.

1 Introduction

Time series analysis and prediction is an important task that can be used in many areas of practice. The task of getting the best prediction to given series may bring interesting engineering applications in wide number of areas like economics, geography or industry. Solution to the problem of obtaining best results in prediction of time series can be based on well-known and simple methods like Winters or Linear method [2]. In this paper we present a tool based on two methods originally developed by members of Institute for Research and Applications of Fuzzy Modeling. The aim of the paper is not to present the details of the methods already published, but to present a tool implementing them. The first method is based on the notion of F-transform devised by the group of Prof. Perfilieva [3]. The second approach use the linguistic rules utilizing fuzzy logic and deduction that is a well-known formalism with very good results in variety of practical applications like industrial ones.

2 F-transform

The core idea of the F-transform technique is a fuzzy partition of the universe. It can be simply presented like set of intervals fulfilling some criteria. It is described in the following definition that is stated in [4].

Definition 1
Let $x_1 < \ldots < x_n$ be fixed nodes within $[a, b]$, such that $x_1 = a$, $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1(x), \ldots, A_n(x)$ defined on $[a, b]$, form a *fuzzy partition* of $[a, b]$ if they fulfil the following conditions for $k = 1, \ldots, n$:

(1) $A_k : [a, b] \longrightarrow [0, 1]$, $A_k(x_k) = 1$;
(2) $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where for the uniformity of denotation, we put $x_0 = a$ and $x_{n+1} = b$;
(3) $A_k(x)$ is continuous;
(4) $A_k(x)$, $k = 2, \ldots, n$, monotonically increases on $[x_{k-1}, x_k]$ and $A_k(x)$, $k = 1, \ldots, n-1$, monotonically decreases on $[x_k, x_{k+1}]$;
(5) for all $x \in [a, b]$

$$\sum_{k=1}^{n} A_k(x) = 1. \qquad (1)$$

The membership functions $A_1(x), \ldots, A_n(x)$ are called *basic functions*.

These partitions form a base for F-transform which lead to the tuple of numbers representing original transformated function. The n-tuple can be obtained using the following notion.

Definition 2
Let $f \in V_l$ be given and $A_1, \ldots, A_n$, $n < l$, be basic functions which constitute a fuzzy partition of $[a, b]$. We say that the n-tuple of real numbers $[F_1, \ldots, F_n]$ is the F-transform of f with respect to $A_1, \ldots, A_n$ if

$$F_k = \frac{\sum_{j=1}^{l} f(p_j) A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)}. \qquad (3)$$

To forecast a time series we will use its F-transform representation and separately forecast the next component Y_{n+1} of the F-transform(of y_t) and a respective residuum. We will consider three methods for the forecasting a component of the F-transform: the F-transform of the second order an extrapolation of the inverse fuzzy transform and a logical deduction [1].

3 Linguistic rules

The theory of linguistic term and variables is well-known approach in the fuzzy logic community. It enables to work with rules containing terms of natural language like small or big and modifiers like very, roughly etc. The rule interpretation is then done by logical deduction based on which is based on fuzzy set theory and fuzzy logic to enable to deduce conclusions on the basis of imprecise description of the given situation using the linguistically formulated fuzzy IF-THEN rules [1].

The usage of this theory within a frame of time series prediction lies in the learning of these rules from the serie and then application to future (predicted) members of the serie. These learning algorithms are already prepared within the LFLC software [1], which is intended to perform logical deduction on linguistic rules. The core of the system serve also for the presented application.

Example

An example fuzzy IF-THEN rule is

IF *the obstacle is near* AND *the speed of the car is high*
THEN *the breaking force is very strong*.

The "obstacle", "speed" and "breaking force" are variables while "near", "high" and "very strong" are expressions characterizing vaguely the magnitude of the variable.

The part before THEN is called the *antecedent* and the part after it the *succedent*. The variables $X_1, \ldots, X_n$ are called *input*, or *independent* variables. The variable Y is called *output*, or *dependent* variable.

The fuzzy IF-THEN rules are usually put together to form the *linguistic description*

$$\mathcal{R}_1 := \text{IF } X_1 \text{ is } \mathcal{A}_{11} \text{ AND } \cdots \text{ AND } X_n \text{ is } \mathcal{A}_{1n} \text{ THEN } Y \text{ is } \mathcal{B}_1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathcal{R}_m := \text{IF } X_1 \text{ is } \mathcal{A}_{m1} \text{ AND } \cdots \text{ AND } X_n \text{ is } \mathcal{A}_{mn} \text{ THEN } Y \text{ is } \mathcal{B}_m$$

4 Time series tool

The software for prediction of time series based on the previously presented formalisms is currently in the development, but first alpha version is now complete. It was implemented on MS-Windows platform in C++ using free GUI library WxWidgets. The predecessor of the tool has been an console application without GUI, but shares the same core like the tool.

The main tasks of the tool are the following:

- Loading and presentation of a prepared file with a time serie in a graph.
- Setting up the methods and parameters for prediction.
- Computation of prediction according to the methods.
- Presentation and selection of predictions in a graph with reports generated during prediction (difference to the original serie etc.)
- Export of selected results.

The application is an SDI (single document interface) application divided into two main windows – Plot and Methods (Fig. 1). Plot window allows opening of user chosen serie through standard file open dialog (File menu). Then it performs desired predictions on loaded serie and presents it in a graph. The graph window contains two basic panels – graphical information panel and text information panel. Graphical information panel enables to present standard graph of the serie and predictions, it can be zoomed (important method is to fit the curve into graph which is connected to 's' key). Some basic graph operations can be obtained by Plot menu. Last menu item - Help - shows the application info and basics of application control (Fig.2).
Text information panel describes performed predictions projected in graphical information panel. It presents information concerning particular prediction in one line (color refers to the same color in graph). The information given could be presented on the following simple example:

No.4 Predictor, name: AvgTrend[k=5] {IntRule[v=9]}, error:0.0105458

It divides into predictor number, prediction description (methods used) – Trend (k = seasons number included for prediction), {method [specification]}, error of prediction. For our example it means:
4-th best prediction, trend is computed from average,[number of seasons included = 5]{logical deduction was used with [variable number = 9]}.

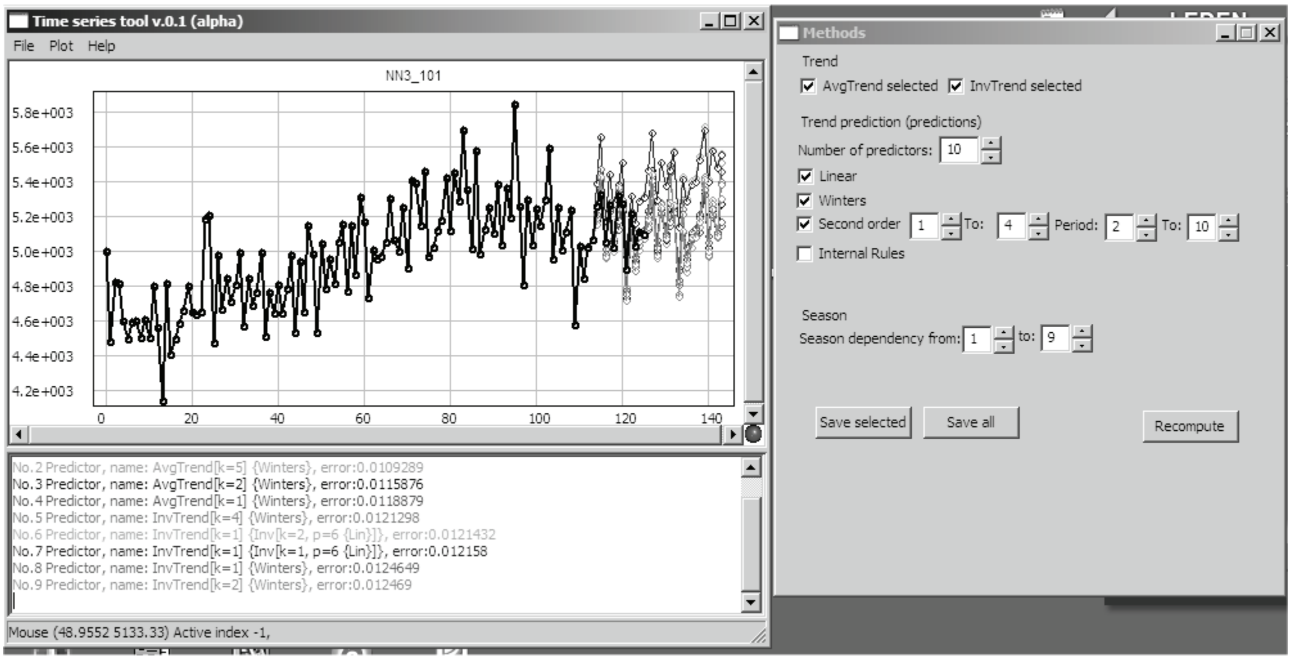

Fig. 1:Time series tool layout

Methods window (Fig. 3) enables to set up the details for prediction process. It consists of four basic parts:

1. Trend computation selection.
2. Method computation selection.
3. Season part computation selection.
4. Operations with application – computation of prediction, export of the selected curve, export of all curves into file representation.

Trend could be selected either to be computed via standard average method or via inverse F-transform. There are basically four possibilities how to predict serie – standard Linear or standard Winters method or new method of logical deduction or second order F-transform. Season computation can be additionally set up for season's dependency.

The input file with a serie should conform to a simple format, which could be observed from the following example:

NN3_101	4998	4480	4824	4814	4602	4499	4594	4600	4507	4606	4503
4801	4564	4142	4818	4408	4496	4587	4656	4799	4652	4638	4650

First item should be name of the serie followed by unlimited number of real numbers delimited by a blank space (TAB, space, etc.).

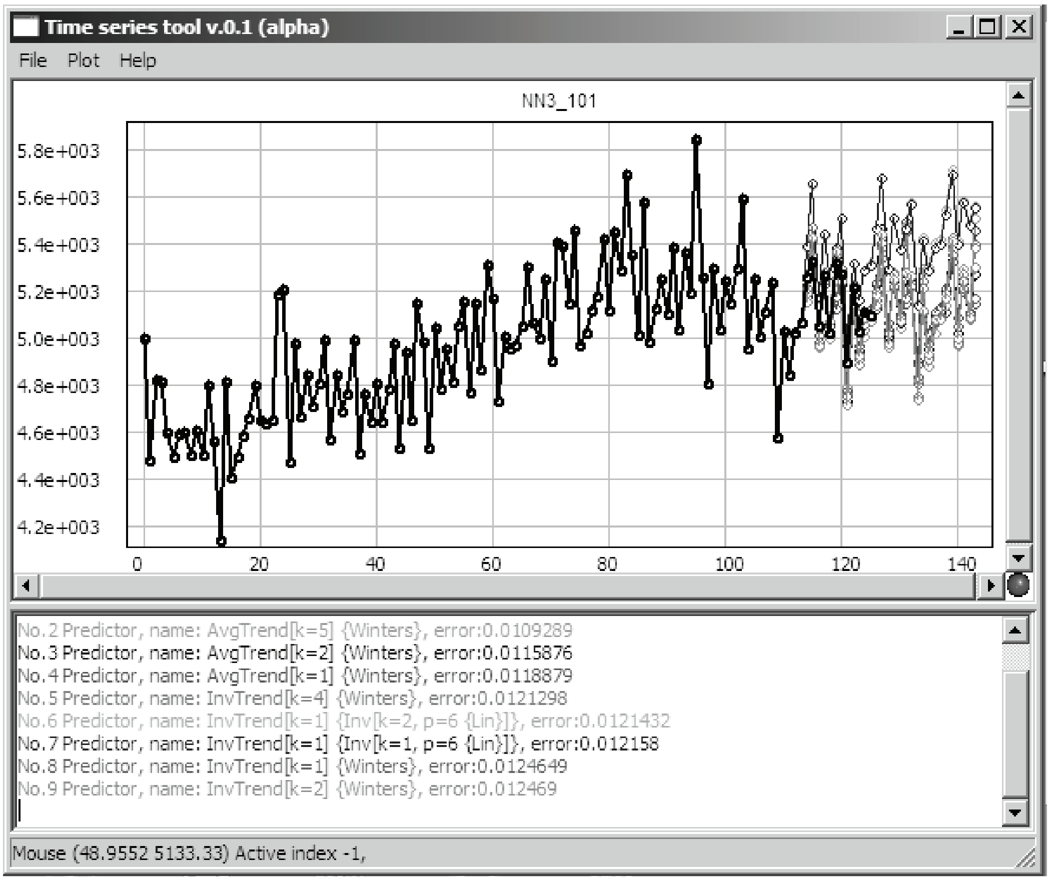

Fig. 2: Plot window

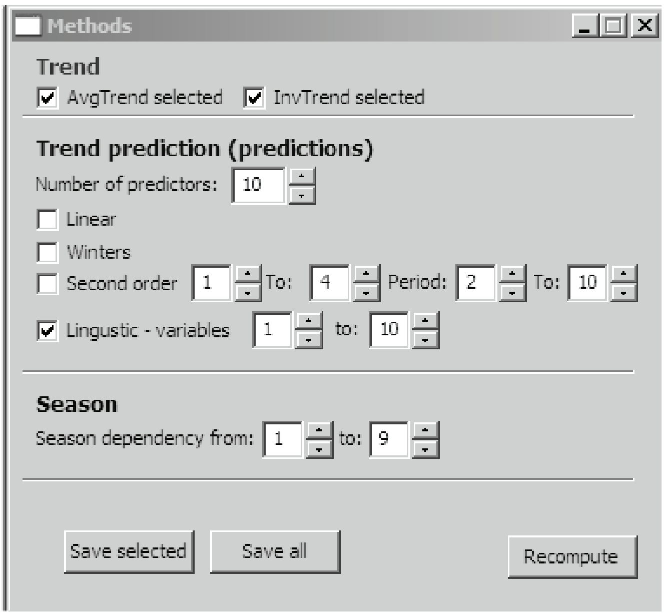

Fig. 3: Methods window

5 Conclusion

The presented application is currently in initial stage of usage and experimentation. We believe it may bring interesting results in comparison with standard methods. Initial experiments show mainly promising values for logical deduction. Application will be soon deployed in its demo version and also there will be additions to the core of methods utilized.

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A NEW SOFTWARE FOR FUZZY METHODS OF MULTIPLE CRITERIA EVALUATION

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Abstract. The paper deals with a new software product FuzzME, which serves for creating fuzzy models of multiple criteria evaluation and decision making. These fuzzy models work with the type of evaluation, which fully corresponds to the paradigm of fuzzy sets theory; all the evaluations occurring in these models have the same clear interpretation - they represent (fuzzy) degrees of fulfillment of the corresponding goals. The software FuzzME can work with both quantitative and qualitative criteria. In the tree of partial goals it is possible to combine arbitrarily two types of aggregation of partial evaluations - a fuzzy weighted average (fuzzy weights, fuzzy evaluations) and a fuzzy expert system. The paper contains an illustrative example based on a real practical problem which was solved in one of the Austrian banks.

Key words and phrases. Fuzzy models, Evaluation, Decision making, Software.

Mathematics Subject Classification. Primary 90B50, 91B06; Secondary 03E72.

1 Introduction

Contemporary social practice more and more often demands the application of mathematical methods of evaluation. The vast majority of evaluation procedures depend on more than one criterion. Besides quantitative criteria, qualitative criteria evaluated expertly must be often also taken into account. To create the evaluation model expert knowledge is often necessary (weights of criteria, partial evaluating functions for quantitative criteria and rule base describing relation between criteria and overall evaluation). For that reason the fuzzy sets theory is the appropriate mathematical instrument for creating multiple criteria evaluation and decision making models. For practical use of the fuzzy evaluation models a user-friendly software is necessary. But a good theoretical base of the applied models of evaluation, which gives among others clear interpretation of input and output values, is important as well.

The most commonly used software for multiple criteria evaluation and decision making based on fuzzy models is FuzzyTECH [7] (even if its main application area is fuzzy control). FuzzyTECH is a general software product that makes it possible to form and use fuzzy expert systems; it includes neural networks algorithms for deriving fuzzy rule bases from data. In the literature [6] interesting applications of this software to evaluation and decision making in the area of business and finance were published.

In 2000 a Czech software company, TESCO SW Inc., developed a software product whose name is NEFRIT. It uses fuzzy methods for multiple criteria evaluation and decision making. Its mathematical bases are described in the book [1]. NEFRIT can work with expert fuzzy evaluations of alternatives according to qualitative criteria; values of quantitative criteria can be either crisp or fuzzy. Evaluating functions for quantitative criteria represent membership functions of partial fuzzy goals. Aggregation is done with the method of weighted average of partial fuzzy evaluations. The weights (crisp, not fuzzy) express shares of particular partial evaluations in the aggregated evaluation. Fuzzy evaluations on all levels of the goals tree express fuzzy degrees of fulfillment of the corresponding goals. NEFRIT does not make it possible to use a fuzzy expert system for evaluation. The software NEFRIT (its demo version is enclosed to the book [1]) was originally developed for the Czech National Bank and was also used by the Czech Tennis Association, by the Czech Basketball Association, etc. At the present time the software is tested by the Supreme Audit Office of the Czech Republic.

The program FuzzME (Fuzzy methods of Multiple criteria Evaluation), which is presented in this paper, is based (analogous to NEFRIT) on the theoretical conception described in [1]. But in contrast with NEFRIT it also admits to define evaluating function by a fuzzy rule base. For the approximate reasoning two algorithms are available - the standard Mamdani algorithm and the generalized Sugeno algorithm (proposed in [1]). For the aggregation of the partial evaluations by the method of weighted average fuzzy weights can be used. The theory of normalized fuzzy weights, ways of their setting (including a method for removing potential inconsistence) and algorithms for calculation of the fuzzy weighted average are taken from [2].

2 Preliminaries

A fuzzy set A on a universal set X is given by its membership function $A : X \to [0, 1]$. $Ker\ A$ denotes a kernel of A, $KerA = \{x \in X \mid A(x) = 1\}$. For any $\alpha \in (0, 1]$, A_α denotes an α-cut of A, $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$. A support A, $Supp\ A$, is defined by $Supp\ A = \{x \in X \mid A(x) > 0\}$. A height of the fuzzy set A, $hgt\ A$, is defined as follows $hgt\ A = \sup\{A(x) \mid x \in X\}$.

A fuzzy number is a fuzzy set C on the set of real numbers $\Re$ which satisfies the following conditions: a) the kernel of C, $Ker\ C$, is not empty, b) the α-cuts of C, C_α, are closed intervals for all $\alpha \in (0, 1]$, c) the support of C, $Supp\ C$, is bounded. A fuzzy number C is called to be defined on $[a, b]$, if $Supp\ C \subseteq [a, b]$.

Real numbers $c^1 \leq c^2 \leq c^3 \leq c^4$ are called significant values of the fuzzy number C, if the following holds: $[c^1, c^4] = Cl(Supp\ C), [c^2, c^3] = Ker\ C$, where $Cl(Supp\ C)$ denotes a closure of $Supp\ C$.

To set uncertain input values in fuzzy models, linear fuzzy numbers are usually used. A fuzzy number C with the significant values $c^1 \leq c^2 \leq c^3 \leq c^4$ is a linear fuzzy number if its membership function is defined as follows

$$C(x) = \begin{cases} 0 & \text{for } x < x_1 \\ \frac{x-x_1}{x_2-x_1} & \text{pro } x_1 \leq x < x_2 \\ 1 & \text{for } x_2 \leq x \leq x_3 \\ \frac{x_4-x}{x_4-x_3} & \text{pro } x_3 < x \leq x_4 \\ 0 & \text{for } x_4 < x. \end{cases} \tag{1}$$

The linear fuzzy number C is uniquely determined by its significant values so we denote $C = (c^1, c^2, c^3, c^4)$. If $c^2 \neq c^3$ the linear fuzzy number C is called trapezoidal. For $c^2 = c^3$ the fuzzy number is called triangular.

For calculations inside the fuzzy models, piecewise linear fuzzy numbers of a class n are suitable to use. (The program FuzzME allows to use piecewise linear fuzzy numbers of an arbitrary class also for modeling input fuzzy values.) A piecewise linear fuzzy number is a fuzzy number which membership function is piecewise linear and is determined by the following sequence of $2n+4$ points $\{(x_1, 0), (x_2, \frac{1}{n+1}), \ldots, (x_{n+1}, \frac{n}{n+1}), (x_{n+2}, 1), (x_{n+3}, 1), (x_{n+4}, \frac{n}{n+1}), \ldots, (x_{2n+3}, \frac{1}{n+1}), (x_{2n+4}, 0)\}$, where $x_1 \leq x_2 \leq \ldots \leq x_{n+1} \leq x_{n+2} \leq x_{n+3} \leq x_{n+4} \leq \ldots \leq x_{2n+3} \leq x_{2n+4}$. For $n = 0$ linear fuzzy numbers are obtained.

Calculations with fuzzy numbers are based on the extension principle. For example, let a continuous function $f : [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \to \Re$ be given, and $f([a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]) = [c, d]$. Let X_i be fuzzy numbers defined on $[a_i, b_i]$, $i = \{1, \ldots, n\}$. Then $Y = f(X_1, X_2, \ldots, X_n)$ is a fuzzy number defined on $[c, d]$, whose membership function is given by the following formula:

$$Y(y) = \max\{\min\{X_1(x_1), \ldots, X_n(x_n)\} \mid y = f(x_1, \ldots, x_n), x_i \in [a_i, b_i], i = 1, \ldots, n\} \tag{2}$$

for $y \in [c, d]$ and 0 elsewhere.

3 Program FuzzME

The mathematical models of the program FuzzME are namely based on the theory and methods of multiple criteria evaluation that was presented in [1] and [5]. The theory of normalized fuzzy weights and the fuzzy weighted average operation have been taken from [2], [4] and [3].

In the program FuzzME, the basic structure of the multiple-criteria evaluation model is expressed by a goals tree. The root of the tree represents the overall goal and each branching corresponds to a partial goal. The goals at the ends of the tree branches are connected either with quantitative or qualitative criteria.

When an alternative is evaluated, evaluations with respect to criteria connected with the ends of the branches are calculated first. Independently of the type of the criterion, each of the evaluations is described by a fuzzy number on the interval $[0, 1]$ and it expresses the degree of fulfillment of the corresponding partial goal.

These partial evaluations are then aggregated according to the type of the node of the tree. Either the fuzzy weighted average method or the fuzzy expert system method is used for the aggregation. For the fuzzy weighted average method the normalized fuzzy weights must be set in advance. They express shares of the goals of the lower level in the goal of the upper level. For the fuzzy expert system the fuzzy rule base must be defined and an inference algorithm (the standard Mamdani algorithm of approximate reasoning or the generalized Sugeno algorithm) must be chosen.

The overall evaluation reflects the degree of fulfillment of the main goal. Language description of the overall evaluation can be obtained by means of the linguistic approximation algorithm that was proposed in [1].

The overall evaluations can be compared in the frame of a given set of alternatives; the best of the alternatives can be chosen. That is why the program FuzzME can be also used as a decision support system.

The import and export of data is supported by the program, too. The program FuzzME is available in the Czech and English version.

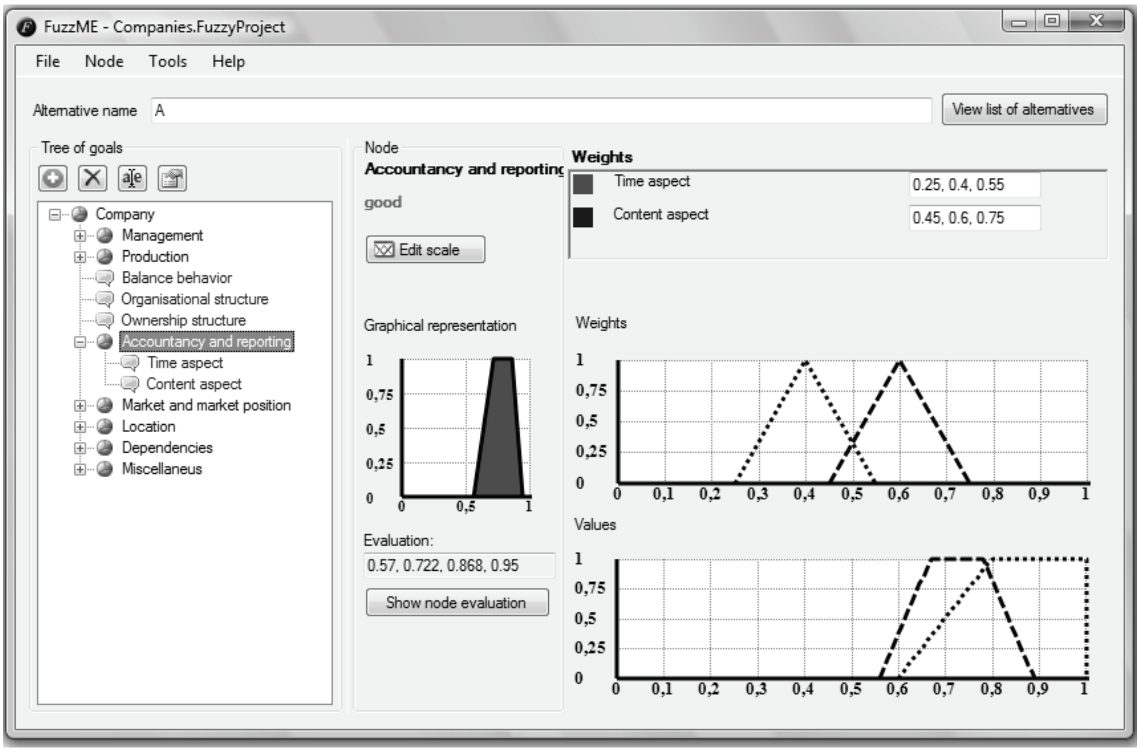

Figure 1: The main window of the program

3.1 Goals tree

The goals tree forms the base structure of the fuzzy model of evaluation created in the program FuzzME. The root of the tree corresponds to the main goal of the evaluation. The main goal is consecutively divided into the goals of the lower and lower levels. The process ends when such goals are reached whose fulfillment can be assessed by means of some known characteristics of alternatives (quantitative or qualitative criteria).

The first step of the evaluation fuzzy model design is to create the structure of the goals tree in the goals tree editor. In the next step, the type of each node in the tree must be specified. For nodes at the ends of the branches the user defines if the node is connected with quantitative or qualitative criterion. For other nodes the user set the type of aggregation - fuzzy weighted average or fuzzy expert system.

3.2 Criteria

In the models of evaluation created by the program FuzzME, qualitative and quantitative criteria can be combined arbitrarily.

3.2.1 Qualitative criteria

Qualitative criteria are evaluated verbally, by means of linguistic variables of special kinds - linguistic scales, extended linguistic scales and linguistic scales with intermediate values.

A linguistic variable is defined as a quintuple $(\mathcal{V}, \mathcal{T}(\mathcal{V}), X, G, M)$, where $\mathcal{V}$ is a name of the variable, $\mathcal{T}(\mathcal{V})$ is a set of its linguistic values, G is a syntactic rule for generating values from $\mathcal{T}(\mathcal{V})$, and M is a semantic rule which maps each linguistic value $\mathcal{C} \in \mathcal{T}(\mathcal{V})$ to its mathematical meaning $C = M(\mathcal{C})$, which is a fuzzy set on X. If the set of the linguistic values is defined explicitly and $X = [a, b]$, we will denote the linguistic variable by $(\mathcal{V}, \mathcal{T}(\mathcal{V}), [a, b], M)$.

A linguistic scale on $[a, b]$ is a linguistic variable $(\mathcal{V}, \mathcal{T}(\mathcal{V}), [a, b], M)$, where $\mathcal{T}(\mathcal{V}) = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ and the meanings of the linguistic values $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$ are modeled by fuzzy numbers $T_1, T_2, \ldots, T_s$ on $[a, b]$ which form a fuzzy partition on $[a, b]$, i.e. for all $x \in [a, b]$ the following holds

$$\sum_{i=1}^{s} T_i(x) = 1, \tag{3}$$

and these fuzzy numbers are numbered according to their ordering.

In the program FuzzME the expert defines a linguistic scale for each qualitative criterion in the fuzzy scale editor. For example, the linguistic scale *Quality of the products* can contain linguistic values *inadequate*, *adequate*, *satisfying*, *good* and *very good*.

The extended scale contains besides elementary terms of the original scale, $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$, also derived terms in the form *from* $\mathcal{T}_i$ *to* $\mathcal{T}_j$, where $i < j, i, j \in \{1, 2, \ldots, s\}$. Expert can e.g. evaluate *the quality of product* by the linguistic term *from good to very good*. The meaning of the linguistic values *from* $\mathcal{T}_i$ *to* $\mathcal{T}_j$ is modeled by $T_i \cup_L T_{i+1} \cup_L \cdots \cup_L T_j$, where $\cup_L$ denotes the union of the fuzzy sets based on the Lukasiewicz disjunction; e.g. $(T_i \cup_L T_{i+1})(x) = \min\{1, T_i(x) + T_{i+1}(x)\}$ for all $x \in \Re$.

The linguistic scale with intermediate values is the original linguistic scale enriched with derived terms *between* $\mathcal{T}_i$ *and* $\mathcal{T}_{i+1}$, $i = 1, 2, \ldots, s-1$. The meaning of the derived term *between* $\mathcal{T}_i$ *and* $\mathcal{T}_{i+1}$ is modeled by the arithmetic average of the fuzzy numbers T_i and T_{i+1}.

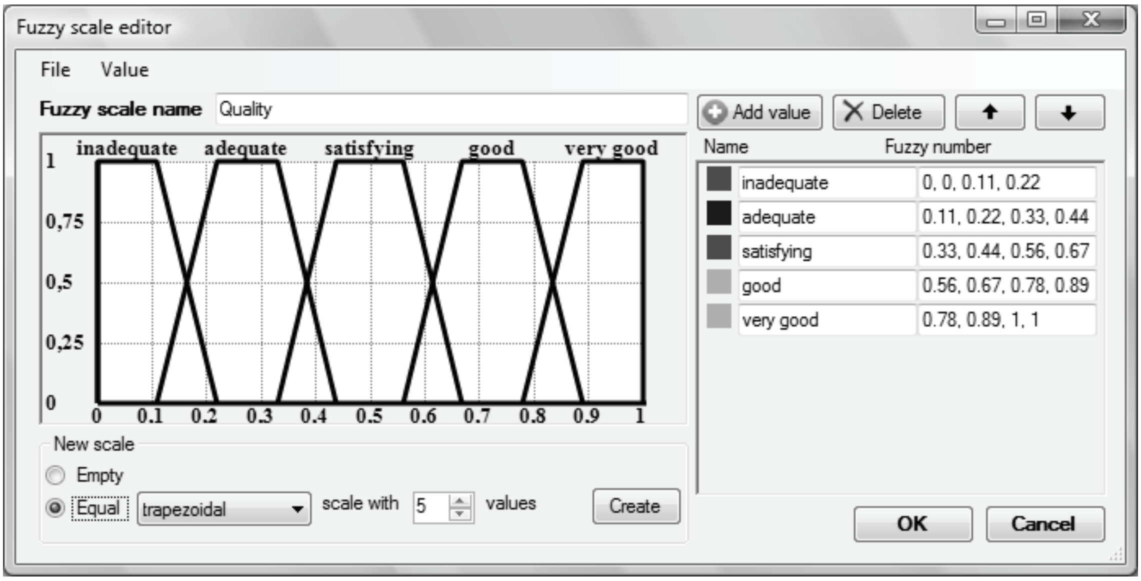

Figure 2: Fuzzy scale editor

Evaluating the alternative in the program FuzzME, the expert selects the particular value of the criterion from drop-down list box. He/she can choose the value from a given standard linguistic scale, extended scale or scale with intermediate values.

The three mentioned structures of linguistic values are also applied in the process of linguistic approximation (see [1]) of the resulting fuzzy evaluations.

3.2.2 Quantitative criteria

The evaluation of an alternative with respect to a quantitative criterion is calculated from the measured value of the criterion by means of the evaluating function defined for the criterion. Program FuzzME admits both crisp and fuzzy values of quantitative criteria. For the fuzzy value of criterion the resulting evaluation is calculated by the extension principle. The evaluating function of a quantitative criterion represents the membership function of the corresponding partial goal.

The evaluating function of a quantitative criterion is set as a fuzzy number in the program FuzzME. For example, if the evaluating function is defined by a linear fuzzy number $F = (f_1, f_2, f_3, f_4)$ then f_1 is the lower limit of the acceptable values of the criterion, f_2 is the lower limit of its fully satisfactory values, f_3 is the upper limit of its fully satisfactory values and f_4 is the upper limit of the acceptable values.

For example, when a bank evaluates profitability of a project, then the evaluating function can be defined by a linear fuzzy number with significant values 10, 30, 500, 500. Values lower than 10% are not satisfying at all (the client would not be able to pay the money back to the bank). Values from 30% to 500% are fully satisfying for the bank (naturally, values greater than 500% are not supposed to occur). This way we can define a monotonous evaluating function, which is the most common in the evaluating models, by a fuzzy number.

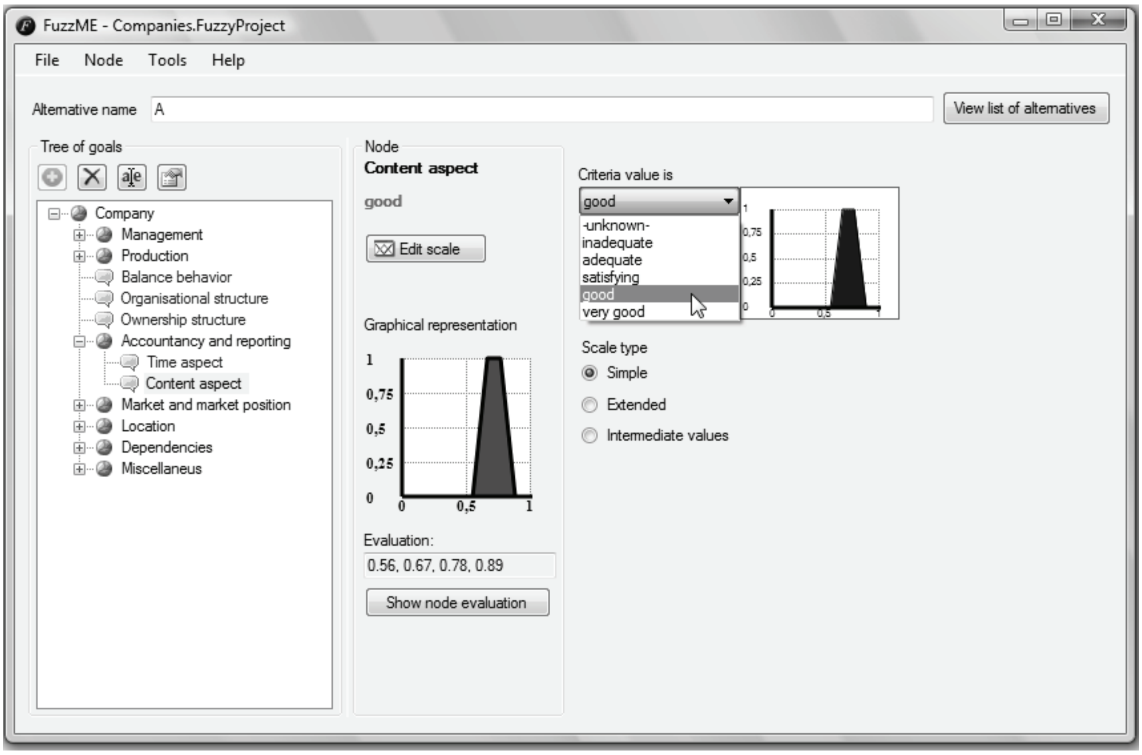

Figure 3: Selecting a linguistic value of the qualitative criterion

3.3 Methods of aggregation

The partial fuzzy evaluations of an alternative are aggregated consecutively according to the structure of the goals tree. With respect to the defined type of the tree node, the fuzzy weighted average method or the fuzzy expert system method is used for the aggregation. Either of the aggregation methods is suitable for a different situation.

The fuzzy weighted average can be used if the goal corresponding with the node of interest can be fully divided into disjunctive goals of the lower level. The fuzzy weights represent uncertain shares of the goals of the lower level in the goal corresponding with the considered node.

If the relationship between the evaluations of the lower level and the evaluation corresponding with the given node is more complex, then in the program FuzzME the relationship is described by a fuzzy rule base of a fuzzy expert system and the approximate reasoning is used for the calculation of the resulting evaluation. The fuzzy expert system is also used in case that the fulfillment of a partial goal at the end of a tree branch depends on several mutually dependent criteria (combinations of their values bring synergic or disynergic effects).

3.3.1 Aggregation by the fuzzy weighted average method

If the aggregation of partial evaluations in the given node of the tree should be calculated by the fuzzy weighted average method, then it is necessary to define normalized fuzzy weights of the corresponding partial goals. To express consistent uncertain weights, a special structure of fuzzy numbers must be used. Fuzzy numbers $V_1, \ldots, V_m$ defined on $[0,1]$ are normalized fuzzy weights if for any $i \in \{1, \ldots, m\}$ and any $\alpha \in [0,1]$ it holds that for any $v_i \in V_{i\alpha}$ there exist $v_j \in V_{j\alpha},\ j = 1, \ldots, m,\ j \neq i$, such that

$$v_i + \sum_{j=1, j\neq i}^{m} v_j = 1. \tag{4}$$

Normalized crisp weights, i.e. the real numbers $v_1, \ldots, v_m$, $v_i \geq 0$, $i = 1, \ldots, m$, $\sum_{i=1}^{m} v_i = 1$, represent a special case of normalized fuzzy weights.

Fuzzy weighted average of fuzzy numbers $U_1, \ldots, U_m$ defined on $[0,1]$ with normalized fuzzy weights $V_1, \ldots, V_m$ is a fuzzy number U on $[0,1]$ with its membership function defined for any $u \in [0,1]$ as follows

$$U(u) = \max\{\min\{V_1(v_1), \ldots, V_m(v_m), U_1(u_1), \ldots, U_m(u_m)\} \mid \sum_{i=1}^{m} v_i u_i = u,\ \sum_{i=1}^{m} v_i = 1,\ v_i \geq 0,\ i = 1, 2, \ldots, m\}.$$

In the program FuzzME the expert sets the weights of the partial goals either by normalized crisp weights or by normalized fuzzy weights. Because generally it is not easy to satisfy the condition of the fuzzy weights normalization, it is sufficient if the weights set by the user are fuzzy numbers on $[0,1]$ and satisfy the following weaker condition

$$\exists v_i \in Ker\, V_i,\ i = 1, \ldots, m : \sum_{i=1}^{m} v_i = 1. \tag{5}$$

The condition means that an m-tuple of normalized crisp weights can be found in the kernels of the fuzzy numbers. The program FuzzME is able to remove the potential inconsistence in such fuzzy weights.

Normalized fuzzy weights and the fuzzy weighted average operation are studied in detail in [2] and [4]. Conditions for verifying normality of fuzzy weights, an algorithm for transformation of fuzzy weights satisfying the condition mentioned above and an algorithm for calculation of the fuzzy weighted average, which all are used in the program FuzzME, can be found there.

3.3.2 Aggregation by the fuzzy expert system

The fuzzy expert system is used if the relationship between the partial evaluations and the overall evaluation is more complicated. The fuzzy rule base can approximate, theoretically with an arbitrary precision, any Borel measurable function. Naturally, the quality of approximation of a real evaluating function by a fuzzy rule base depends on the expert's knowledge of the relationship.

If the fuzzy rule base models the relation between values of criteria and the corresponding partial goal then the evaluation function is of the following form

If $\mathcal{K}_1$ is $\mathcal{A}_{1,1}$ and $\mathcal{K}_2$ is $\mathcal{A}_{1,2}$ and ... and $\mathcal{K}_m$ is $\mathcal{A}_{1,m}$, then $\mathcal{C}$ is $\mathcal{U}_1$
If $\mathcal{K}_1$ is $\mathcal{A}_{2,1}$ and $\mathcal{K}_2$ is $\mathcal{A}_{2,2}$ and ... and $\mathcal{K}_m$ is $\mathcal{A}_{2,m}$, then $\mathcal{C}$ is $\mathcal{U}_2$
....
If $\mathcal{K}_1$ is $\mathcal{A}_{n,1}$ and $\mathcal{K}_2$ is $\mathcal{A}_{n,2}$ and ... and $\mathcal{K}_m$ is $\mathcal{A}_{n,m}$, then $\mathcal{C}$ is $\mathcal{U}_n$

where for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$:
$(\mathcal{K}_j, \mathcal{T}(\mathcal{K}_j), V_j, M)$ are linguistic variables representing the criteria $\mathcal{K}_j$,
$\mathcal{A}_{i,j} \in \mathcal{T}(\mathcal{K}_j)$ are their linguistic values,
$(\mathcal{C}, \mathcal{T}(\mathcal{C}), [0, 1], M)$ is a linguistic variable representing the evaluation of alternatives,
$\mathcal{U}_i \in \mathcal{T}(\mathcal{C})$ are its linguistic values.
All the linguistic variables used in the expert system are usually linguistic scales.

In the program FuzzME rule bases are defined expertly. For any combination of linguistic values of the criteria the expert defines the resulting evaluation by selecting the proper linguistic value from a given evaluating linguistic scale.

The resulting fuzzy evaluation is calculated either by the standard Mamdani fuzzy inference or by the generalized Sugeno fuzzy inference.

In the case of the Mamdani fuzzy inference, the degrees of correspondence between the input fuzzy vector $(A_1^{'} \times A_2^{'} \times \cdots \times A_m^{'})$ and the mathematical meanings of the left sides of the rules are calculated first. For each i, $i = 1, \ldots, n$, the degree of correspondence h_i is calculated in the following way

$$\begin{aligned} h_i &= hgt[(A_1^{'} \times \cdots \times A_m^{'}) \cap (A_{i,1} \times \cdots \times A_{i,m})] = \\ &= \min\{hgt(A_1^{'} \cap A_{i,1}), \ldots, hgt(A_m^{'} \cap A_{i,m})\}. \end{aligned} \tag{6}$$

Then the output fuzzy values $U_i^{'}$, $i = 1, \ldots, n$, corresponding with the given input fuzzy vector are calculated as follows

$$\forall y \in [0, 1] : U_i^{'}(y) = \min\{h_i, U_i(y)\}. \tag{7}$$

The final fuzzy evaluation of the alternative is given as the union of all the fuzzy evaluations calculated in the previous step, i.e.

$$U^{'} = \bigcup_{i=1}^{n} U_i^{'}. \tag{8}$$

Generally, the result obtained by the Mamdani inference algorithm need not be a fuzzy number. So, for further calculations in the frame of the fuzzy model it must be approximated by a fuzzy number.

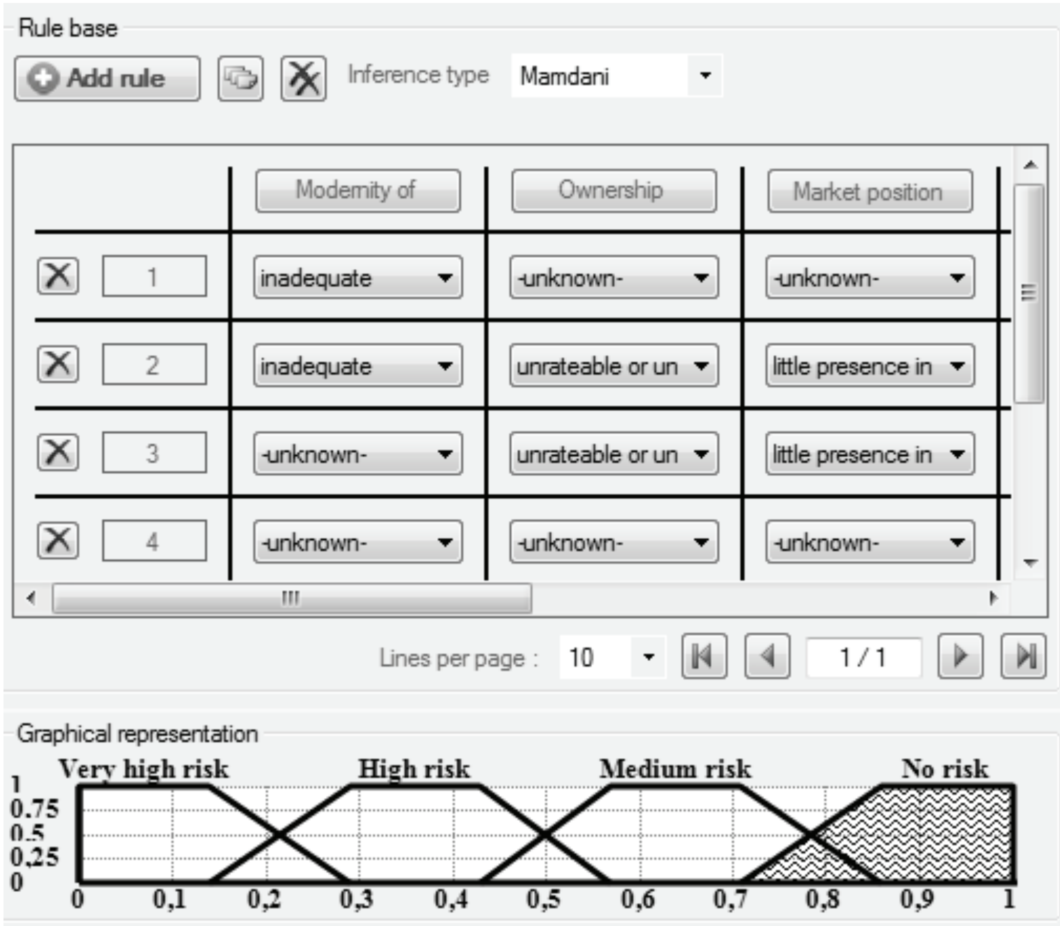

Figure 4: Fuzzy rule base editor

The advantage of the generalized Sugeno inference algorithm (see [1]) is that the result is always a fuzzy number. In the first step of the Sugeno algorithm, the degrees of correspondence h_i, $i = 1, \ldots, n$, are calculated in the same way as in the Mamdani fuzzy inference algorithm.

The resulting fuzzy evaluation U is then computed as a weighted average of the fuzzy evaluations $U_i,\ i = 1, 2, \ldots, n$, which model the mathematical meanings of the linguistic evaluations on the right sides of the rules, with the weights h_i. This is done by the following formula

$$U = \frac{\sum\limits_{i=1}^{n} h_i.U_i}{\sum\limits_{i=1}^{n} h_i}. \tag{9}$$

3.4 The overall fuzzy evaluations, the optimum alternative

The result of the consecutive aggregation of the partial fuzzy evaluations is the overall fuzzy evaluation of the given alternative. Obtained overall fuzzy evaluations of alternatives are fuzzy numbers on $[0,1]$. They express uncertain degrees of fulfillment of the main goal by the particular alternatives.

The program FuzzME compares alternatives according to the centers of gravity of their overall fuzzy evaluations. A center of gravity of a fuzzy number U on $[0,1]$ which is not a real number is defined as follows

$$t_U = \frac{\int_0^1 U(x).x\,dx}{\int_0^1 U(x)\,dx}. \tag{10}$$

If U is a real number, then $t_U = U$. A fuzzy number U is greater than a fuzzy number V if $t_U > t_V$. In the program FuzzME, the optimum alternative from a given set of alternatives is the one which has the greatest center of gravity of all of the overall fuzzy evaluation.

At present, the program FuzzME is aimed at solving multiple criteria evaluation problems. For a good quality of the choice of the optimum alternative it is necessary to include other methods of ordering of the fuzzy evaluations in the program in the future. Some approaches are proposed in [1] and further research in this area is planned.

3.5 Import and export of data

For fuzzy models of evaluation created in the frame of the program FuzzME, the criteria values of alternatives can be either set directly or imported e.g. from Excel. Similarly resulting evaluations can be exported to the Excel for their further processing.

4 Example

The program FuzzME was tested on a soft-fact-rating problem of one of the Austrian banks. The problem was solved in co-operation with the Technical University in Vienna. The fuzzy model of evaluation, created in the frame of the program FuzzME, represents a part of the evaluation of creditability of companies carried out by the bank - the evaluation according to soft (qualitative) data, which complements the evaluation according to hard (quantitative) data.

In total, 62 companies were evaluated by the fuzzy model. The goals tree of the model contained 27 qualitative criteria.

During the testing, two approaches were compared - the original soft-fact-rating model used by the bank and the fuzzy model created in the frame of the program FuzzME.

The original evaluation model used intuitive discrete numeric scales with linguistic descriptors for the evaluation according to the particular qualitative criteria. The aggregation of partial evaluation was done by a standard weighted average.

By testing with the program FuzzME the used linguistically described numeric scales were analyzed. It was found out that in some cases there is not just a perfect correspondence between the linguistic and numerical scales. Two fuzzy models were formed. The first one used equable fuzzy scales representing a simple fuzzification of the original numerical scales. The other one worked with fuzzy values which tried to model as well as possible the linguistic descriptors used in the original evaluation model. The results of the two models were quite different.

Simultaneously the normalized crisp weights were replaced by normalized fuzzy weights which correspond better to the expert's knowledge about the importance of the criteria.

The comparison of the two approaches showed that the solid theoretical basis of the evaluation fuzzy models formed in the program FuzzME improves the quality of resulting evaluations of alternatives. Positive experience with such fuzzy models of evaluation could persuade the todays opponents of the soft-fact-rating in the future. The proposed fuzzy evaluations models don't have the imperfections that the opponents of the soft-fact-rating criticize legitimately.

The subsequent discussion of the results of the given soft-fact-rating problem showed that there exist values of criteria and combinations of values of criteria which signalize a substantial danger that the company will go bankrupt or at least it will be difficult for it to acquit the debt. Nevertheless, the overall evaluation calculated by the weighted average need not be so bad. That is why besides the evaluation of the companies described above another evaluation, "risk rate of the company", was calculated. A fuzzy expert system was applied for the calculation. The particular rules of the base described the dangerous combinations of the criteria values and assign them the corresponding risk rate. In the case of high risk degree of a company the bank would not grant a credit even if the weighted average of the company evaluation is not bad.

5 Conclusion

The software FuzzME, which is presented in this paper, has several positive features. The essential one is the solid theoretical basis of the methods contained in the program. The mathematical potential of the software is a result of many years of research. The implemented methods were tested on solving real problems. Well-elaborated theory of multiple criteria evaluation, which forms the mathematical basis of the program FuzzME, provides a clear interpretation of all steps of the evaluation process and brings understandableness of the methodology to the user.

The mathematical basis of the implemented methods of evaluation corresponds very well with the fuzzy set theory paradigm. The type of the evaluation, fuzzy evaluation representing the degree of fulfillment of a defined goal (or the fuzzy membership degree of the alternative to the fuzzy set of fully satisfying alternatives), fits into the paradigm.

The program FuzzME is user-friendly. The positive features of the software product showed themselves by solving the mentioned soft-fact-rating problem.

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ON SOME PROPERTIES OF FUZZY CONTROLLERS

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Abstract. Some kind of a generalized fuzzy controller is introduced. Special attention is paid to the information boundedness principle and the interaction property.

Key words and phrases. relevancy transformation operator, information boundedness principle, fuzzy controller, interaction.

Mathematics Subject Classification. Primary 03E72, 93C42; Secondary 47S40.

1 Introduction

A standard controller performs a mapping

$$\varphi : \mathcal{X} \longrightarrow Y,$$

where $\mathcal{X}$ is the input space and $\mathcal{Y}$ is the output space. Fuzzy controller [2, 4, 5] maps fuzzy subsets of an input space $\mathcal{X}$ onto fuzzy subsets of an output space $\mathcal{Y}$. Such control process is given by a control function

$$\Phi : \mathcal{F}(\mathcal{X}) \longrightarrow F(Y),$$

where the symbols $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}(\mathcal{Y})$ stand for the sets of all fuzzy subsets of $\mathcal{X}$ and $\mathcal{Y}$ respectively. The sets $\mathcal{X}$ and $\mathcal{Y}$ are usually supposed to be convex subsets of finite-dimensional real vector spaces. Despite of some imprecision, inputs of fuzzy controllers are often crisp. Then fuzziness is restricted to the computations inside a core of fuzzy controllers, while they communicate with surroundings through crisp values of inputs and outputs. Of course, crisp inputs can be fuzzified or they can be expressed as singletons. The output can be also fuzzy and a defuzzification is needed. In this paper we will not pay attention to the fuzzification and defuzzification processes. Note that we identify fuzzy sets with their membership functions. Consider the fuzzy rule base consisting of a set of if-then rules in the form:

$$If\ X\ is\ A_1\ then\ Y\ is\ C_1$$

$$If\ X\ is\ A_2\ then\ Y\ is\ C_2$$

$$\ldots$$

$$If\ X\ is\ A_n\ then\ Y\ is\ C_n$$

where $X, A_1, \ldots, A_n \in \mathcal{F}(\mathcal{X})$ and $Y, C_1, \ldots, C_n \in \mathcal{F}(\mathcal{Y})$. The rules in a rule base express the expert's knowledge and can be obtained by various methods (experiences of experts, analysis of some models, preference structures, etc.) [2, 5, 3]. Of course, we expect that the rule base is consistent in some sense and fulfills some requirements. We mention some of them [2, 5, 4] :

(i) For any $i \in \{1, 2, , n\}$ there exists $x_i \in \mathcal{X}$ such that $A_i(x_i) = 1$ (normality of antecedents)

(ii)

$$\bigcup_i Supp\ A_i = \mathcal{X}$$

(completeness), where $Supp\ A_i = \{x \in \mathcal{X} | A_i(x) > 0\}$,

(iii) $(A_i = A_j) \Longrightarrow (C_i = C_j)$ (consistency),

(iv) $\Phi(A_i) = C_i$ for $i = 1, 2, , n$ (interaction),

The property **(iv)** is sometimes called "interpolation". It says that any rule antecedent as an input of a fuzzy controller should produce the corresponding consequent as an output.

The compositional rule of inference says that the output $Y \in \mathcal{F}(\mathcal{Y})$ can be obtained by the composition of the input $X \in \mathcal{F}(\mathcal{X})$ and fuzzy relation $R \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$, i.e.,

$$R : \mathcal{X} \times \mathcal{Y} \longrightarrow [0, 1] .$$

It means that

$$Y = \Phi(X) = X \circ R$$

such that for all $y \in \mathcal{Y}$

$$Y(y) = \sup_{x \in \mathcal{X}} \left(T \left(R\left(x, y\right), X(x)\right)\right)$$

where T is a continuous t-norm. In the case of Mamdani-Assilian controller [2, 5, 7, 4] we have

$$R\left(x, y\right) = \max_i \left(T\left(A_i(x), C_i(y)\right)\right) .$$

Then

$$Y(y) = \sup_{x \in \mathcal{X}} \left(T\left(\max_i \left(T\left(A_i(x), C_i(y)\right)\right), X(x)\right)\right)$$

for all $y \in \mathcal{Y}$, or

$$Y(y) = \max_i \left(T\left(r_i\left(X, A_i\right), C_i\left(y\right)\right)\right) .$$

The values

$$r_i\left(X, A_i\right) = \sup_{x \in \mathcal{X}} \left(T\left(X(x), A_i(x)\right)\right) ,$$

$i = 1, 2, \ldots, n$, are degrees of the overlapping of the input X and the antecedents A_i. They are called the firing values of the rules for a given input X. In this case we can divide the process

of obtaining of a fuzzy output Y for a given fuzzy input X into 3 steps:
Step 1: Obtain firing values $r_1, r_2, \ldots, r_n$ of individual rules for a given input $X \in \mathcal{F}(\mathcal{X})$.
Step 2: Derive outputs of individual rules ($i = 1, 2, , n$):

$$Y_i(y) = T\left(r_i, C_i\left(y\right)\right).$$

for all $y \in \mathcal{Y}$
Step 3: Obtain the global output for all $y \in \mathcal{Y}$ by

$$Y(y) = \max_i \left(Y_i\left(y\right)\right).$$

In this paper we shall try to introduce a generalized fuzzy controller with fuzzy inputs and fuzzy outputs; GFC for short. We expect that Mamdani-Assilian controllers should be particular examples of it. We will also investigate some reasonable additional properties of GFC.

To generalize *Step 2* we will use a relevancy transformation operator (RET operator, for short) which is a generalization both fuzzy conjunction and fuzzy implication. It was originally introduced by R. Yager [9, 10, 8, 6, 7].

Definition 1.1 *Let $e \in [0,1]$ be a given element. A binary operation*

$$Ret : [0,1]^2 \longrightarrow [0,1]$$

is called the relevancy transformation (RET) operator with respect to the element e if it satisfies the following axioms:

(r1) *$Ret(1, a) = a$ and $Ret(0, a) = e$ for all $a \in [0,1]$,*

(r2) *$Ret(r, a_1) \leq Ret(r, a_2)$ for all $a_1, a_2 \in [0,1]$ such that $a_1 < a_2$ and $r \in [0,1]$,*

(r3) *if $a \geq e$, then $Ret(r_1, a) \leq Ret(r_2, a)$ for all $r_1, r_2 \in [0,1]$ such that $r_1 < r_2$,*

(r4) *if $a \leq e$, then $Ret(r_1, a) \geq Ret(r_2, a)$ for all $r_1, r_2 \in [0,1]$ such that $r_1 < r_2$.*

Note that the axiom **(r1)** means that the effective rule output is equal to the consequent of a rule if the rule is fully fired, and the rule output does not distinguish between elements of output space if the rule is not fired. The element e is related to the following aggregation; it should be its neutral element. The last two conditions are called a consistency in the antecedent argument. Note that a RET operator is a fuzzy conjunction for $e = 0$ and a fuzzy implication for $e = 1$. In [9] one can find more about philosophical background of mentioned properties.

Example 1.2 *Let $e \in [0,1]$ be a given element. Define*

$$h(r, a) = ra + (1 - r)e,$$

$r, a \in [0,1]$. Then h is a RET operator with respect to the element e. It is called Product RET operator.

We will use an aggregation function [1] as a generalization of the maximum operator (*Step 3*).

Definition 1.3 *A function*

$$Agg : \bigcup_{m \in N} [0,1]^m \longrightarrow [0,1]$$

is called the aggregation function if

(a1) $Agg(x) = x$ *for all* $x \in [0,1]$,

(a2) $Agg(0,\dots,0) = 0,\ Agg(1,\dots,1) = 1,$

(a3) $Agg(x_1, x_2, \dots, x_m) \leq Agg(y_1, y_2, \dots, y_m)$ *if* $x_1 \leq y_1,\ x_2 \leq y_2, \dots,\ x_m \leq y_n$.

An element $e \in [0,1]$ is called the neutral element of an aggregation function Agg if

$$Agg(x_1, \dots, x_{i-1}, e, x_{i+1}, \dots, x_m) = Agg(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$$

for all $x_1, \dots, x_m \in [0,1], m \in N$.

2 Generalised fuzzy controller

Now consider a generalized fuzzy controller (GFC) with fuzzy inputs $X \in \mathcal{F}(\mathcal{X})$ and fuzzy outputs $Y \in \mathcal{F}(\mathcal{Y})$:

$$\Theta = (RB, Fir, Ret, Agg),$$

where

- $RB = (A_i, C_i)_{i=1}^n$ is a rule base consisting of a set of simple if-then rules with normal antecedents $A_i \in \mathcal{F}(\mathcal{X})$ and consequents $C_i \in \mathcal{F}(\mathcal{Y})$, $i = 1, 2, , n$. Suppose that

$$\bigcup_i Supp\ A_i = \mathcal{X}.$$

- $Fir : \mathcal{F}(\mathcal{X}) \times \mathcal{F}(\mathcal{Y}) \longrightarrow [0,1]$ assigns a firing value r_i to any fuzzy input $X \in \mathcal{F}(\mathcal{X})$ and the antecedent of the i-th rule ($i = 1, 2, \dots, n$), i. e.,

$$r_i = Fir(X, A_i).$$

  Assume that the operation Fir is nondecreasing in both fuzzy arguments. For simplicity, we denote:

$$Fir(X, RB) = (r_1, r_2, \dots, r_n).$$

  Suppose that for all normal $X \in \mathcal{F}(\mathcal{X})$ it holds:

$$Fir(X, X) = 1, \quad Fir(X, RB) \neq (0, 0, \dots, 0)$$

  and, moreover, the n-tuple $Fir(X, RB)$ contains at most once 1. It ensures that for any normal input at least one rule is fired and at most one rule is fully fired. It implies that $Fir(A_i, A_j) < 1$ for $i \neq j$.

- $Ret : [0,1]^2 \longrightarrow [0,1]$ is a RET operator with respect to a given special element $e \in [0,1]$. The individual output
$$Y_i = Ret(r_i, C_i)$$
of each rule is given pointwisely by
$$Y_i(y) = Ret\left(r_i, C_i(y)\right)$$
for all $y \in \mathcal{Y}$ and $i = 1, 2, \ldots, n$.

- $Agg : [0,1]^n \longrightarrow [0,1]$ is a restriction of an aggregation function with neutral element e. Emphasize that the element e is the special element for the operator Ret and the neutral element for the aggregation function Agg simultaneously.

The overall output Y for a given normal fuzzy input X is given by a control function

$$\Phi_\Theta : \mathcal{F}(\mathcal{X}) \longrightarrow F(Y)$$

such that for all $y \in \mathcal{Y}$ hold:

$$Y(y) = Agg\left(Ret\left(r_i, C_i(y)\right)\right).$$

Note that if the operator Ret is a t-norm ($e = 0$), the aggregation function Agg is the maximum operator and

$$r_i = Fir(X, A_i) = \sup_{x \in \mathcal{X}} T\left(X(x), A_i(x)\right)$$

for $i = 1, 2, \ldots, n$, we obtain a Mamdani - Assilian controller.

3 Properties of GFC

To have a reasonable performance of a generalized fuzzy controller we study conditions under which this controller fulfills some additional requirements. The information boundedness principle (IBP for short) for individual rule says that the knowledge obtained as a result of inference process should not have more information than that contained in the consequent of the rule [10, 6]. It means that GFG fulfills IBP if the operator Ret has the property:

$$Infor\left(Ret(r_i, C_i)\right) \leq Infor(C_i)$$

for all $i = 1, 2, \ldots, n,$, where r_i is a firing value of the i-th rule with the consequent C_i and $Infor$ is some measure of information. The stronger form of (IBP) can be written in the form

$$Infor\left(Ret(r_1, C_i)\right) \leq Infor\left(Ret(r_2, C_i)\right)$$

whenever $r_1 < r_2,\ r_1, r_2 \in [0,1]$. Many information measures have been proposed attached to fuzzy sets (measure of imprecision, Shannon's entropy etc)[4]. If we consider the finite universe $\mathcal{X}, \mathcal{Y}$ with cardinalities $|X| = |Y| = m$ then each fuzzy set A is represented by m-tuple $(a_1, a_2, \ldots, a_m)$ and we can define a special type of information measure which is called a linear specificity [6, 9, 10].

Definition 3.1 *Let* $1 \geq w_2 \geq w_3 \cdots \geq w_m \geq 0$ *be given constants for which* $w_2 + w_3 + \cdots + w_m = 1$. *If* $1 \geq a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$, *then a mapping* $LSp : [0,1]^m \longrightarrow [0,1]$ *given by*

$$LSp(a_1, a_2, \ldots, a_m) = a_1 - \sum_{i=2}^{m} w_i a_i$$

is called the linear specificity measure.

Now, we are looking for RET operators which fulfill IBP with respect to a linear specificity as an information measure. The next proposition gives a full solution of this problem [6].

Theorem 3.2 *A relevancy transformation operator*

$$Ret : [0,1]^2 \longrightarrow [0,1]$$

satisfies the stronger form of (IBP) with respect to a linear specificity if and only if the operator Ret is 2-increasing.

Note that a mapping $f : [0,1]^2 \longrightarrow [0,1]$ is 2-increasing if

$$f(x_1, y_1) + f(x_2, y_2) \geq f(x_1, y_2) + f(x_2, y_1)$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

The requirement of interaction in GFC says that an antecedent of any rule as an input of GFC produces the corresponding consequent as an output. This requirement is generally violated in many fuzzy controllers, also in Mamdani-Assilian controllers. We try to find conditions under which our GFC fulfills the interaction property:

$$\Phi_\Theta(A_i) = C_i$$

for $i = 1, 2, \ldots, n$. The next theorem gives some sufficient conditions [7].

Theorem 3.3 *Consider a generalized fuzzy controller*

$$\Theta = (RB, Fir, Ret, Agg)$$

such that for all $r, b \in [0,1]$, $0 \leq r \leq c$ *holds:*

$$Ret(r, b) = e,$$

where Ret is a relevancy transformation operator with respect to the given element e *and*

$$c = \max \{ Fir(A_j, A_k),\ k \neq j,\ k, j \in \{1, 2, \ldots, n\} \} < 1.$$

Then

$$\Phi_\Theta(A_i) = C_i$$

for $i = 1, 2, , n$.

Theorem 3.3 offers a way how to construct GFC with interaction property. For given rule base RB and given mapping Fir we find out a constant

$$c = \max \{Fir\,(A_j, A_k)\,,\ k, j \in \{1, 2, \ldots, n\}\, k \neq j,\ \}\,.$$

We expect that $c < 1$. If not, we must change the rule base or the operation Fir. Then we choose a RET operator with respect to a given element e satisfying $Ret(r, b) = e$ for all $r, b \in [0, 1]$, $0 \leq r \leq c$ and some appropriate aggregation function with e as a neutral element (e. g., uninorm [10]). The obtained GFC has interaction property, as well. Sumarising Theorem 3.2 and Theorem 3.3 we obtain:

Corollary 3.4 *Consider GFC*

$$\Theta = (RB, Fir, Ret, Agg)$$

such that

- *the operator Ret is 2-increasing,*
- *for all* $r, b \in [0, 1]$, $0 \leq r \leq c$ *holds:*

$$Ret(r, b) = e.$$

Then Θ *has interaction property and IBP for individual rules holds.*

The next example gives a possibility of a construction of a RET operator with respect to any special element $e \in [0, 1]$ having both mentioned properties.

Example 3.5 *Let* $c, e \in [0, 1]$ *be given elements. Define* $h : [0, 1]^2 \longrightarrow [0, 1]$ *by*

$$h(r, b) = \begin{cases} e & if\ r \leq c \\ b + (b - e)(r - 1)/(1 - c) & elsewhere \end{cases}$$

Then h *is a RET operator with respect to the element* e *and fulfills* $h(r, b) = e$ *for all* $r, b \in [0, 1]\,, r \leq c$. *Moreover* h *is 2-increasing and so, it fulfills IBP.*

4 Conclusion

We have introduced a generalized fuzzy controller which is a generalization of Mamdani-Assilian fuzzy controller. We have investigated some of its properties. Finally, some techniques of possible compositions of fuzzy controllers were suggested.

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GEODESIC MAPPINGS ONTO WEYL MANIFOLDS

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Abstract. In this paper we study geodesic mappings of special n-dimensional manifolds with affine connection onto Weyl manifolds. It is proved that non projective flat Weyl manifolds ($n > 8$) do not admit nontrivial geodesic mappings onto a generalized recurrent (include symmetric and recurrent) manifold with affine connection.

Key words and phrases. Geodesic mapping, manifold with affine connection, Weyl manifold, generalized recurrent space.

Mathematics Subject Classification. Primary 53B05.

1 Introduction

The theory of special mappings is an interesting part of differential geometry of manifolds with affine connection and (pseudo-) Riemannian spaces, see [1]-[26]. Geodesic mappings of manifolds with affine connection and Riemannian spaces were studied by many authors (e.g., [1, 3, 12, 17, 20, 22]). In [26] geodesic mappings between Kähler-Weyl spaces were investigated by G.Ç. Yıldırım and G.G. Arsan.

N.S. Sinyukov had demonstrated in [21] and [22] that non projectively flat symmetric and recurrent equiaffine manifolds do not admit non-trivial geodesic mappings onto Riemannian manifolds. In works by V. Fomin [4, 5], J. Mikeš [7, 8, 9, 10, 11, 12, 13], M. Prvanovič [18], V.S. Sobchuk [23, 24], P. Venzi [25], and others, geodesic mappings are studied for manifolds with more general conditions of recurrency.

In this paper we study geodesic mappings of special manifolds with affine connection and with general conditions of recurrency onto Weyl manifolds.

For the research in this paper we will make use of tensorial analysis in local form, all used functions are continuous and sufficiently differentiable. The dimension n of the studied spaces is larger than two, unless stated otherwise. All spaces are linearly connected.

We use notions from the theory of manifolds with affine connection as in the monographs and reviews [1]-[22].

2 Geodesic mappings of manifolds with affine connection

Consider a diffeomorphism $f\colon A_n \to \bar{A}_n$ between manifolds with affine connection in a *common coordinate system* x, i.e. the point $M \in U \subset A_n$ and its image $f(M) \in f(U) \subset \bar{A}_n$ have the same coordinates $x = (x^1, x^2, \ldots, x^n)$; the corresponding geometric objects in $\bar{A}_n$ will be marked with a bar. For example, ∇ and $\bar{\nabla}$ are the affine connection of A_n and $\bar{A}_n$, respectively.

Definition 2.1 ([3, 12, 17, 22]) The diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a *geodesic mapping* if f maps any geodesic of A_n into a geodesic of $\bar{A}_n$.

The mapping from A_n onto $\bar{A}_n$ is *geodesic* if and only if, in the common coordinate system x with respect to the mapping, the Levi-Civita equations

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i \tag{1}$$

hold, where $\psi_i(x)$ is a covector, δ^h_i is the Kronecker delta. If $\psi_i \not\equiv 0$, then a geodesic mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

Applying the formula (1) we find a relationship for the curvature and Ricci tensors of A_n and $\bar{A}_n$:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_i(\psi_{kj} - \psi_{jk}) + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}, \tag{2}$$
$$\bar{R}_{ij} = R_{ij} + (n-1)\,\psi_{ij} + (\psi_{kj} - \psi_{jk}),$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$, the comma " , " denotes the covariant derivative on ∇.

The Weyl tensor of projective curvature

$$\begin{aligned} W^h_{ijk} &= R^h_{ijk} + \frac{1}{n+1}\delta^h_i(R_{jk} - R_{kj}) \\ &\quad - \frac{1}{(n+1)(n-1)}\left[(n\,R_{ij} + R_{ji})\delta^h_k - (n\,R_{ik} + R_{ki})\delta^h_j)\right] \end{aligned} \tag{3}$$

is invariant under geodesic mappings of manifolds with affine connection, i.e. $\bar{W} = W$,

$$\bar{W}^h_{ijk} = W^h_{ijk}. \tag{4}$$

3 Geodesic mappings of generalized recurrent manifolds

In works by Fomin [4, 5] and Mikeš [12] geodesic mappings are studied for manifolds with more general conditions of recurrency.

Under a *generalized recurrent* manifold with affine connection we understand A_n in which

$$R^h_{ijk,l} = \varphi_l R^h_{ijk} + \nu_j R^h_{ikl} - \nu_k R^h_{ijl} + \delta^h_i A_{jkl} + \delta^h_j B_{ikl} - \delta^h_k B_{ijl} + \delta^h_l C_{ijk}, \tag{5}$$

holds, where φ, ν, A, B, C are some tensors.

Let us note that these conditions are satisfied by *symmetric*, *recurrent*, and also *projective symmetric* and *projective recurrent* spaces, which are characterized by the conditions

$$\begin{array}{ll} \nabla R = 0, & R^h_{ijk,l} = 0; \\ \nabla R = \varphi \otimes R, & R^h_{ijk,l} = \varphi_l R^h_{ijk}; \\ \nabla W = 0, & W^h_{ijk,l} = 0; \\ \nabla W = \varphi \otimes W, & W^h_{ijk,l} = \varphi_l W^h_{ijk}, \end{array} \tag{6}$$

where R is the curvature, W is the Weyl projective curvature, and φ is a non vanishing linear form.

Theorem 3.1 *Riemannian manifolds $\bar{V}_n$ $(n > 4)$ with nonconstant curvature do not admit nontrivial geodesic mappings onto manifolds with affine connection A_n which satisfy* (5)*.*

For infinite-dimensional spaces this result was partially obtained by Fomin [4]. If A_n is equiaffine and $A_{jkl} = 0$ holds then the theorem is valid also for $n = 3,\ 4$, see [10]. Sinyukov proved this theorem for equiaffine symmetric manifolds and semisymmetric recurrent manifolds in [21, 22]:

Theorem 3.2 *Riemannian manifolds $\bar{V}_n$ $(n > 2)$ with nonconstant curvature do not admit non trivial geodesic mappings onto symmetric equiaffine manifolds.*

Theorem 3.3 *Riemannian manifolds $\bar{V}_n$ $(n > 2)$ with nonconstant curvature do not admit non trivial geodesic mappings onto semisymmetric recurrent equiaffine manifolds.*

Let us prove the more general theorem:

Theorem 3.4 *Non projective flat Weyl manifolds $\bar{W}_n$ $(n > 8)$ do not admit nontrivial geodesic mappings onto a manifold with affine connection A_n which satisfies* (5)*.*

Let us recall that Weyl manifolds are generalization of Riemannian manifolds.

Definition 3.5 A manifold A_n with affine connection is called *Weyl manifold W_n* if there is a regular form g on A_n for which the following holds:

$$\nabla g = w \otimes g, \tag{7}$$

where w is a linear form.

In local notation: $g_{ij,k} = w_k g_{ij}$. If w is a gradient (or locally a gradient) then W_n is conformal (= conformally equivalent) to some Riemannian manifold. For the definition of conformal mapping see [3, 12, 17, 22].

By the inspiration of Yıldırım and Arsan [26] we get

Theorem 3.6 *A manifold A_n admits a geodesic mapping onto a Weyl manifold $\bar{W}_n$ with metric $\bar{g}$ if and only if the following equations are satisfied:*

$$\bar{g}_{ij,k} = 2\varphi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{8}$$

or equivalently, for any vector fields $X, Y, Z \in \mathcal{X}(A_n)$:

$$\nabla_Z \bar{g}(X,Y) = 2\varphi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z). \tag{9}$$

Here "," is the covariant derivative relative to the connection ∇ on A_n, ψ, φ are one-forms, and ψ_i, φ_i denote their components. Moreover, the Weyl form of $\bar{W}_n$ has the following shape:

$$\bar{w} = \varphi - \psi. \tag{10}$$

The *proof* is analogous to the proof of the analogous Levi-Civita Theorem for geodesic mappings of Riemannian spaces, so we restrict ourselves only to its formulation. ∎

Let us *prove* Theorem 3.4. Let A_n be such that (5) holds, and let A_n admit a geodesic mapping onto a Weyl manifold $\bar{W}_n$, in which the metric form $\bar{g}$ satisfies the condition $\bar{\nabla}\, \bar{g} = \bar{w} \otimes \bar{g}$. The corresponding integrability conditions take the form

$$\bar{g}_{i\alpha} \bar{R}^{\alpha}_{jkl} + \bar{g}_{j\alpha} \bar{R}^{\alpha}_{ikl} = (\bar{w}_{l|k} - \bar{w}_{k|l})\, \bar{g}_{ij}, \tag{11}$$

where $\bar{R}^h_{ijk}$ is the curvature tensor of $\bar{W}_n$, "$|$" is the covariant derivative in $\bar{W}_n$. Let us raise the indices i and j by means of the dual tensor of the metric, i.e. by $\bar{g}^{ij}$:

$$\bar{g}^{i\alpha} \bar{R}^{j}_{\alpha kl} + \bar{g}^{j\alpha} \bar{R}^{i}_{\alpha kl} = (\bar{w}_{l|k} - \bar{w}_{k|l})\, \bar{g}^{ij}. \tag{12}$$

Using (2), let us express the curvature tensor of $\bar{W}_n$ in the formula (12). We get

$$\bar{g}^{i\alpha} R^{j}_{\alpha kl} + \bar{g}^{j\alpha} R^{i}_{\alpha kl} = b_{kl}\, \bar{g}^{ij} + \delta^i_k a^j_l + \delta^j_k a^i_l - \delta^i_l a^j_k - \delta^j_l a^i_k, \tag{13}$$

where b_{kl} and a^i_k are components of tensors. Let us note that the equations (8) are equivalent to

$$\bar{g}^{ij}_{\ ,k} = -2\varphi_k \bar{g}^{ij} - \psi_\alpha \bar{g}^{\alpha i} \delta^j_k - \psi_\alpha \bar{g}^{\alpha j} \delta^i_k\, . \tag{14}$$

Taking the covariant derivative of (13) with respect to x^m and using (14) we obtain

$$\Lambda^{(i} R^{j)}_{mkl} + \bar{g}^{\alpha(i} R^{j)}_{\alpha kl,m} = b_{klm}\, \bar{g}^{ij} + \delta^i_k a^j_{lm} + \delta^j_k a^i_{lm} - \delta^i_l a^j_{km} - \delta^j_l a^i_{km} + \delta^i_m c^j_{kl} - \delta^j_m c^i_{kl},$$

where $\Lambda^i = \bar{g}^{i\alpha}\psi_\alpha$, $b_{klm}, a^i_{km}, c^i_{kl}$ are certain tensors. From the last formula we express the derivative of the curvature tensor by means of the formulas (5). After some calculations we get

$$\Lambda^{(i} R^{j)}_{mkl} = B_{klm}\, \bar{g}^{ij} + \delta^i_k A^j_{lm} + \delta^j_k A^i_{lm} - \delta^i_l A^j_{km} - \delta^j_l A^i_{km} + \delta^i_m C^j_{kl} - \delta^j_m C^i_{kl}, \tag{15}$$

where $A^i_{km}, B_{klm}, C^i_{kl}$ are certain tensors.

From (15) it follows that if $n > 8$, then $B_{klm} = 0$.

It was proved that for $n > 2$ when $W \neq 0$ there exists a coordinate system x in which $R^1_{223} \neq 0$, see the following Theorem 4.2. One by one we insert into (15):

$$i = 1, \dots, n,\ j = 1,\ m = k = 2,\ l = 3;$$
$$i = j = k = 1,\ l = 3,\ m = 2;$$
$$i = j = m = 1,\ l = 3,\ k = 2;$$
$$i = j = k = 1,\ l = m = 2$$

and we can see that $\Lambda^i = 0$ ($\Leftrightarrow \psi_i = 0$) holds.

By a detailed analysis, we can check that if A_n is equiaffine and $\bar{W}_n$ is Riemannian then $B_{klm} = 0$. And if this is the case, we can restrict ourselves to the case $n > 2$. ■

Since we do not suppose that A_n is semisymmetric, the last results can be considered as a generalization of the results from the previous subsection above.

4 Characterization of projectively flat manifolds

In this part we find a new property of projectively-Euclidean manifolds which can be viewed as a generalization of properties that have been found earlier by Schouten and Struik [20].

First we prove the following theorem [15]:

Theorem 4.1 *Let $A^h_{ijk}(x)$ be a tensor of type $(1, 3)$ which satisfies the following equation:*

$$A^h_{ijk} + A^h_{ikj} = A^h_{ijk} + A^h_{jki} + A^h_{kij} = 0. \tag{16}$$

Let $A^1_{223} = 0$ (or $A^1_{234} = 0$) hold in all local coordinate systems.

Then the tensor A^h_{ijk} has the following form:

$$A^h_{ijk} = \delta^h_i(A_{jk} - A_{kj}) + \delta^h_j A_{ik} - \delta^h_k A_{ij}, \tag{17}$$

where A_{ij} is a certain tensor of type $(0, 2)$.

Proof. By the coordinate transformation $x'^h = x'^h(x^1, x^2, \cdots, x^n)$, the components of the tensor A^h_{ijk} are transfomed according to

$$A'^h_{ijk} = A^\alpha_{\beta\gamma\delta} A^h_\alpha B^\beta_i B^\gamma_j B^\delta_k \tag{18}$$

where $A^h_i \stackrel{\text{def}}{=} \partial_i x'^h$; $\left\|B^h_i\right\| \stackrel{\text{def}}{=} \left\|A^h_i\right\|^{-1}$. It is easy to see that if $A^1_{223} = 0$ (or $A^1_{234} = 0$) holds in any coordinate system then

$$(a)\ A^h_{iij} = 0; \quad \text{(b)}\ A^h_{ijk} = 0 \tag{19}$$

is satified in any coordinate system for any distinct indices h, i, j, k. We use the following transformations of coordinates:

$$x'^p = x^p + r x^q, \quad x'^s = x^s, \quad s \neq p. \tag{20}$$

Here p and q, respectively, are fixed distinct indices, and r is any real constant. Thus A^h_i and B^h_i are in the form (we used (18))

$$A^h_i = B^h_i = \delta^h_i; \quad A^p_q = -B^p_q = r, \tag{21}$$

provided either $h \neq p$, or $i \neq q$. Let us express the components of the tensor A in a new coordinate system which is determined by the transformations (20):

$$\begin{array}{ll}(a) & A'^{h}_{pqk} = A^h_{pqk} + rA^h_{qqk};\\ (b) & A'^{h}_{ppk} = A^h_{ppk} + r(A^h_{pqk} + A^h_{qpk}) + r^2A^h_{qqk};\\ (c) & A'^{q}_{pjk} = A^q_{pjk} + r(A^q_{qjk} - A^p_{pjk}) - r^2A^p_{qqk};\\ (d) & A'^{q}_{ipk} = A^q_{ipk} + r(A^q_{iqk} - A^p_{ipk}) - r^2A^p_{iqk};\\ (e) & A'^{q}_{ppk} = A^q_{ppk} - r(A^p_{ppk} - A^q_{qpk} - A^q_{pqk}) - \\ & \qquad - r^3A^p_{qqk} + r^2(A^q_{qqk} - A^p_{pqk} - A^p_{qpk}).\end{array} \tag{22}$$

In the last formulas, differently marked indices are actually different. We do not use the Einstein summation convention for the indices p and q neither here nor further. Using (22a), it is easy to see that (19b) follows from (19a).

Now let us prove the converse. Let (19a) hold, i.e. $A^h_{iik} = 0$ in any coordinate system for different h, i, k. Then from (22b) it follows

$$A^h_{ijk} + A^h_{jik} = 0,$$

where h, i, j, k are different. Let us alternate the last formula in the indices j and k. Using (16) we get (19b) for the tensor A. Further from (22d) and (22) we find

$$A^q_{qjk} = A^p_{pjk} \quad \text{and} \quad A^q_{iqk} = A^p_{ipk},$$

where $p, q \neq j, k$. Hence it follows that

$$A^p_{pjk} = B_{jk}; \quad A^p_{ipk} = A_{ik} \tag{23}$$

(for all indices p, j, k with $p \neq j, k$) form a geometric object.

Using (19b), (23) where r is any real constant, from (22e) we get

$$A^p_{ppk} = B_{pk} + A_{pk} \tag{24}$$

for the indices p, k with $p \neq k$. We check that (19), (23) and (24) can be expressed as follows (h, i, j, k are arbitrary)

$$A^h_{ijk} = \delta^h_i B_{jk} + \delta^h_j A_{ik} - \delta^h_k A_{ij}. \tag{25}$$

We symmetrize in the indices i, j, k and make sure that

$$B_{jk} = A_{jk} - A_{kj}.$$

Now we can see that (17) holds. It remains to prove that A_{ij} is a tensor of type $(0, 2)$. Contracting (17) we get

$$A_{kj} = \frac{1}{n^2 - 1}(nA^\alpha_{jk\alpha} + A^\alpha_{kj\alpha}). \tag{26}$$

Obviously, components of a tensor stand on the right hand side of (26), hence A_{kj} is a type $(0, 2)$ tensor, which completes the proof of Theorem 4.1. ■

The following can be verified:

Theorem 4.2 *Let A_n be a manifold with affine connection. Let the components of its curvature tensor satisfy in any coordinate system the following:*

$$\text{either} \quad (a) \quad R^1_{223} = 0, \qquad \text{or} \quad (b) \quad R^1_{234} = 0. \tag{27}$$

Then A_n is a projectively flat manifold.

a) It is possible to replace the formula (27) by one of the following formulas:

$$\text{(a)} \quad R^h_{iik} = 0; \qquad \text{or} \quad \text{(b)} \quad R^h_{ijk} = 0, \tag{28}$$

where h, i, j, k, respectively, are distinct indices.

b) It is possible to replace the curvature tensor in the Theorem 4.2 by a Weyl tensor of the projective curvature, or by a Yano tensor of the sectional curvature.

c) If we replace the curvature tensor by the Weyl tensor of conformal curvature then $\bar{A}_n$ will be conformally Euclidean (A_n is a Riemannian space).

Proof. By Theorem 4.1, the curvature tensor of A_n for which the assumptions of the Theorem 4.2 hold, satisfies

$$R^h_{ijk} = \delta^h_i(A_{jk} - A_{kj}) + \delta^h_j A_{ik} - \delta^h_k A_{ij}.$$

These equations have the character of a tensor of a projectively flat manifold [20], as we wanted to show. ■

Theorem 4.2 strengthens the results by Schouten and Struik [20], and it can be used to reduce the reasoning by I.P. Egorov [2], his concept of rank of moving groups of Riemannian manifolds and manifolds with affine connnection, but also by Sinyukov [22], his concept of a degree of mobility groups V_n with respect to geodesic mapppings.

This theorem allows to find a new exact estimation of the first lacuna in a distribution of a degree of mobility groups $\bar{A}_n$ with respect to geodesic mappings onto Riemannian manifolds.

Further Mikeš with the help of Moldobayev [16] has found an exact estimation of the first lacuna in a distribution of a degree of a complete group of conformal mappings and so an exact estimation of the first lacuna of a degree of a complete group of conformal mappings of second order approximation.

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INVARIANT OBJECTS BY HOLOMORPHICALLY PROJECTIVE MAPPINGS OF PARABOLICALLY KÄHLER SPACES

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Abstract. In this paper we study holomorhically projective mappings between parabolically Kähler spaces. Our aim is to find objects invariant with respect to these mappings. Namely, we introduce objects analogous to Thomas parameters of geodesic mappings and proof their invariance.

Key words and phrases. Holomorhically projective mapping, parabolically Kähler spaces, invariant object.

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1 Introduction

We consider holomorphically projective mappings of parabolically-Kählerian spaces and define holomorphically projective flat parabolically-Kählerian spaces. Our aim is to find objects invariant with respect to these mappings. Namely, we introduce objects analogous to Thomas parameters of geodesic mappings and proof their invariance.

Many authors studied geodesic and holomorphically projective mappings of Kählerian spaces and their generalizations [1]−[23]. Some facts from the theory of holomorphically projective mappings of parabolically-Kählerian spaces $K_n^{o(m)}$ were published in [4], [11]–[19], [22].

A (pseudo-) Riemannian space $K_n^{o(m)}$ is said to be *parabolically-Kählerian space* if together with a metric tensor $g_{ij}(x)$ it possesses a type $(1,1)$ tensor field F of rank $m \geq 2$ which is nilpotent, $F^2 = 0$, covariant constant with respect to the metric connection ∇, $\nabla F = 0$, and related to the metric tensor by $g(X, FY) + g(FX, Y) = 0$. That is, an affinor structure $F_i^h(x)$ of rank $m \geq 2$ is given satisfying in local coordinates the relations

$$a)\quad F_\alpha^h F_i^\alpha = 0, \qquad b)\quad g_{i\alpha} F_j^\alpha + g_{j\alpha} F_i^\alpha = 0, \qquad c)\quad F_{i,j}^h = 0, \tag{1}$$

where comma denotes covariant derivation.

2 Holomorphically projective mappings of parabolically-Kählerian spaces

The following criteria from the papers [12, 15] hold for holomorphically projective mappings from a parabolically-Kählerian space $K_n^{o(m)}$ onto a parabolically-Kählerian space $\bar{K}_n^{o(m)}$.

A curve defined by the equations $x^h = x^h(t)$ with tangent vector field λ, $\lambda^h = dx^h/dt$, is called *analytically planar curve* of the parabolically-Kählerian space $K_n^{o(m)}$ if, being translated, the tangent vector remains in the distribution spanned by the tangent vector λ and its conjugate $F\lambda$, i.e. the conditions

$$d\lambda^h/dt + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho_1(t)\lambda^h + \rho_2(t)\lambda^\alpha F^h_\alpha$$

are satisfied. Here Γ^h_{ij} is the Christoffel symbols and ρ_1, ρ_2 are functions of the argument t.

A diffeomorphism f of $K_n^{o(m)}$ onto $\bar{K}_n^{o(m)}$ is a *holomorphically projective mapping*, if it transforms all analytically planar curves of $K_n^{o(m)}$ into analytically planar curves of $\bar{K}_n^{o(m)}$.

Consider a fixed diffeomorphism f: $K_n^{o(m)} \to \bar{K}_n^{o(m)}$, both spaces being referred to the so-called general coordinate system x with respect to this mapping. It means a coordinate system such that two corresponding points $M \in K_n^{o(m)}$ and $f(M) \in \bar{K}_n^{o(m)}$ have equal coordinates $x = (x^1, x^2, \dots, x^n)$; due to the diffeomorphism, atlases can be chosen in an appropriate way so that the underlying manifolds carrying the structures are identified; the corresponding geometric objects in $\bar{K}_n^{o(m)}$ will be marked with a bar. For example, Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of the Christoffel symbols on $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$, respectively.

Structures of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$ are preserved under f, i.e. $\bar{F}^h_i(x) = F^h_i(x)$. Among others, the structure F^h_i is covariantly constant, and $\bar{g}_{i\alpha}F^\alpha_j + \bar{g}_{j\alpha}F^\alpha_i = 0$ holds.

It is proved in [12, 15] that a parabolically-Kählerian space $K_n^{o(m)}$ admits a *holomorphically projective mapping* f onto a parabolically-Kählerian space $\bar{K}_n^{o(m)}$ if and only if the following conditions (in the common coordinate system x) hold:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i, \tag{2}$$

where φ_i is a covector, $\psi_i = \varphi_\alpha F^\alpha_i$, and $\psi_i(x)$ is a gradient, i.e. there is a function $\psi(x)$, such that $\psi_i(x) = \partial\psi(x)/\partial x^i$.

If $\varphi_i \not\equiv 0$ then a holomorphically projective mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

The condition (2) is equivalent to

$$\bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi_j\bar{g}_{ik} + \varphi_i\bar{g}_{j\alpha}F^\alpha_k + \varphi_j\bar{g}_{i\alpha}F^\alpha_k. \tag{3}$$

Under a holomorphically projective mapping f: $K_n^{o(m)} \to \bar{K}_n^{o(m)}$, the following conditions hold:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \psi_{ij}\delta^h_k - \psi_{ik}\delta^h_j + \varphi_{ij}F^h_k - \varphi_{ik}F^h_j - (\varphi_{jk} - \varphi_{kj})F^h_i, \tag{4}$$

where R^h_{ijk} and $\bar{R}^h_{ijk}$ are Riemannian tensors of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$,

$$\varphi_{ij} = \varphi_{i,j} - \psi_i\varphi_j - \varphi_i\psi_j, \quad \psi_{ij} = \varphi_{\alpha j}F^\alpha_i \quad (= \psi_{ji} = \psi_{i,j} - \psi_i\psi_j). \tag{5}$$

3 Invariant objects of holomorphically projective mappings

Assume a holomorphically projective mapping $K_n^{o(m)} \to \bar{K}_n^{o(m)}$ between parabolically-Kählerian spaces. Then the equations (2) are satisfied. Contracting these equations we can express the covector ψ_i:

$$\psi_i = \frac{1}{n+2}\left(\bar{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha}\right). \tag{6}$$

Moreover, from (6) we obtain

$$\psi_i = \frac{1}{n+2}\,\frac{\partial}{\partial x^i}\ln\sqrt{\left|\frac{\bar{g}}{g}\right|}$$

since

$$\Gamma^\alpha_{i\alpha} = \frac{\partial}{\partial x^i}\ln\sqrt{|g|} \quad \text{and} \quad \bar{\Gamma}^\alpha_{i\alpha} = \frac{\partial}{\partial x^i}\ln\sqrt{|\bar{g}|}\,,$$

where

$$g = \det(g_{ij}) \quad \text{and} \quad \bar{g} = \det(\bar{g}_{ij}).$$

Substituting (6) we get after some rearranging:

$$\bar{G}^h_{ij} = G^h_{ij} + \varphi_i F^h_j + \varphi_j F^h_i \tag{7}$$

where

$$G^h_{ij} = \Gamma^h_{ij} - \frac{1}{n+2}\,\delta^h_i\Gamma^\alpha_{j\alpha} - \frac{1}{n+2}\,\delta^h_j\Gamma^\alpha_{i\alpha}, \tag{8}$$

and $\bar{G}^h_{ij}$ are introduced analogously.

Since $F^h_i \neq 0$, there exists a bivector $\varepsilon^i\eta_h$ such that $F^h_i\varepsilon^i\eta_h = 1$. This bivector is generated on a common underlying manifold of both structures $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$, hence it is in fact independent of holomorphically projective mappings between $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$. Contracting (7) with $\varepsilon^i\varepsilon^j\eta_h$ we get

$$\varphi_\alpha\varepsilon^\alpha = \frac{1}{2}\left(\bar{G}^\alpha_{\beta\gamma}\varepsilon^\beta\varepsilon^\gamma\eta_\alpha - G^\alpha_{\beta\gamma}\varepsilon^\beta\varepsilon^\gamma\eta_\alpha\right), \tag{9}$$

and then we contract (7) with $\varepsilon^j\eta_h$ to obtain

$$\varphi_i = \bar{G}^\alpha_{i\beta}\varepsilon^\beta\eta_\alpha - \frac{1}{2}\bar{G}^\alpha_{\beta\gamma}\varepsilon^\beta\varepsilon^\gamma\eta_\alpha\cdot F^\kappa_i\eta_\kappa - \left(G^\alpha_{i\beta}\varepsilon^\beta\eta_\alpha - \frac{1}{2}G^\alpha_{\beta\gamma}\varepsilon^\beta\varepsilon^\gamma\eta_\alpha\cdot F^\kappa_i\eta_\kappa\right). \tag{10}$$

Substituting (10) to (7) we find that the tensor

$$\begin{aligned} T^h_{ij} = \Gamma^h_{ij} \quad & - \frac{1}{n+2}\,\delta^h_i\Gamma^\alpha_{j\alpha} - \frac{1}{n+2}\,\delta^h_j\Gamma^\alpha_{i\alpha} \\ & - F^h_i\left(G^\alpha_{j\beta}\varepsilon^\beta\eta_\alpha - \frac{1}{2}G^\alpha_{\beta\gamma}\varepsilon^\beta\varepsilon^\gamma\eta_\alpha\cdot F^\kappa_j\eta_\kappa\right) \\ & - F^h_j\left(G^\alpha_{i\beta}\varepsilon^\beta\eta_\alpha - \frac{1}{2}G^\alpha_{\beta\gamma}\varepsilon^\beta\varepsilon^\gamma\eta_\alpha\cdot F^\kappa_i\eta_\kappa\right) \end{aligned} \tag{11}$$

is invariant with respect to holomorphically projective mappings, that is,

$$T^h_{ij} = \bar{T}^h_{ij}. \tag{12}$$

Apparently, the formula (2) is a system of linear equations with respect to components ψ_i and φ_i. Hence, the aforementioned solution is determined uniquely.

Due to this, the object T^h_{ij} is also unique. However, this object is not a tensor but it is an analogy of Thomas' parameters which are invariants under geodesic mappings, see [5, 20, 21]. So we have verified the following.

Theorem 3.1 *Geometric objects T defined by the formula (11) are invariant with respect to holomorphically projective mappings between parabolically-Kählerian spaces.*

Theorem 3.2 *Equality of the parameters T in a common coordinate system is a necessary and sufficient condition for a mapping $f : K^{o(m)}_n \to \bar{K}^{o(m)}_n$ to be holomorphically projective.*

4 Holomorphically projective flat parabolically-Kählerian space

A parabolically-Kählerian space $K^{o(m)}_n$ is said to be *holomorphically projective flat* if it admits a holomorphically projective mapping onto a flat space, i.e. the space with vanishing Riemannian tensor.

The following was proved [13, 18].

Theorem 4.1 *The parabolically-Kählerian space $K^{o(m)}_n$ is holomorphically projective flat if and only if the following conditions are true for the Riemannian tensor:*

$$R_{hijk} = c\,(2\,F_{hi}F_{jk} + F_{hj}F_{ik} - F_{hk}F_{ij})$$

where $c = \text{const}$, $F_{ij} = g_{i\alpha}F^\alpha_j$.

On the other hand, if a manifold is flat then in an affine coordinate system x components of the connection vanish, $\Gamma^h_{ij}(x) = 0$. Therefore in this coordinate system, also components of the object T vanish,

$$\Gamma^h_{ij}(x) = 0.$$

Accounting invariance of the object T we get

Theorem 4.2 *In any holomorphically projective flat parabolically-Kählerian space there exists a coordinate system x in which*

$$T^h_{ij}(x) = 0.$$

Note that metrics of holomorphically projective flat parabolically-Kählerian spaces were constructed in [13, 18]. A metric tensor g and structure F of these spaces in some Riemannian coordinate system $(y^1, y^2, \ldots y^n)$ at a point x_o have the following form:

$$g_{ij} = \overset{o}{g}_{ij} - c\,F_iF_j, \tag{13}$$

where $F_i = \overset{o}{F}_{i\alpha} y^\alpha$, $\overset{o}{F}_{ij}$ are the components of tensor F_{ij} at x_o.

Note that for a given point x_o of holomorphically projective flat parabolically-Kählerian space $K_n^{o(m,c)}$ the metric and structure tensors may be simultaneously reduced to the form:

$$\overset{o}{g}_{ij} = \begin{pmatrix} 0 & 0 & b_{ab} \\ 0 & \overset{*}{e} & 0 \\ b_{ab}^T & 0 & 0 \end{pmatrix} \quad \text{and} \quad \overset{o}{F}{}_i^h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_m & 0 & 0 \end{pmatrix},$$

where b_{ab}^T is a transposed matrix b_{ab}, $a, b = \overline{1, m}$, E_m is the identity matrix,

$$b_{ab} = \begin{pmatrix} 0 & 1 & & & & & 0 \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & 0 & & & & -1 & 0 \end{pmatrix} \quad \text{and} \quad \overset{*}{e} = \begin{pmatrix} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_{n-2m} \end{pmatrix}, \quad e_a = \pm 1.$$

Thus, we proved the following theorem.

Theorem 4.3 *In the holomorphically projective flat parabolically-Kählerian space $K_n^{o(m,c)}$ there exists a coordinate system y in which the metric tensor has the form (13).*

Without loss of generality, in Theorem 4.1 it is sufficient to consider the cases $c = 0, \pm 1$, that is, spaces $K_n^{o(m,0)}$, $K_n^{o(m,+1)}$ and $K_n^{o(m,-1)}$.

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VISUALIZATION OF TRIVARIATE NURBS VOLUMES

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Abstract. In this paper we focus on one particular set of free-form objects and its visualization. We extended approach for modeling curves and surfaces and prepared trivariate volumes based on Bezier and B-spline expressions. Our main goal is to visualize given parametric volumes. The visualization is done by approximation of volumes by net of isoparametric curves or surfaces and by boundary evaluation of volumes. Rendering is driven by OpenGL platform and there are used several techniques for enhanced visual effects like transparency. We also present practical output based on our implementation of proposed solutions.

Key words. Geometric modelling, NURBS volumes, rendering, OpenGL

1 Introduction

Free-form curves and surfaces are well known and described part of mathematical background in the field of Computer-Aided Geometric Design [2], [3]. In many cases, the parametric expression with polynomial functions is used. For better representation of objects like sphere or cone, natural extension in the form of rational function can be applied [3]. There are also many works on visualization of these free form objects. One uses approximation of object by polylines or triangular meshes. Different visualization option is raytracing, where most important part is to find intersection of line and given objects. To find these of intersections, we can use several algorithms like root finding or subdivision (Bezier clipping).

In our research, we are trying to extend polynomial curves and surfaces into trivariate polynomial objects (volumes) based on Bezier expression [5]. We can also prepare extension in the form of rational volumes and rational splines. For splines, we will use well-known spline representation, B-splines. These volumes can be used in many parts of geometric design, for example as grid for three-dimensional deformations, for scattered data interpolation or for representation of solid objects [7]. In design, it is very important to visualize objects. We will propose some basic options how to show these objects [6]. For rendering 3D objects on 2D screen, we will use Open GL graphics library, because it is supported on many platforms and in many hardware solutions [4].

2 Trivariate volumes

General trivariate object is defined by trivariate function F and the domain D for that function. Precisely:

$$TO : R^3 -> E^3$$
$$TO(u,v,w) = F(u,v,w)$$
$$(u,v,w) \in D$$

In the field of geometric modeling, polynomial or piecewise polynomial (spline) function is most commonly used. We will work with only this representation defined by net of control points.

The rational Bézier tetrahedral volume is defined with a degree, domain, control net of points, and for each point one real number (weight). The degree is a positive integer number n and means degree of polynomials used to blend given points, The domain is a nondegenerated tetrahedron $ABCD$ in E^3 which is used for addressing of barycentric coordinates and control net with weights is tetrahedral structure of points in E^3, that can be written following way:

$$V_{\mathbf{i}} \in E^3; w_{\mathbf{i}} \in R$$
$$\mathbf{i} = (i,j,k,l); |\mathbf{i}| = i + j + k + l = n;$$
$$i.j,k,l \geq 0$$

Let us have point U from the domain and let $\mathbf{u}=(u,v,w,t);u+v+w+t=1$ are barycentric coordinates of point U ($U=uA+vB+wC+tD$) with respect to $ABCD$. Now we can define point of rational Bézier tetrahedral volume $RB^n(\mathbf{u})$ with recursive de Casteljau algorithm:

$$V_{\mathbf{i}}^0(\mathbf{u}) = V_{\mathbf{i}}; w_{\mathbf{i}}^0(\mathbf{u}) = w_{\mathbf{i}};$$
$$w_{\mathbf{i}}^r(\mathbf{u}) = uw_{\mathbf{i}-\mathbf{e}_1}^{r-1}(\mathbf{u}) + vw_{\mathbf{i}-\mathbf{e}_2}^{r-1}(\mathbf{u}) +$$
$$+ ww_{\mathbf{i}-\mathbf{e}_3}^{r-1}(\mathbf{u}) + tw_{\mathbf{i}-\mathbf{e}_4}^{r-1}(\mathbf{u});$$
$$w_{\mathbf{i}}^r(\mathbf{u})V_{\mathbf{i}}^r(\mathbf{u}) = uw_{\mathbf{i}-\mathbf{e}_1}^{r-1}(\mathbf{u})V_{\mathbf{i}-\mathbf{e}_1}^{r-1}(\mathbf{u}) +$$
$$+ vw_{\mathbf{i}-\mathbf{e}_2}^{r-1}(\mathbf{u})V_{\mathbf{i}-\mathbf{e}_2}^{r-1}(\mathbf{u}) + ww_{\mathbf{i}-\mathbf{e}_3}^{r-1}(\mathbf{u})V_{\mathbf{i}-\mathbf{e}_3}^{r-1}(\mathbf{u}) +$$
$$+ tw_{\mathbf{i}-\mathbf{e}_4}^{r-1}(\mathbf{u})V_{\mathbf{i}-\mathbf{e}_4}^{r-1}(\mathbf{u});$$
$$RB^n(\mathbf{u}) = V_{\mathbf{0}}^n(\mathbf{u})$$

where $r=1,\ldots,n$; $|\mathbf{i}|=n-r$ and e_1=(1,0,0,0); e_2=(1,0,0,0); e_3=(1,0,0,0); e_4=(1,0,0,0). From this definition the analytical expression of rational Bézier tetrahedra can be evaluated. So, for barycentric coordinates $\mathbf{u}$ of any point U from domain we have:

$$RB^n(\mathbf{u}) = \frac{\sum_{|\mathbf{i}|=n} w_{\mathbf{i}} V_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u})}{\sum_{|\mathbf{i}|=n} w_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u})}$$

where

$$B_{\mathbf{i}}^n(\mathbf{u}) = \frac{n!}{i!\,j!\,k!\,l!} u^i v^j w^k t^l$$

are generalized Bernstein polynomials.

Rational Bézier tensor volume is defined with three degrees n, m, o and a control net of points and for each point a real number (weight). The degrees are positive integers and represent the degree of blending polynomials in each direction, the domain is nondegenerated box *ABCDEFGH* in E^3 and the control net with weights is box structure of points in E^3, that can be written following way:

$$V_{\mathbf{i}} \in E^3; w_{\mathbf{i}} \in R$$
$$\mathbf{i} = (i,j,k); n \ge i \ge 0; m \ge j \ge 0; o \ge k \ge 0$$

Assume that we have point U from domain and let **u** are coordinates of U with respect to *ABCDEFGH*, so $\mathbf{u}=(u,v,w); 0\le u,v,w \le 1$. Now we can define point of Bézier tensor volume $RB^{n,m,o}(\mathbf{u})$ with analytical expression:

$$RB^{n,m,o}(\mathbf{u}) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{o} w_{(i,j,k)} V_{(i,j,k)} B_i^n(u) B_j^m(v) B_k^o(w)}{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{o} w_{(i,j,k)} B_i^n(u) B_j^m(v) B_k^o(w)}$$

where

$$B_i^n(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

are simple Bernstein polynomials. For this type of volume also exists de Casteljau algorithm, but it is less generalizable.

Non-uniform rational B-spline (NURBS) volumes are a natural extension of NURBS curves and surfaces used in geometric modeling. Parameters that define NURBS volume are similar to the surface case, but there are new parameters due to new added parameter (direction). NURBS volume is defined with :

- three degrees d_u, d_v, d_w,
- three non-decreasing knot vectors $(u_0, u_1, \ldots u_{mu})$, $(v_0, v_1, \ldots, v_{mv})$, $(w_0, w_1, \ldots, w_{mw})$,
- three-dimensional net of control points $V_{i,j,k}$ in E^3; $0<=i<=n_u$; $0<=j<=n_v$; $0<=k<=n_w$;
- for each control point $V_{i,j,k}$ real number (weight) $p_{i,j,k}$
- domain $<u_{du}, u_{nu}+1)$x$< v_{dv}, v_{nv}+1)$x$< w_{dw}, w_{nw}+1)$
- $m_u=n_u+d_u+1$, $m_v=n_v+d_v+1$, $m_w=n_w+d_w+1$

Then NURBS volume is given analytically as

$$S(u,v,w) = \frac{\sum_{i=0}^{n_u}\sum_{j=0}^{n_v}\sum_{k=0}^{n_w} p_{i,j,k} V_{i,j,k} N_i^{d_u}(u) N_j^{d_v}(v) N_k^{d_w}(w)}{\sum_{i=0}^{n_u}\sum_{j=0}^{n_v}\sum_{k=0}^{n_w} p_{i,j,k} N_i^{d_u}(u) N_j^{d_v}(v) N_k^{d_w}(w)}$$

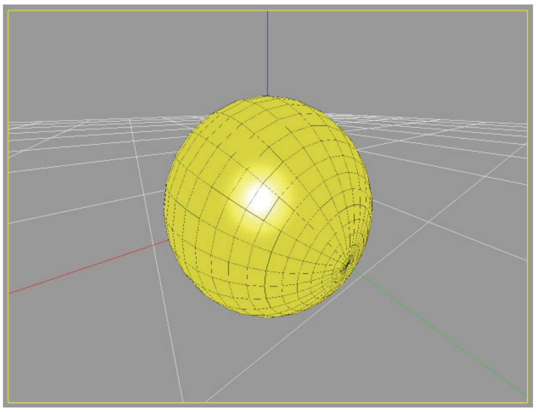
where $N_i^{d_u}(u), N_j^{d_v}(v), N_k^{d_w}(w)$ are B-spline blending functions defined on particular knot vectors with given degrees. Parameters u,v,w are from domain, $u \in <u_{d_u}, u_{n_u+1}), v \in <v_{d_v}, v_{n_v+1}), w \in <w_{d_w}, w_{n_w+1})$.

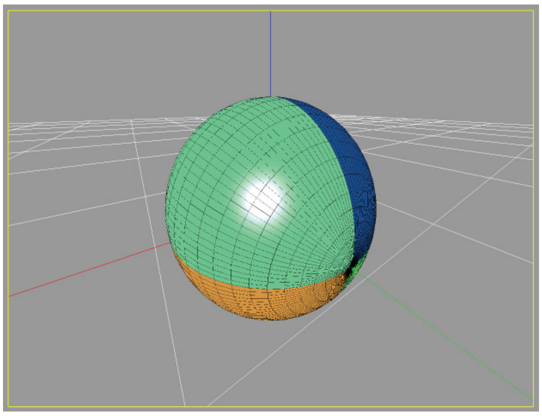
3 Visualization

In this section, we will describe basic solutions for visualization of trivariate free-form objects. Because we are dealing with parametric description of objects, many algorithms can be used in polynomial, rational or spline cases. There also exists simple algorithm for converting

NURBS volume into set of Bezier volumes. Conversion algorithm is based on well-known algorithm of knot insertion for NURBS objects. This process is illustrated in Figure 1b.

Because trivariate volumes represent also interior of the object, we have to create some insight into interior. This leads to some approximations or loss of visualization on the boundary. In some algorithms, only boundary of volume will be determined.





a) b)

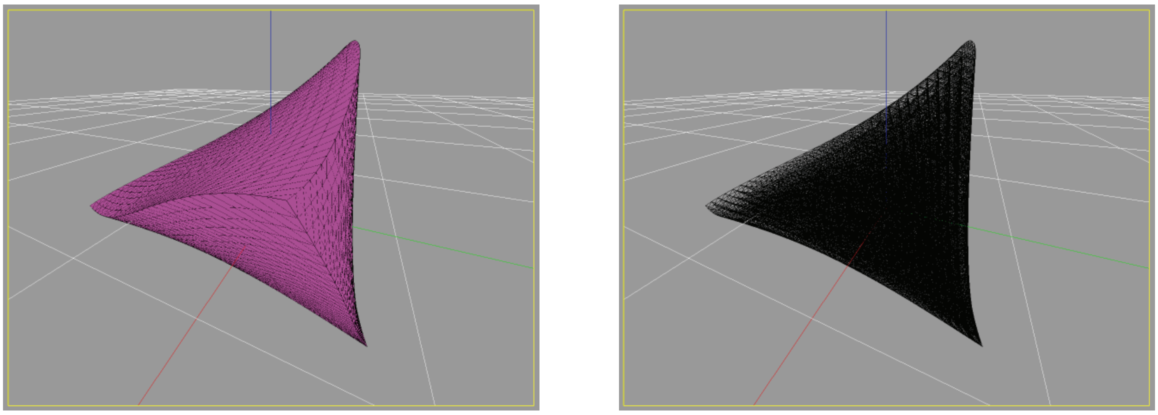
Figure 1: Sphere displayed with isoparametric curves and surfaces: a) Simple NURBS volume, b) Volume decomposed to four Bezier volumes.

3.1 Isoparametric curves

Isoparametric curves are curves generated from parametric function that defines parametric volume. This curve is univariate free-form object derived from trivariate function by making two of three parameters as constants. For example in the case of Bezier tensor volume, we have following set of curves in *u* direction:

$$\{C_{v,w}(u), v, w \in <0,1>; C_{v,w}(u) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{o} w_{(i,j,k)} V_{(i,j,k)} B_i^n(u) B_j^m(v) B_k^o(w)}{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{o} w_{(i,j,k)} B_i^n(u) B_j^m(v) B_k^o(w)}\}$$

This set contains infinite number of curves, so we have to pick some finite number of sample curves. The basic choices are curves when we uniformly sample interval <0,1> for values v and w. This process of generating isoparametric curves can be repeated in v and w direction. Figure 2b shows result of visualization of several curves for all directions in the case of Bezier tetrahedral volume.



a) b)

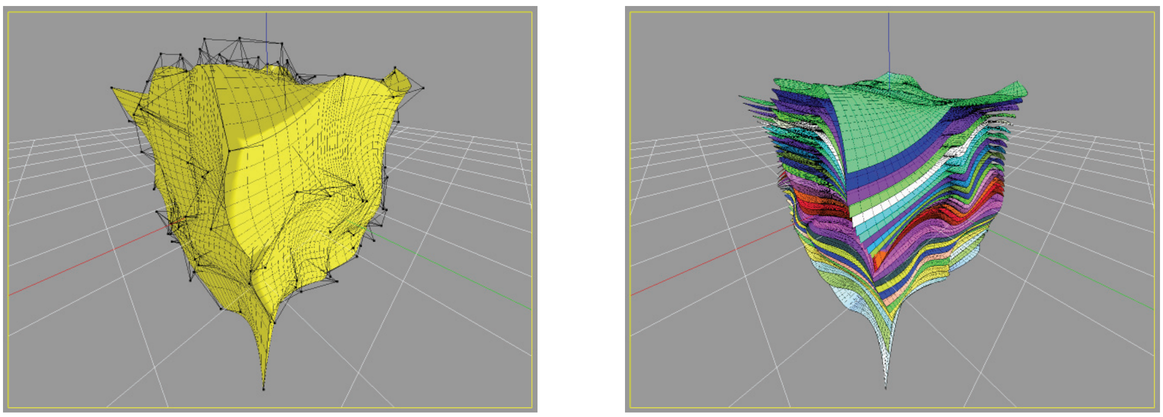
Figure 2: a) Bezier tetrahedra, b) Bezier tetrahedral with displayed isoparametrical curves

3.2 Isoparametric surfaces

Just like the case of isoparametric curves, isoparametric surfaces are generated from volume parametric expression by putting one of the parameters as constant. Because this constant parameter is from continuous interval, we will get infinite number of surfaces. We need to choose some finite number of representative surfaces, this is done by simple uniform sampling of definition interval. Precisely for rational Bezier tensor volume we will choose and visualize following surfaces for *u* direction (for *v*, *w* direction, surfaces can be determined similarly):

$$\{C_u(v,w), u \in <0,1>; C_u(v,w) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{o} w_{(i,j,k)} V_{(i,j,k)} B_i^n(u) B_j^m(v) B_k^o(w)}{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{o} w_{(i,j,k)} B_i^n(u) B_j^m(v) B_k^o(w)}\}$$

For whole volume rendering it is not necessary to render isoparametric surfaces in all directions, it is often sufficient to choose only one direction.



a) b)

Figure 3: a) NURBS volume with random control points, b) NURBS volume displayed using isoparametric surafces

3.3 Boundary evaluation

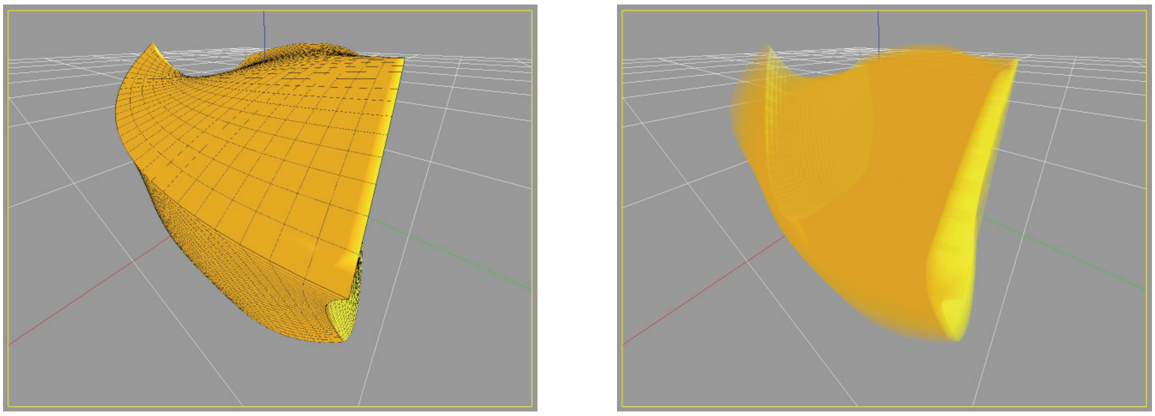
In many applications, visualization of volume interior is not necessary. The boundary of volume is sufficient visualization element. The boundary of trivariate object consist of boundary isoparametric curves (for extremal parameters) together with part of solids where Jacobi determinant is zero:

$$J(U) = \begin{vmatrix} \frac{\partial RB}{\partial u}(U) & \frac{\partial RB}{\partial u}(U) & \frac{\partial RB}{\partial u}(U) \\ \frac{\partial RB}{\partial v}(U) & \frac{\partial RB}{\partial v}(U) & \frac{\partial RB}{\partial v}(U) \\ \frac{\partial RB}{\partial w}(U) & \frac{\partial RB}{\partial w}(U) & \frac{\partial RB}{\partial w}(U) \end{vmatrix} = 0$$

When combined these two options, we can render whole boundary of trivariate object, as in Figures 1, 3a.

3.4 Transparency

It is possible to render all isoparametric surfaces for three sampled parameters presented in section 3.2. In that case, only approximated boundary of whole volume will be visible (Figure X). For insight into interior of object, we can use transparency of generated isoparametric surfaces. Rendering of transparency is done using basic features of OpenGL rendering engine. Figure 4 shows NURBS volume rendered using enabled transparency.



a) b)

Figure 4: a) NURBS volume with random control points, b) NURBS volume displayed using isoparametric surafces

4 Results & Future Work

We have implemented all proposed visualization options of trivariate objects. Our own system called GeomForge can model and visualize volumes in easy and efficient way. All figures presented in this paper were renderer using this system.

We have tested frame rates of visualization on AMD Sempron system with 1.61GHz and 1GB size of memory. Used graphics card was ATI Radeon X700 with 128MB of graphics memory. Table 1 shows these rates for various objects and settings.

In the future, we want to focus on raytracing methods using approximation methods and Bezier clipping algorithm. We have also some unsolved features like exact sorting of geometry for proper transparency of NURBS volumes.

Model	Isoparametric curves	Isoparametric surfaces	Transparency
Sphere	59 FPS	161 FPS	51 FPS
Torus	60 FPS	140 FPS	51 FPS
Random	23 FPS	80 FPS	19 FPS

Table 1: Frame rates of visualization options for several NURBS volumes.

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METRIZATION OF LINEAR CONNECTIONS

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Abstract. We contribute to the following problem: given a manifold endowed with a linear connection, decide whether the connection arises from some metric tensor. Compatibility condition for a metric is given by a system of ordinary differential equations. We present an algorithm that provides a solution in general and show how the process can be simplified in dimension two. We also propose one application in the calculus of variations: for a particular type of second order system of ODEs, which define geodesics of a linear connection, components of a metric compatible with the connection play the role of variational multipliers.

Key words and phrases. Riemannian geometry, manifold, connection, metric, metrization, curvature, differential equations.

Mathematics Subject Classification. Primary 53B05, 53B20; Secondary 53B21, 53C05.

1 Algorithm for metrization

The metrization problem (MP) for linear connections, i.e. the problem to find necessary and sufficient conditions for a given symmetric connection on an n-dimenisonal manifold (smooth or analytic) to be the Riemannian (= Levi-Civita) connection of some metric, belongs to the oldest and difficult problems of classical differential geometry. We can either demand a metric be classical Riemannian (i.e. positive definite) or pseudo-Riemannian (non-degenerate).

Although the problem is also of intrinsic mathematical importance, a strong and natural motivation comes from theoretical physics, cf. explanations in [1] etc.: In the gauge-theoretic framework, the linear (= affine) connections correspond to gauge potentials in other gauge theories, and it is quite natural to suppose that the metric is not given but it follows from field equations or some other natural considerations.

In the favourable case when the solution for MP exists, it depends on the dimension n of the manifold; only the case $n = 2$ is easy.

As well known, the Levi-Civita connection is uniquely determined by zero torsion and the requirement that the scalar product of tangent vectors with respect to g should be preserved under the parallel transport w.r.t. ∇ along all curves, another speaking, g should be covariantly constant,

$$\nabla g = 0. \tag{1}$$

In local coordinates, this condition reads

$$\frac{\partial g_{ij}}{\partial x^k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk} = 0. \tag{2}$$

Components (Christoffel's symbols of the second type) Γ^i_{jk} of the Levi-Civita connection are related to components of the metric by the well-known formula

$$\Gamma^\ell_{ik} = \frac{1}{2} g^{\ell j}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j}\right) \tag{3}$$

where (g^{ij}) is the inverse of the matrix (g_{ij}), $g^{is}g_{sj} = \delta^i_j$; the conditions (1), (2) and (3) are equivalent.

In 1920', L.P. Eisenhart and O. Veblen started to study geometries that arise on an (analytic, C^ω) n-manifold where a system of curves called "paths" is given; as a motivation coming from gravitation theory, let us mention free-fall trajectories as an example of a "preferred family" of curves.

The question formulated in [6] is a bit more general; in a free paraphrase:

On a manifold (M, ∇) *with linear connection, find a covariantly constant (i.e. satisfying* $\nabla g = 0$*) symmetric type* $(0, 2)$ *tensor field* g.

Note that even if such a tensor g exists it might not be a metric if its rank is not maximal.

The authors in [6] also mentioned earlier attempts of H. Weyl, [21], and A.S. Eddington, [4], who preferred "generalization of Levi-Civita's concept of infinitesimal parallelism to the natural idea of paths".

Paths (in favourable case, geodesic paths) arise as the family of solutions of a system of differential equations

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0, \quad i, j, k \in \{1, \ldots, n\} \tag{4}$$

where $\Gamma^i_{jk}(x^1, \ldots, x^n)$ are analytic functions of the coordinates in the manifold.

The classical approach is of analytic character: possible solutions of the system corresponding to the condition $\nabla g = 0$ (for unknowns g_{ij}) are discussed; for the purpose of non-degenerate metric, under the implicite assumption $\det(g_{ij}) \neq 0$. Application of higher order covariant derivatives and tensor methods give rise to integrability conditions. Then algebraic methods can be applied.

Necessary conditions for metrizability can be given in the form of an infinite system of linear equations in $\frac{1}{2}n(n+1)$ functions g_{ij}

$$g_{sj}R^s_{ik\ell} + g_{is}R^s_{jk\ell} = 0, \tag{5}$$

$$g_{sj}R^s_{ik\ell;m_1;\dots;m_r} + g_{is}R^s_{jk\ell;m_1;\dots;m_r} = 0 \quad 1 \le r \tag{6}$$

where

$$R^i_{hjk} = \frac{\partial \Gamma^i_{kh}}{\partial x^j} - \frac{\partial \Gamma^i_{jh}}{\partial x^k} + \sum_s \left(\Gamma^i_{js}\Gamma^s_{kh} - \Gamma^i_{ks}\Gamma^s_{jh}\right) \tag{7}$$

are components of the curvature tensor. Hence coefficients in (6) are functions in $\Gamma's$ and their partial derivatives. In a coordinate-free form, $g(R(X,Y)Z,W) + g(Z,R(X,Y)W) = 0$, $g(\nabla^r R(X,Y;Z_1;\dots;Z_r)(Z),W) + g(Z,\nabla^r R(X,Y;Z_1;\dots;Z_r)(W)) = 0$ for all X,Y,Z,W,Z_1, $\dots,Z_r \in \mathcal{X}(M)$, $1 \le r < \infty$. In order that the connection shall be metrizable the system (5), (6) must have non-trivial solution, i.e. it must possess at least a 1-dimensional solution space (over the ring $\mathcal{F}(M)$ of smooth functions); the above linear conditions must stabilize for some positive integer r.

In general, $0 \le \operatorname{rank}(g_{ij}) \le n$ holds. It might happen that the maximum q of ranks of all possible solutions of (5)-(6) is less than n; if this is the case then (M,∇) is non-metrizable (Example 2.1, 2.2; even the case $q = 0$ might come). There exist non-metrizable n-dimensional manifolds with linear connection for any $n \ge 2$. There are in fact more non-metrizable (M,∇) than metrizable ones. The existence problem for compatible metrics has been considered by various authors, e.g. [7], [8] ($n = 2$), [9], [10] ($n = 4$), [14, p. 75] ($n = 2$) etc., and various methods have been used. For 2-manifolds, the metrization problem was solved e.g. in [1], [16], [17].

If the curvature tensor vanishes then the connection is surely metrizable. An obvious condition for existence of a non-degenerate compatible metric is that the system up to some r^{th} stage of differentiation should possess at least a 1-dimensional solution space. The following result saying that in the favourable case, it is sufficient to take $r = 1$, may be used in examples:

Theorem 1.1 [6, p. 23] (a free paraphrase), *Let (M,∇) be given, with local coordinates (x^i) and ∇ symmetric. If the system (5) admits a non-trivial solution (g_{ij}) and all elements of the solution space satisfy also the system*

$$g_{sj}R^s_{ik\ell;m} + g_{is}R^s_{jk\ell;m} = 0 \tag{8}$$

(that is, (6) for $r = 1$) then locally, there exists a covariantly constant symmetric type $(0,2)$ tensor field g with components (g_{ij}).

Proof. Recall the main facts used in the proof, which support us by a method how to find compatible metrics in concrete examples. Suppose that the system (5) is solvable and that any solution of (5) satisfies (6). Let $\langle G^{(1)},\dots,G^{(p)}\rangle$ be a basis of the solution space. Then any solution g can be written in the form

$$g = \sum_{\alpha=1}^{p} \varphi^{(\alpha)} G^{(\alpha)} \tag{9}$$

where $\varphi^{(\alpha)}$ are some functions on M. Due to (8), $G^{(\alpha)}_{sj;m}R^s_{ik\ell} + G^{(\alpha)}_{is;m}R^s_{jk\ell} = 0$, $\alpha = 1,\dots,p$ holds. That is, the covariant derivatives $G^{\alpha}_{sj;m}$ satisfy (5), too, and hence can be expressed by means of generators (the coefficients can be calculated),

$$G^{(\alpha)}_{ij;k} = \sum_{\beta=1}^{p} \mu_k^{(\alpha\beta)} G^{(\beta)}_{ij}, \quad \alpha = 1, \dots, p. \tag{10}$$

Since second covariant derivatives satisfy the so-called Ricci indentity $G^{(\alpha)}_{ij;k\ell} - G^{(\alpha)}_{ij;\ell k} = G^{(\alpha)}_{sj} R^s_{ik\ell} + G^{(\alpha)}_{is} R^s_{jk\ell}$, and the right hand sides vanish for our $G^{(\alpha)}_{ij}$, we get $G^{(\alpha)}_{ij;k\ell} - G^{(\alpha)}_{ij;\ell k} = 0$, and further (after some evaluations) we obtain

$$\frac{\partial \mu_k^{(\alpha\beta)}}{\partial x^\ell} - \frac{\partial \mu_\ell^{(\alpha\beta)}}{\partial x^k} + \sum_{\gamma=1}^{p} \left(\mu_k^{(\alpha\gamma)} \mu_\ell^{(\gamma\beta)} - \mu_\ell^{(\alpha\gamma)} \mu_k^{(\gamma\beta)} \right) = 0. \tag{11}$$

If g of the form (9) shall satisfy $\nabla g = 0$ then the φ's must satisfy the equations

$$\frac{\partial \varphi^{(\alpha)}}{\partial x^k} + \sum_{\beta=1}^{p} \varphi^{(\beta)} \mu_k^{(\alpha\beta)} = 0, \quad \alpha = 1, \dots, p, \quad k = 1, \dots n. \tag{12}$$

But according to (11), the system (12) is completely integrable, hence there exist functions $\varphi^{(1)}, \dots, \varphi^{(p)}$ which, by means of (9), determine a compatible (pseudo-)Riemannian metric.

The proof of Theorem 1.1 provides an algorithm how to proceed in examples. We demonstrate it below. Moreover, if the solution of (5) is just "one-dimensional" over $\mathcal{F}(M)$ (i.e. it is determined uniquely up to a multiple by function of local coordinates), another speaking, if all solutions are of the form $g = \varphi(x^1, \dots, x^n) \cdot G$ where G is a fixed non-trivial solution, then the solution procedure is simplified considerably. Instead of (10), we have the conditions $G_{ij;k} = \mu_k G_{ij}$ for suitable functions $\mu_k(x^i)$, $k = 1, \dots, n$; it means that G should be recurrent, that is, there must exist a one-form μ with μ_k as components such that $\mu \otimes G = \nabla G$. Instead of (11), we have the integrability conditions $(k, \ell = 1, \dots, n)$

$$\frac{\partial \mu_k}{\partial x^\ell} - \frac{\partial \mu_\ell}{\partial x^k} + \mu_k \mu_\ell - \mu_\ell \mu_k = 0. \tag{13}$$

The system (12) is reduced to

$$\frac{\partial \varphi}{\partial x^k} + \varphi \mu_k = 0, \quad k = 1, \dots, n. \tag{14}$$

If the conditions (13) hold then there exists a function $f(x^1, \dots, x^n)$ such that for $k = 1, \dots, n$, $\frac{\partial f}{\partial x^k} = \mu_k$ holds, that is, μ is exact (=gradient), $\mu = df$, and (14) reads

$$\frac{\partial \varphi}{\partial x^k} + \varphi(x^1, \dots, x^n) \frac{\partial f}{\partial x^k} = 0, \quad k = 1, \dots, n. \tag{15}$$

Then (15) is completely integrable, and all solutions are just $\varphi = e^{-f}$ with $\mu = df$. Finally, each $g = e^{-f} \cdot G$ is covariantly constant, $\nabla g = 0$.

Let us demonstrate this method by a simple example.

Example 1.2 Let us given a system

$$\begin{aligned} \ddot{x}^1 + 2(\cot x^2)\dot{x}^1\dot{x}^2 = 0, \\ \ddot{x}^2 - 2(\sin x^2 \cdot \cos x^2)(\dot{x}^1)^2 = 0. \end{aligned} \tag{16}$$

Paths correspond to the linear connection with non-vanishing components

$$\Gamma^1_{12} = \cot x^2, \quad \Gamma^2_{11} = -\sin x^2 \cos x^2, \quad R^1_{212} = 1, \quad R^2_{112} = -\sin^2 x^2; \quad \nabla R = 0.$$

The solution system $\{\varphi G; \varphi(x^1, x^2) \text{ function}\}$ for (5) is generated by the form

$$G = \sin^2 x^2 \, dx^1 \otimes dx^1 + dx^2 \otimes dx^2;$$

note that G is the first fundamental form of a unit sphere $\mathbb{S}^1$ parametrized in spheric coordinates. We check $\nabla G = 0$, $G_{11;1} = G_{11;2} = 0$, $\mu_1 = \mu_2 = 0$, $f = \text{const}$, $e^{-f} = \text{const}$. Hence all covariantly constant metrics are $\{cG; c \in \mathbb{R}\}$. That is, G is unique up to scaling by constants.

As a consequence of the above considerations, we get

Theorem 1.3 *A manifold (M, ∇) with symmetric connection and the curvature tensor $R\,(R^i_{hjk})$ is locally metrizable if and only if (5) has non-degenerate solution, and any solution of (5) satisfies (8).*

As we have seen, in principle, it is possible to give an algorithm, known already to classics of differential geometry, [6], [5] that provides a solution; the algorithm may be implemented to a simple computer program. However, this algorithm yields only a prescriptive solution. That is why some authors tried, at least in low dimensions, to express necessary and sufficient conditions in terms of components of the given connection (and their partial derivatives).

2 Metrization of affine 2-manifolds (revisited)

Let us pay attention to the case when $n = 2$ in details. As a matter of fact, it is the only truly easy case of MP.

As already mentioned we can use the above classical algorithm of Eisenhart and Veblen, as in the above Example 1.2.

2.1 Direct approach

In the simplest cases of dimenison two, we are often able to decide about solution of $\nabla g = 0$ directly:

Example 2.1 ([6, p. 122]) On $\mathbb{R}^2$ with coordinates $x = (x^1, x^2)$, assume the system of ODEs

$$\begin{aligned} \ddot{x}^1 + (x^1 - x^2)(\dot{x}^1)^2 = 0, \\ \ddot{x}^2 + (x^1 - x^2)(\dot{x}^2)^2 = 0. \end{aligned} \tag{17}$$

Curves $c(s) : I \to \mathbb{R}^2$ (parametrized by arc length), which are solutions of the system, represent the family of geodesics of a (symmetric) linear connection ∇ with components $\Gamma^1_{11} = \Gamma^2_{22} = x^1 - x^2$, $\Gamma^i_{jk} = 0$ otherwise, or equivalently,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} &= (x^1 - x^2)\frac{\partial}{\partial x^1}, \\ \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^2} &= (x^1 - x^2)\frac{\partial}{\partial x^2}, \quad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0 \quad \text{otherwise.} \end{aligned}$$

Let us solve the system of equations arising from the condition $\nabla g = 0$ for (smooth) functions $g_{ij}(x^1, x^2)$, which should be components of a symmetric functional matrix $G = (g_{ij})$ (in short we write ∂_k instead of $\frac{\partial}{\partial x_k}$):

$$\begin{aligned} \partial_1 g_{11} &= (x^1 - x^2) g_{11}, & \partial_2 g_{11} &= 0, & & \\ \partial_1 g_{12} &= 0, & \partial_2 g_{12} &= (x^1 - x^2) g_{12}, & & g_{ij} = 0 \text{ for all } i, j, \\ \partial_1 g_{22} &= 0, & \partial_2 g_{22} &= (x^1 - x^2) g_{22}; & & \text{max rank} = q = 0. \end{aligned}$$

Hence $G = 0$, the only solution is trivial, the connection is not metrizable.

Example 2.2 ([2]) The system

$$\ddot{x}^1 = 0, \quad \ddot{x}^2 = -2\dot{x}^1\dot{x}^2 \tag{18}$$

defines on $\mathbb{R}^2$ (or on the cylinder $\mathbb{R} \times \mathbb{S}^1$, or on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$) a symmetric linear connection ∇ which is not a metric one. In fact, the only non-zero Christoffels are $\Gamma^2_{12} = \Gamma^2_{21} = 1$, i.e. ∇ can be introduced also by

$$\nabla_{X_1} X_1 = \nabla_{X_2} X_2 = 0, \quad \nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_2, \quad X_i = \frac{\partial}{\partial x_i}.$$

Solving directly the corresponding system of differential equations we get

$$\begin{aligned} \partial_1 g_{11} &= 0, & \partial_2 g_{11} &= 2g_{12}, & & G = (g_{ij}) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R}, \\ \partial_1 g_{12} &= g_{12}, & \partial_2 g_{12} &= g_{22}, & & \\ \partial_1 g_{22} &= 2g_{22}, & \partial_2 g_{22} &= 0, & & \text{max rank} = q = 1. \end{aligned}$$

Hence the corresponding tensor is degenerate, and no compatible metric exists in $(\mathbb{R}^2, \nabla)$.

2.2 Tensor methods

By a detailed analysis of the system (5), (6) of the integrability conditions we find that for a nowhere flat two-manifold with linear connection, necessary and sufficient conditions for existence of a compatible metric can be formulated in terms of components of the Ricci tensor, which we prefer here. Note that essentially the same "very classical" approach was used by Kuo-Shung Cheng and Wei-Tou Ni in [1], without any reference to Veblen's and Eisenhart's results. In [1], necessary and sufficient conditions are given, but without any explanation of their geometric interpretation concerning properties of the connection or the underlying manifold. In the proof given in [1], there are some gaps and missprints (e.g. in formulas (28), (35) from [1]); we hope to succeed in improving them. Note that some kind of answer, dealing with various tensors related to the curvature tensor, can be found in [14].

In local coordinates,

$$g_{ij;k} = \tfrac{\partial g_{ij}}{\partial x^k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk}. \tag{19}$$

Hence the formula $\nabla g = 0$ reads

$$\frac{\partial g_{ij}}{\partial x^k} = g_{sj}\Gamma^s_{ik} + g_{is}\Gamma^s_{jk}. \tag{20}$$

If $n = 2$, (20) takes the form of six PDEs in three variables g_{11}, g_{12}, g_{22}

$$\begin{aligned} \partial_1 g_{11} =&\ 2\left(\Gamma^1_{11}g_{11} + \Gamma^2_{11}g_{12}\right), & \partial_1 g_{22} =&\ 2\left(\Gamma^1_{12}g_{12} + \Gamma^2_{12}g_{22}\right), \\ \partial_2 g_{11} =&\ 2\left(\Gamma^1_{12}g_{11} + \Gamma^2_{12}g_{12}\right), & \partial_2 g_{22} =&\ 2\left(\Gamma^1_{22}g_{12} + \Gamma^2_{22}g_{22}\right), \end{aligned}$$

$$\begin{aligned} \partial_1 g_{12} =&\ \Gamma^1_{12}g_{11} + \left(\Gamma^1_{11} + \Gamma^2_{12}\right)g_{12} + \Gamma^2_{11}g_{22}, \\ \partial_2 g_{12} =&\ \Gamma^1_{22}g_{11} + \left(\Gamma^1_{12} + \Gamma^2_{22}\right)g_{12} + \Gamma^2_{12}g_{22}. \end{aligned} \tag{21}$$

Partial differentiating of (21) followed by substituion for partial derivatives $\partial_k g_{ij}$ from (19) gives

$$\frac{\partial^2 g_{sk}}{\partial x^r \partial x^j} = g_{s\ell}\left(\frac{\partial \Gamma^\ell_{rk}}{\partial x^j} + \Gamma^\ell_{jt}\Gamma^t_{rk}\right) + g_{k\ell}\left(\frac{\partial \Gamma^\ell_{rs}}{\partial x^j} + \Gamma^\ell_{jt}\Gamma^t_{rs}\right) + g_{\ell t}\left(\Gamma^t_{jk}\Gamma^\ell_{rs} + \Gamma^\ell_{rk}\Gamma^t_{js}\right).$$

Substituting for components of the curvature tensor and accounting interchangeability of second partial derivatives, we get, as a necessary condition, that g_{ij} must satisfy a homogeneous system of linear algebraic equations (5). In dimension two, the Ricci tensor $R_{hj} = \sum_k R^k_{hjk}$ has components

$$\begin{aligned} R_{11} &= -R^2_{121} = R^2_{112}, & R_{12} &= -R^1_{112} = R^1_{121}, \\ R_{21} &= -R^2_{221} = R^2_{212}, & R_{22} &= -R^1_{212} = R^1_{221}. \end{aligned}$$

Recall that on a 2-dimensional pseudo-Riemannian manifold, the Ricci tensor Ric is always proportional to the metric tensor,

$$\mathrm{Ric} = K \cdot g \tag{22}$$

where K denotes the sectional curvature (or Gaussian curvature, respectively, for embedded submanifolds in the positive definite case), [17]. Employing the Ricci tensor, we get a homogeneous system of three linear algebraic equations in three unknowns g_{11}, g_{12}, g_{22},

$$\begin{aligned} R_{12}g_{11} - R_{11}g_{12} \qquad\qquad &= 0, \\ R_{22}g_{11} + (R_{12} - R_{21})g_{12} - R_{11}g_{22} &= 0, \\ R_{22}g_{12} - R_{21}g_{22} &= 0. \end{aligned} \tag{23}$$

The system (23) has a non-trivial solution iff its matrix has vanishing determinant on M, i.e. if and only if the following is satisfied:

$$0 = \begin{vmatrix} R_{12} & -R_{11} & 0 \\ R_{22} & R_{12} - R_{21} & -R_{11} \\ 0 & R_{22} & -R_{21} \end{vmatrix} = (R_{12} - R_{21}) \cdot \det(R_{ij}). \tag{24}$$

If $R \equiv 0$ then (24) holds identically, and we obtain no new condition for g_{ij}; (23) has three-parameter solution system.

If $R(x) \neq 0$ in one point $x \in M$ ("x is a regular point of R") then from continuity, $R \neq 0$ on some neighborhood $U \subset M_2$ of x. Note that the family of points with non-vanishing curvature form an open subset in M; in the following, let us investigate this non-flat part.

So suppose $R \neq 0$ on M (the manifold is "no-where flat"). Then also the Ricci tensor is non-vanishing.

Due to (22) regularity of the Ricci tensor is a necessary condition for metrizability of a nowhere flat 2-manifold[1], [17].

Under the regularity assumption, $\det(R_{ij}) \neq 0$, we obtain $R_{12} = R_{21}$, symmetry of Ric, as second necessary condition. Accounting this condition, the system (23) reads

$$\begin{pmatrix} R_{12} & -R_{11} & 0 \\ R_{22} & 0 & -R_{11} \\ 0 & R_{22} & -R_{12} \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{25}$$

If $x \in M_2$ is a fixed point such that $R(x) \neq 0$, $\det(R_{ij}(x)) \neq 0$ then the following cases might occur

(a) $R_{11}(x) = 0$ (or $R_{22}(x) = 0$, or even both are equal zero), but $R_{12}(x) \neq 0$,
(b) $R_{12}(x) = 0$, but $R_{11}(x) \cdot R_{22}(x) \neq 0$,
(c) $R_{ij}(x) \neq 0$ for $i, j \in \{1, 2\}$.

In each of these three subcases, there exists a non-vanishing subderterminant of order two, hence the coefficient matrix is of rank two. So under the assumptions $R \neq 0$, Ric symmetric, the coefficient matrix has rank two, and the general solution of (25) is one-dimensional (over $\mathcal{F}(M_2)$), $g_{ij}(x) = k(x) \cdot R_{ij}(x)$, $k(x) : M \to \mathbb{R} - \{0\}$, or shortly,

$$g_{11} : g_{12} : g_{22} = R_{11} : R_{12} : R_{22}. \tag{26}$$

This solution satisfies the system of differential equations (21). If we prescribe initial data $x_o \in M$, $K(x_o) \neq 0$, and set $k(x) = \frac{1}{K(x_o)}$ then the solution is unique, $g_{ij} = \frac{1}{K(x_o)} R_{ij}$ (the choice of a constant has a geometric meaning of a "constant scale change" of the metric). In the case (c), and only in this case, we can proceed as in [1]. We can calculate the ratios

$$\frac{\partial_k g_{ij}}{g_{ij}} = \frac{\partial \ln |g_{ij}|}{\partial x^k} = \Gamma^s_{ik} \frac{R_{sj}}{R_{ij}} + \Gamma^s_{jk} \frac{R_{is}}{R_{ij}} \tag{27}$$

from (21), derive (necessary and sufficient) integrability conditions for the first four of these equations (namely with $i = j = 1$ and $i = j = 2$),

$$\begin{aligned} \partial_2(\Gamma^1_{11} + \tfrac{R_{12}}{R_{11}}\Gamma^2_{11}) - \partial_1(\Gamma^1_{12} + \tfrac{R_{12}}{R_{11}}\Gamma^2_{12}) &= 0, \\ \partial_1(\Gamma^2_{12} + \tfrac{R_{12}}{R_{22}}\Gamma^1_{12}) - \partial_2(\Gamma^2_{22} + \tfrac{R_{12}}{R_{22}}\Gamma^1_{22}) &= 0. \end{aligned} \tag{28}$$

In [1], after some evaluation, besides symmetry of the Ricci tensor, $R_{12} = R_{21}$, further necessary conditions are obtained, originally formulated in components of the curvature tensor, which in our notation read

$$\begin{aligned} R_{12}R_{11;1} &= R_{11}R_{12;1}, \quad & R_{12}R_{11;2} &= R_{11}R_{12;2}, \\ R_{12}R_{22;1} &= R_{22}R_{12;1}, \quad & R_{12}R_{22;2} &= R_{22}R_{12;2}. \end{aligned} \tag{29}$$

[1] Such geometric arguments seem to be missing in [1].

This family of conditions can be rewritten as

$$\begin{aligned}\varrho_1 &= R_{11;1} : R_{11} = R_{12;1} : R_{12} = R_{22;1} : R_{22},\\ \varrho_2 &= R_{11;2} : R_{11} = R_{12;2} : R_{12} = R_{22;2} : R_{22}.\end{aligned} \tag{30}$$

Obviously, (30) is equivalent to recurrency: $\nabla\mathrm{Ric} = \varrho \otimes \mathrm{Ric}$, $\varrho = \varrho_i dx^i$, $i = 1, 2$ (it was realized in [16], [17]). If the integrability conditions $R_{12} = R_{21}$ and (29) are satisfied, that is, if the (non-degenerate) Ricci tensor is symmetric and recurrent, then (29) hold, (28) can be integrated[2], and the solution of (21) can be given in the following (rather complicated) form, [1]:

$$\begin{aligned} g_{11} = \ & \exp(2\textstyle\int_{(x_0^1,x_0^2)}^{(x^1,x^2)}(\Gamma^1_{11} + \frac{R_{12}}{R_{11}}\Gamma^2_{11})\mathrm{d}x^1 \\ & +(\Gamma^1_{12} + \frac{R_{12}}{R_{11}}\Gamma^2_{12})\mathrm{d}x^2 + c_1), \end{aligned} \tag{31}$$

$$\begin{aligned} g_{12} = \ & \exp(2\textstyle\int_{(x_0^1,x_0^2)}^{(x^1,x^2)}(\Gamma^1_{11} + \Gamma^2_{12} + \frac{R_{11}}{R_{12}}\Gamma^1_{12} - \frac{R_{22}}{R_{12}}\Gamma^2_{11})\mathrm{d}x^1 \\ & +(\Gamma^1_{12} + \Gamma^2_{22} + \frac{R_{11}}{R_{12}}\Gamma^1_{22} - \frac{R_{22}}{R_{12}}\Gamma^2_{12})\mathrm{d}x^2 + c_2), \end{aligned} \tag{32}$$

$$\begin{aligned} g_{22} = \ & \exp(2\textstyle\int_{(x_0^1,x_0^2)}^{(x^1,x^2)}(\Gamma^2_{12} + \frac{R_{12}}{R_{22}}\Gamma^1_{12})\mathrm{d}x^1 \\ & +(\Gamma^2_{22} + \frac{R_{12}}{R_{22}}\Gamma^1_{22})\mathrm{d}x^2 + c_3) \end{aligned} \tag{33}$$

where the constants c_1, c_2, c_3 must be selected in order to satisfy (23); the choice of c_2 is free (and corresponds to the constant scale change of the compatible metric), c_1 and c_3 depend on c_2.

Partial answer, concerning nowhere flat 2-manifolds with linear connection, can be formulated as follows:

Theorem 2.3 *Local necessary and sufficient condition for a nowhere flat symmetric connection ∇ on M_2 be metrizable are: the Ricci tensor* $\mathrm{Ric}\,(R_{ij})$ *of ∇ should be*
non-degenerate ($\det R_{ij} \neq 0$),
symmetric ($R_{ij} = R_{ji}$),
recurrent, $\nabla\mathrm{Ric} = \omega \otimes \mathrm{Ric}$ *for some one-form ω.*
If ω is exact (=gradient), i.e. $\omega = df$ for some function f, then compatible metrics exist globally, one of the representants being $g = e^{-f}\,\mathrm{Ric}$*, the other differ upto a scalar multiple (in this sense, g is "unique"). If M_2 is simply connected, a compatible g exists globally.*

In the case of "regular" curvature R, also the results from [12] are relevant.

2.3 Real analytic case

Note that if $n = 2$, and the manifold (M_2, ∇) is real analytic, connected and simply connected, and we are interested in positive definite metrics only then the decision procedure from [13] can be used, see also [18], [19], based on the so-called holonomy algebras.

[2]Note that sufficiency of the reached system of conditions $R_{12} = R_{21}$, (29) is only asserted in [1], without any detailed proof; the possibilities (a) and (b) are not discussed at all.

2.4 Locally but not globally metrizable examples

In practice, we can try to find all compatible metrics on (open) nowhere flat parts, and then try to glue the metrics together, using suitable metrics on flat parts; we may succeed or not, obstructions might arise on the boundary of flat parts. Let us give an example of this kind (see also [15]). At the same time, the example shows that in the C^∞ class, the situation is more complicated than in the real analytic case.

Example 2.4 On $M_2 = \mathbb{R}^2$ endowed with an atlas $\{\mathbb{R}^2, \mathrm{id}\}$ and global coordinates (x^1, x^2), assume a (global) symmetric linear connection ∇ defined componentwise as follows:

$$\Gamma^1_{12} = \Gamma^1_{21} = \begin{cases} \frac{h'(x^2)}{h(x^2)} & \text{if } x^2 > 0, \\ 0 & \text{if } -1 \le x^2 \le 0, \\ \frac{f'(x^2)}{f(x^2)} & \text{if } x^2 < -1, \end{cases}$$

$$\Gamma^2_{11} = \begin{cases} h'(x^2) \cdot h(x^2) & \text{if } x^2 > 0, \\ 0 & \text{if } -1 \le x^2 \le 0, \\ f'(x^2) \cdot f(x^2) & \text{if } x^2 < -1, \end{cases}$$

$\Gamma^i_{jk} = 0$ otherwise, where $f, g \in C^\infty(\mathbb{R})$ are smooth real functions in one variable satisfying [3]

$$\begin{array}{ll} h(t) \neq 0 \text{ for all } t > 0, & \lim_{t\to 0} h(t) = 1, \\ f(t) \neq 0 \text{ for all } t < -1, & \lim_{t\to -1} h(t) = a \in \mathbb{R} - \{-1, 1\}, \\ \lim_{t\to 0} \frac{d^s h(t)}{dt^s} = 0 \text{ for all } s \in \mathbb{N}, & \lim_{t\to 0} \frac{d^s f(t)}{dt^s} = 0. \end{array}$$

Components of the curvature and Ricci tensors are

$$R^1_{112} = -R^1_{121} = R_{12} = 0, \qquad R^2_{221} = -R^2_{212} = R_{21} = 0,$$

$$R^1_{212} = -R^1_{221} = R_{22} = \begin{cases} \frac{-h''(x^2)}{h(x^2)} & \text{if } x^2 > 0, \\ 0 & \text{if } -1 \le x^2 \le 0, \\ \frac{-f''(x^2)}{f(x^2)} & \text{if } x^2 < -1, \end{cases}$$

$$R^2_{121} = -R^2_{112} = R_{11} = \begin{cases} h''(x^2) \cdot h(x^2) & \text{if } x^2 > 0, \\ 0 & \text{if } -1 \le x^2 \le 0, \\ f''(x^2) \cdot f(x^2) & \text{if } x^2 < -1. \end{cases}$$

The Ricci tensor is symmetric on M. Consider the disjoint decomposition $M = N_I \cup N_{II} \cup N_{III}$ (corresponding to definition domains of respective formulas for components of the connection) where

$$N_I = \{(x^1, x^2); -1 \le x^2 \le 0\},$$

$$N_{II} = \{(x^1, x^2); x^2 > 0\}, \qquad N_{III} = \{(x^1, x^2); x^2 < -1\}.$$

[3] In [15], the condition $a \neq -1$ was forgotten.

On N_I, $R = \mathrm{Ric} = 0$, the restricted connection $\nabla|N_I$ is flat, hence metrizable; we can choose even its signature. On N_{II}, the restricted Ricci tensor is recurrent. In fact, since $R_{11;1} = R_{22;1} = 0$ put $\varrho_1(x) = 0$. Further compute $R_{11;2} = (h^{(3)} \cdot h - h'' \cdot h')(x^2)$, $R_{22;2} = -\frac{h^{(3)} \cdot h - h'' \cdot h'}{h^2}(x^2)$, and set $\varrho_2(x) = \frac{h^{(3)} \cdot h - h'' \cdot h'}{h'' \cdot h}(x^2)$. Then $\varrho = \varrho_2(x)dx^2$ satisfies $\nabla \mathrm{Ric} = \varrho \otimes \mathrm{Ric}$. Let $\beta \in \mathbb{R} - \{0\}$, $g_{II} = (h(x^2))^2(dx^1)^2 - (dx^2)^2$; the most general form for compatible metrics on $(N_{II}, \nabla|N_{II})$ is βg_{II} (Lorentzian metrics). Due to properties of h, for any $\beta \neq 0$, there is a unique continuous extension of βg_{II} on N_I, namely $\beta((dx^1)^2 - (dx^2)^2)$, such that ∇ is the Levi-Civita connection of the extended metric on $N_I \cup N_{II}$. Similarly, Ric is recurrent on N_{III}; compatible metrics are γg_{III}, $\gamma \neq 0$, $g_{III} = (f(x^2))^2(dx^1)^2 - (dx^2)^2$; each γg_{III} has a continuous extension $\gamma(a^2(dx^1)^2 - (dx^2)^2)$ on N_I such that the restriction of ∇ is the Levi-Civita connection of the extended metric on $N_I \cup N_{III}$. But $a^2 \neq 1$ holds (since $a \neq -1$, $a \neq 1$); hence there is no global metric on the entire M compatible with the given connection ∇.

The situation when the manifold with linear connection is locally but not globally metrizable can arise also on now-where flat manifolds; an example on n-torus is given in [2]. Obstructions have topological reasons.

3 Application

Let us mention the relationship of our problem to the Calculus of Variations. The so-called Inverse Problem (IP) of the calculus of variations is: if a system $\ddot{x}^i = f^i(t, x^k, \dot{x}^k)$, $i, k = 1, \ldots, n$ of second order differential equations (SODEs) is given, find - sufficiently differentiable - Lagrangian functions $L(t, x^k, \dot{x}^k)$ and a multiplier matrix $g_{ij}(t, x^k, \dot{x}^k)$ such that

$$g_{ij}(\ddot{x}^i - f^i) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}.$$

Given a system of second order ODEs of a particular type

$$\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \qquad k = 1, \ldots, n, \tag{34}$$

that is, second derivatives can be expressed as quadratic forms in first derivatives, we can use the above theory for deciding whether the system (34) is derivable from a Lagrangian. In fact, provided $\det(g_{ij}) \neq 0$, the system (34) is equivalent to the system

$$g_{mi}(\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k) = 0, \quad i, m = 1, \ldots, n. \tag{35}$$

Another speaking, MP can be viewed as a particular case of IP, where $f^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$ (that is, f^i are quadratic forms in components of velocities, with coefficients depending only on components of positions) in the particular case when the multipliers are time- and velocities-independent. We can assume that the coefficients in (34), the functions $\Gamma^k_{rs}(x)$, are components of a symmetric linear connection ∇ on some neighborhood $U \subset \mathbb{R}^n$. If ∇ is (locally) metrizable, and $g_{ij}(x)$ (with $\det(g_{ij}(x)) \neq 0$ at any $x \in U$) are components of some non-degenerate metric g compatible with ∇ on U, then (34) and (35) are equivalent, hence the functions $g_{ik}(x)$ can be taken as the desired variational multipliers. One of particular Lagrangians comming from MP (and solving IP) is

$$L = T = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j, \tag{36}$$

the kinetic energy. There might exist multipliers of a more general form $g_{ik}(t, x, \dot{x})$, [3], depending on “time, positions and velocities”, which might bring more complicated Lagrangians.

Acknowledgement

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SURFACE MODELLING BY MEANS OF MINKOWSKI SUM

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Abstract. Main ideas of Minkowski sum of two point sets are mentioned in the presented paper, with respect to their utilisation in geometric modelling of surface patches. Several examples are illustrated, in which sum of convex point sets is calculated and visualised. Several applications of Minkowski sum in modelling of zonotopes and surface patches in the Euclidean space created as sums of two curves determined by point functions are given.

Key words. Minkowski sum, zonotop, modelling of surfaces

Mathematics Subject Classification: Primary 51N25; Secondary 53A056.

1 Introduction

Minkowski sum of two point sets is a binary geometric operation defined on point subsets of the Euclidean space, which can be interpreted in different ways. This sum was defined in 1903 by Hermann Minkowski. The most common interpretation of Minkowski sum is its representation by means of vector sum of the position vectors of all points in the given sets – point subsets of the Euclidean space. Next most frequently appearing interpretation of Minkowski sum is a continuous movement of one set operand on the boundary of the other one without any change of orientation. Applications of Minkowski sum can be found in computer graphics, robotics, in finding algorithms for dense placement of planar figures and determination of offsets, in geometric modelling, in CAD, and many others. Minkowski sum of two curves is presented in the paper, as sum of two continuous point subsets of the Euclidean space, which is the tool for surface modelling. Algorithms is based on vector representations of geometric figures in the role of operands, sum of the two given figures is a geometric figure in the space determined by vector function.

2 Definition and properties of Minkowski sum

Let A and B be two point sets in the n-dimensional Euclidean space E^n.

Definition 1. Minkowski sum of two point sets A and B in the Euclidean space is a point set, which is the sum of all points from the set A with all points from the set B, i.e. set of points

$$A \oplus B = \{a + b; a \in A, b \in B\}. \tag{2.1}$$

Definition 2. Minkowski sum of two point sets *A* and *B* inj the Euclidean space is the set

$$A \oplus B = \bigcup_{b \in B} A^b , \tag{2.2}$$

where A^b is the set *A* translated by the vector *b*

$$A^b = \{a + b; a \in A\}. \tag{2.3}$$

Minkowski sum is also called a binary dilatation of the et *A* by the set *B*. Minkowski sum is sometimes also denoted, not exactly and correctly, as a convolution of point sets *A* and *B*.
Let us mention some basic properties of the Minkowski sum of point sets *A*, *B*, *C*, *D* in the Euclidean space, which are used most frequently in its calculations.

1. Minkowski sum of convex sets is also a convex set.
2. Minkowski sum is a commutative operation, it holds: $A \oplus B = B \oplus A$.
3. Minkowski sum is an associative operation, it holds: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.
4. It holds: $(A \cup B) \oplus (C \cup D) = (A \oplus C) \cup (A \oplus D) \cup (B \oplus C) \cup (B \oplus D)$.
5. Minkowski sum of the union of two sets is a union of the Minkowski sums of the separate sets. Let P_i and Q_j for $i, j \in N$ are sets in the Euclidean space, then it holds:

$$\bigcup_i P_i \oplus \bigcup_j Q_j = \bigcup_{i,j} P_i \oplus Q_j \tag{2.4}$$

Property 5. is very useful, for determination of the Minkowski sum of non-convex sets, especially. These must be first distributed onto a union of a finite number of disjunctive convex sets, for which Minkowski sums can be calculated, and then it is possible to utilise property 5 for calculation of their union.

3 Several simple examples

The most commonly used examples of Minkowski sum are sums of convex planar closed polygons. For any two closed convex planar polygons *P* and *Q* with number of vertices *m* and *n* in th given order holds the following: Minkowski sum is a convex polygon with $m + n$ vertices, which can be determined by a simple calculation. This can be described as follows. The same orientation can be attached to both polygons, e.g. a positive one, therefore all sides will be defined as oriented line segments – vectors with given polar angles, determining their order in a sequence. Unifying both sequences of oriented sides of polygons *P* and *Q* into one sequence *S* ordered according to size of the polar angles we obtain an ordered sequence of oriented line segments. Positioning the oriented line segments in one plane so that the end point of each of them is the start point of the line segment following in the given sequence *S* e will obtain a convex polygon, which is the Minkowski sum of the polygons *P* and *Q*.

Let us work with two simple simplexes – triangles determined by their vertices

$$A = \{ (1, 0), (0, 1), (0, -1)\}, B = \{ (0, 0), (1, 1), (1, -1)\}. \quad (3.1)$$

For the set A we receive sequence $S_1 = \{(1, 0) - (0, -1), (0, 1) - (1, 0), (0, -1) - (0, 1)\}$,
For the set B sequence $S_2 = \{(1, 1) - (1, -1), (0, 0) - (1, 1), (1, -1) - (0, 0)\}$.
Sequence, which is the union of sequences S_1 and S_2, and which is ordered with respect to the size of polar angles formed by the oriented segments is in the form

$$S=\{(1,0) - (0,-1), (1,1) - (1,-1), (0,1) - (1, 0), (0, 0) - (1,1), (0,-1) - (0, 1), (1,-1) - (0, 0)\}=$$

$$= \{(1, 1), (0, 2), (-1, 1), (-1, -1), (0,-2), (1,-1)\} \quad (3.2)$$

Minkowski sum of sets A and B is a set

$$A \oplus B = \{(1, 0), (2, 1), (2, -1), (0, 1), (1, 2), (1, 0), (0, -1), (1, 0), (1, -2)\}, \quad (3.3)$$

Having the shape of hexagon, see in the fig. 1.
From geometric point of view the Minkowski sum of triangles is such hexagon, which fills in the plane one of the triangles moving on the sides of the other triangle.

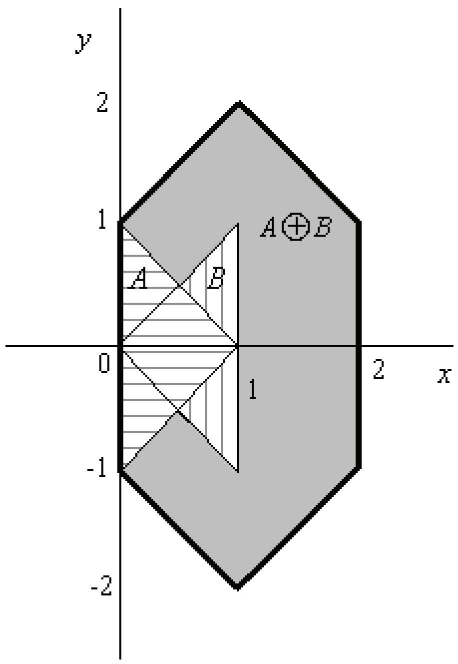


Fig. 1 Minkowski sum of two triangles

Minkowski sum of two line segments is a parallelogram or line segment, provided both added line segments, subsets of the space E^1, are parallel. Minkowski sum of three non-complanar line segments is a parallelepiped. Minkowski sum of n line segments in the n-dimensional space, which are in one plane (space hyperplane), is a figure called **zonotop**, or. **zonohedron**, which has, equally as parallelogram or parallelepiped, with all opposite sides parallel and congruent. Facets of zonotop are zonotops themselves, but of a lesser dimension.

As an example of a four dimensional zonotop can serve, e.g. **tesseract**, which is the Minkowski sum of four perpendicular and congruent line segments. Tesseract, also called as 8-cell or regular octachoron, is a four-dimensional analogue of a cube, which is also a three-dimensional analogue of a square. Tesseract can be also described as a regular convex 4-polytop, whose boundary consists of eight cells in the form of a cube. Generalisation of a cube into the dimension higher than 3 is generally described as "hypercube", "*n*-cube" or "regular polytop". Tesseract is therefore a four-dimensional hypercube or 4-cube, tetracube. According to Oxford English Dictionary, both names of *tesseract* appeared for the first time in the year 1888, when it was cited in the book *A New Era of Thought* by Charles Howard Hinton. He created it from the Greek words "τέσσερεις ακτίνες" ("four rays"), in correspondence to four edges meeting in each vertex of the figure. At any vertex of the tesseract there are meeting four cubes, six squares and four edges. It consists overall from 8 cells - cubes, 24 facets - squares, 32 edges and 16 vertices.

Example of a zonotop, which is a Minkowski sum of five line segments parallel to five lines passing through the origin of the co-ordinate system is a truncated 5-cell, which consists of five truncated eight-facets cells, as illustrated in fig. 2, in the middle.

Next from the presented zonotopes is a Minkowski sum of six line segments joining opposite end points of twelve permutations of vector (+1,−1,0,0), truncated 24-cell, as illustrated in the fig. 2, on the right, called also permutohedron.

Any permutohedron is a zonotop. **Permutohedron** of rank n is $(n-1)$-dimensional polytop in the n-dimensional space, with vertices originated as permutations of coordinates $(1, 2, 3,..., n)$.

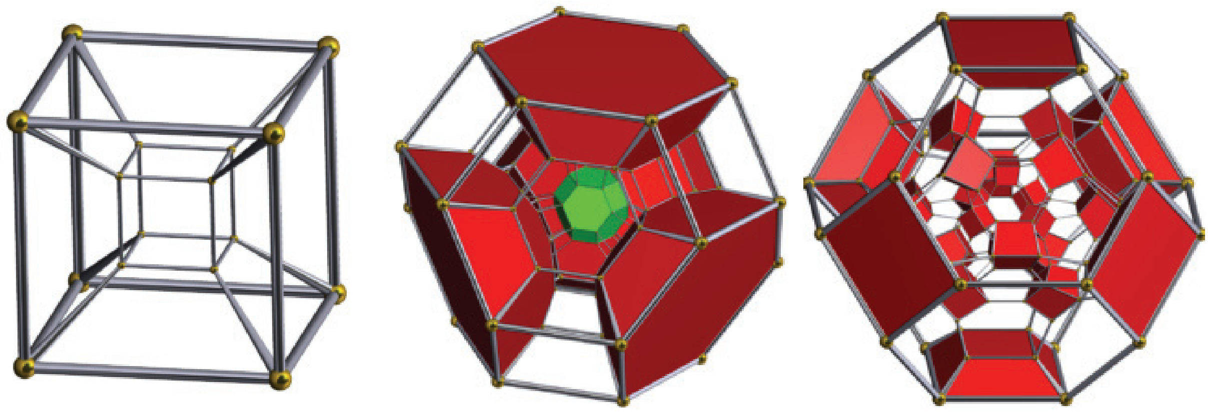


Fig. 2 Tesseract (on the left), truncated 5-cell (in the middle), truncated 24-cell (on the right).

4 Modelling of curve segments and surface patches

Let there are defined curve segments k and h in the space E^3 determined by the vector functions parametrized on the unit interval

$$\mathbf{r}(u) = (xr(u), yr(u), zr(u)), u \in \langle 0,1 \rangle \tag{4.1}$$

$$\mathbf{s}(v) = (xs(v), ys(v), zs(v)), v \in \langle 0,1 \rangle \tag{4.2}$$

Sum of curves k and h is a patch of translation surface, which can be created by translation of one curve alongside the other curve

$$k \oplus h = \chi \tag{4.3}$$

And is determined by vector function defined on the unit square in the form

$$\mathbf{p}(u,v) = \mathbf{r}(u) + \mathbf{s}(v) = (xr(u) + xs(v), zr(u) + zs(v), yr(u) + ys(v)), (u,v) \in \langle 0,1 \rangle^2 \tag{4.4}$$

Surface patch illustrated in the fig. 3 on the left serves as a simple example of the Minkowski sum of two sinusoidal curves, which is represented by equation

$$\mathbf{p}(u,v) = (2\pi u, 2\pi v, \sin 2\pi u + \sin 2\pi v), (u,v) \in \langle 0,1 \rangle^2 . \tag{4.5}$$

Surface patch on the right in the fig. 3 is created as the Minkowski sum of two exponential curves, and its vector equation can be written in the form

$$\mathbf{p}(u,v) = (au, bv, e^u + e^v), (u,v) \in \langle 0,1 \rangle^2 . \tag{4.6}$$

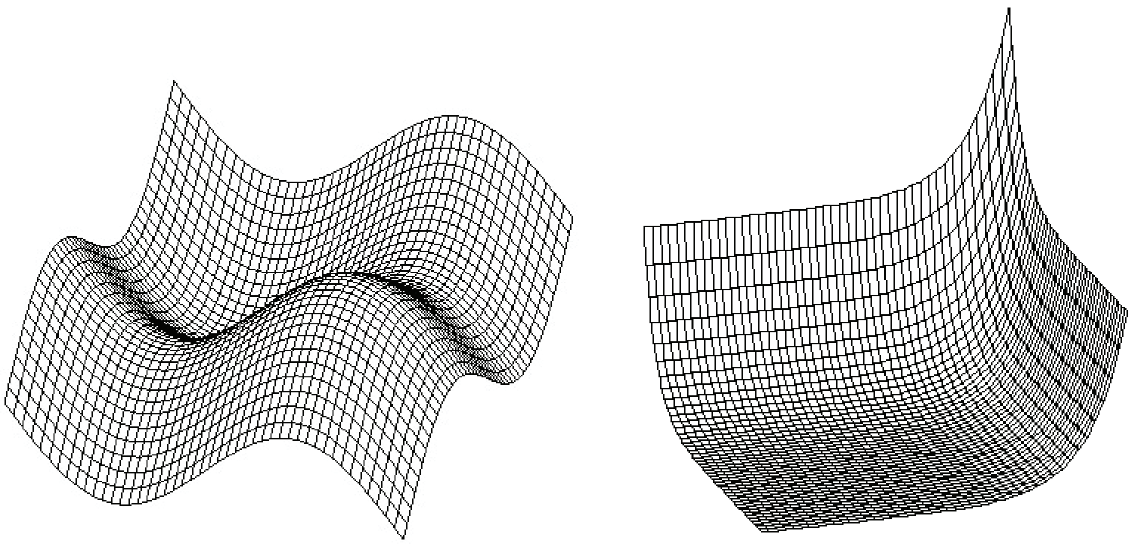


Fig. 3 Patch of a sinusoid-sinusoidal surface and exponential-exponential surface.

Minkowsi sum of an ellipse and one pitch of a cylindrical helix is an elliptic-cylindrical-helical surface presented in the fig. 4 on the left, which is determined by vector equation

$$\mathbf{p}(u,v) = (m + a\cos 2\pi u + r\cos 2\pi v, r\sin 2\pi v, n + b\sin 2\pi u + dv), (u,v) \in \langle 0,1 \rangle^2 , \tag{4.7}$$

where a and b are semi-axes and $S = [m, 0, n]$ is centre of ellipse located in the coordinate plane xz, and cylindrical helix is determined by the radius r and pitch d. Patch surface on the fig. 4 is defined by the same parameters but the ellipse is located in the coordinate plane xy with the centre in $S = [m, n, 0]$, and its vector equation is

$$\mathbf{p}(u,v) = (m + a\cos 2\pi u + r\cos 2\pi v, n + b\sin 2\pi u + r\sin 2\pi v, dv), (u,v) \in \langle 0,1 \rangle^2 \qquad (4.8)$$

Minkowski sum of an ellipse and one pitch of a conical helix is an elliptic-conical-helical surface presented in the fig. 5 on the left, which is determined by vector equation

$$\mathbf{p}(u,v) = (m + a\cos 2\pi u + r(1-v)\sin 2\pi v, n + b\sin 2\pi u + r(1-v)\cos 2\pi v, dv), (u,v) \in \langle 0,1 \rangle^2, \quad (4.9)$$

in which ellipse with semi-axes a and b and centre $S = [m, n, 0]$ is located in the coordinate plane xy and conical helix is determined by radius r and pitch v. Surface patch in the fig. 5 on the right is an elliptic-spherical-helical surface created form the same ellipse and spherical helix determined by radius r, with the vector equation

$$\mathbf{p}(u,v) = (m + a\cos 2\pi u + \sin \pi v \sin 2\pi v, n + b\sin 2\pi u - r\sin \pi v \cos 2\pi v, r\cos \pi v), (u,v) \in \langle 0,1 \rangle^2 \qquad (4.10)$$

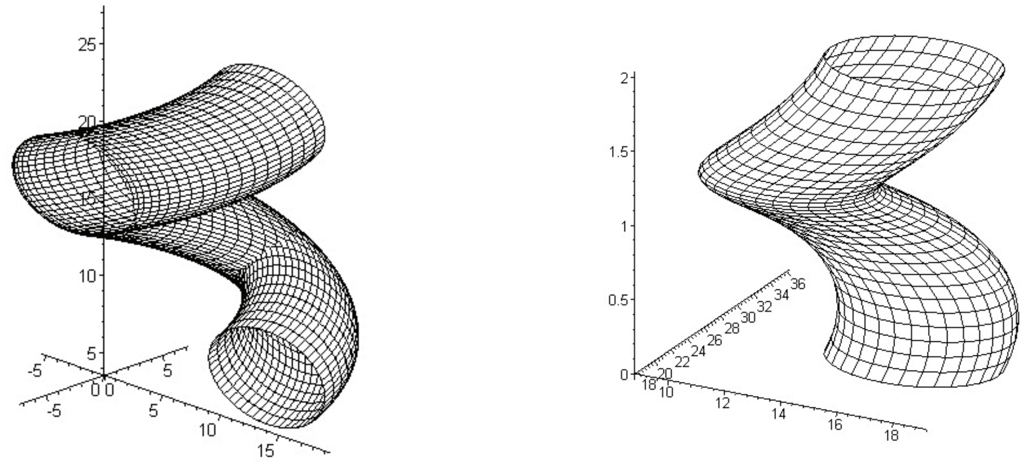


Fig. 4 Patches of elliptic-helical surfaces.

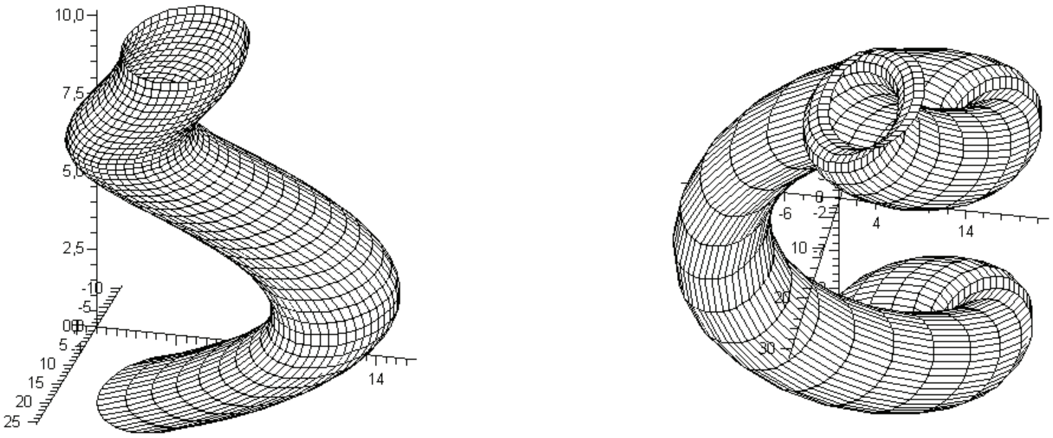


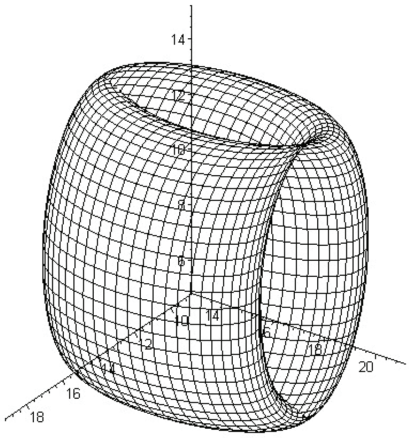
Fig. 5 Minkowski sum of ellipse and conical helix, and ellipse and spherical helix.

In fig.6 on the left a view of the patch of an elliptic-elliptic surface is presented, determined by vector equation

$$\mathbf{p}(u,v) = (m_1 + a_1 \cos 2\pi u + m_2 + a_2 \cos 2\pi v, n_2 + b_2 \sin 2\pi v, n_1 + b_1 \sin 2\pi v), (u,v) \in \langle 0,1 \rangle^2, \quad (4.11)$$

in which ellipses are determined similarly than in previous examples in the coordinate planes *xz* and *xy*, while on the right of the figure view of the patch of parabolic-parabolic surface is given, determined by 2 parabolas located also in coordinate planes *xz* and *xy*. Vector equation of this surface patch appears in the following form

$$\mathbf{p}(u,v) = (a_1 u + a_2 v, b_2 (v^2 - v), b_1 (u^2 - u)), (u,v) \in \langle 0,1 \rangle^2 \quad (4.12)$$



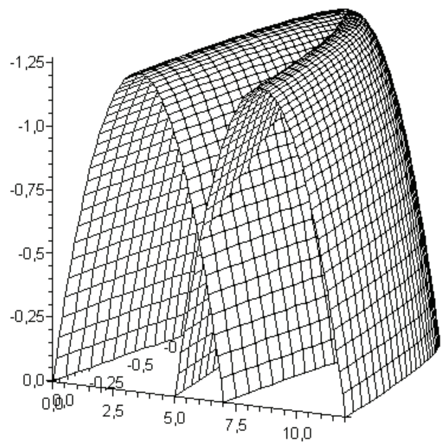


Fig. 6 Minkowski sum of two ellipses and two parabolas.

5 Applications

Minkowski sum plays an important role in many algorithms of 2D and 3D computer graphics; it is also the basis of the sweep modelling of surfaces and solids. Minkowski sum is also used in determining the trajectory of rigid motion of object in-between obstacles, for calculations of optimal trajectory and for configurations of the active working space of a robot, which is the set of reachable positions. In an easy model accepting only translational movement of object in the plane, when the position of the object can be determined uniquely with respect to a given reference point the working space is a Minkowski sum of the set of obstacles and the moving object.

In the computer aided control systems and technology solutions of material elaboration, the Numerical-Control machines programming is based on principles of Minkowski sum, where sum of the geometric model of the cutting tool and its trajectory, which is a planar curve, enables to determine the form of the cutting edge on material. Presented problems of determination of the equidistant to a given figure boundary form a specific field of computer graphics, the offsetting theory.

In figures 7 – 10, some more illustrations of surface patches that are modelled as Minkowski sums of 2 curves are presented. Sum of two planar curves located in one osculating plane is a planar figure, surface patch determined as a part of plane. In addition to

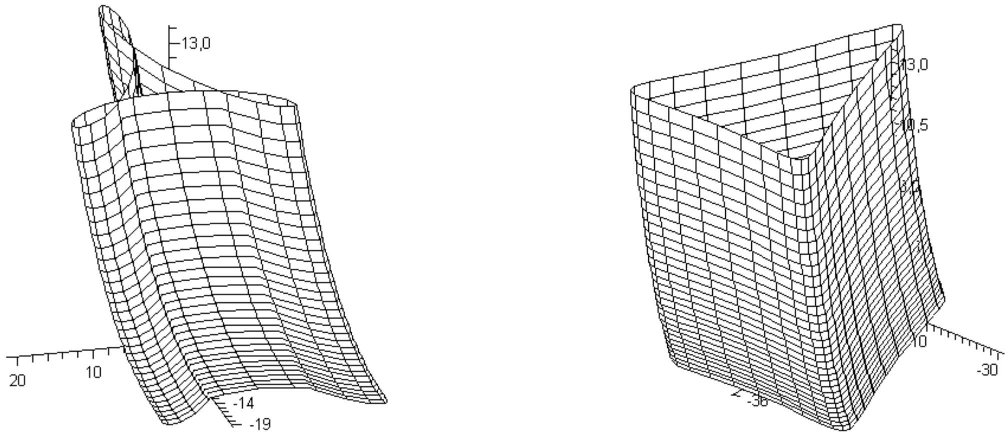


Fig. 7 Minkowski sum of parabola and Steiner hypocycloids.

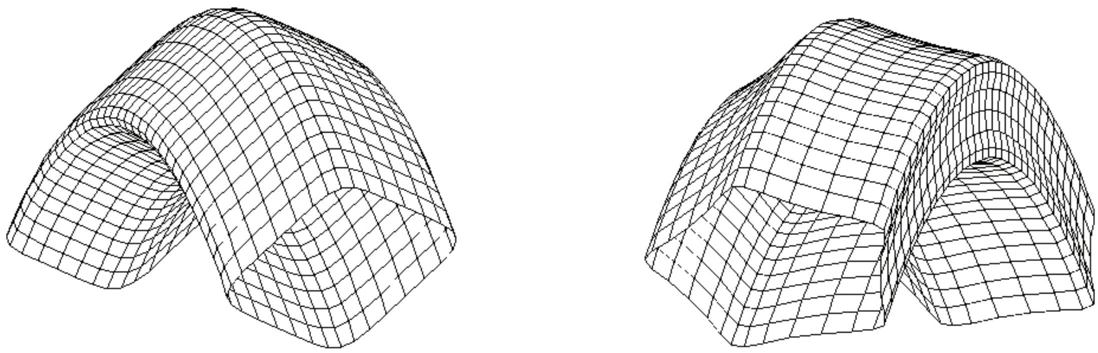


Fig. 8 Minkowski sum of parabola and epicycloidal curves.

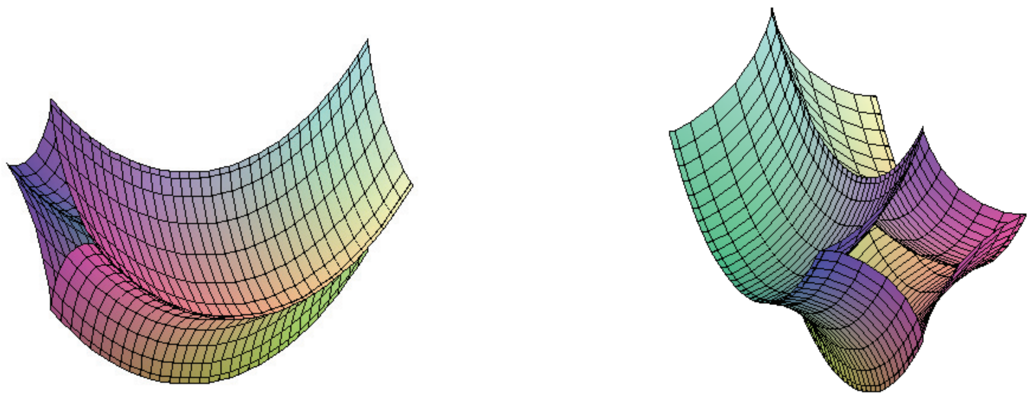


Fig. 9 Minkowski sum of chain curve and asteroid.

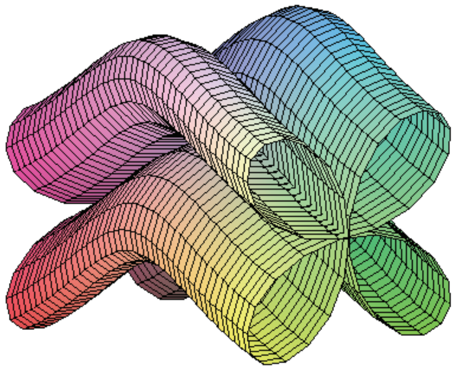


Fig. 10 Minkowski sum of versiére and shamrock curve.

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MATHEMATICAL MODELLING FOR DEVELOPMENT OF EGOCENTRIC VIRTUAL ENVIRONMENTS

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Abstract. The aim of this work is to introduce a novel solution for Virtual Reality - Cognitive Behaviour Therapy based on Virtual EgoCentric Holistic Environments. This is an alternative to the classical virtual environments (VEs) currently used and it is built on acquired user based information. The high-fidelity system is accompanied with several important attributes which will stimulate the human senses such as vision and hearing. An intensive study of these senses is leading us to answer an important question, what level of realism is required to provide effective immersive experience for the end users. We will demonstrate the potentials of the virtual egocentric holistic environments on three projects: virtual reality for the efficient treatment of infants with feeding difficulties, virtual reality therapeutic intervention for social anxiety, virtual reality for treatment of flying ants phobia. All these projects are multi-sensory and require visuals and audio to provide an appropriate level of realism.

Key words and phrases. Virtual reality, computer graphics, modeling, psychology, virtual environment, cognitive behavior therapy.

Mathematics Subject Classification. Primary 68N30; Secondary 62-07, 62K20

1 Introduction

Cognitive behavioral therapy is a structured, brief psychotherapy which is comprised of many techniques, based on research evidence, and recognized as a treatment of choice for many psychological conditions [30]. One of the CBT techniques is exposure. This approach draws from the classical conditioning work of Pavlov, and also operant conditioning [25], research demonstrating that anxiety can be associated with neutral or mildly anxiety provoking stimuli.

The fear is then maintained by avoidance of that stimulus or approaching the stimulus with safety seeking behaviors. The use of exposure therapy with specific phobias has been an evidence based treatment for a considerable time [23]. Exposure therapy relies on habituation of the fight or flight response when presented with the feared stimuli - this habituation can occur fairly quickly, but may take up to 40 minutes in some cases. In order for the therapy to work, treatment must be prolonged to habituation [27], practiced regularly [24], and be as realistic as possible [15]. The use of virtual reality techniques to help clients become familiar with feared stimuli is not new and a number of cases are described in [30], but it has rarely if at all been undertaken in the context of exposure protocols. Most of the virtual environments used in [30] are non-interactive with low resolution and artificial non-realistic sound. Multi-modal high-fidelity virtual environments attempt to recreate a real scene including many of the sensory inputs a user would experience in the real world. This has been termed "there reality", providing the perceptual response from the user as if he/she were actually "there" in the real environment [33]. There has been a significant amount of research into recreating real places using virtual reality, see for example [29], although much of this has concentrated on only one sensory stimulus, typically the visuals. Physically-based high-fidelity graphics model the way in which light interacts with objects in the real world including aspects of the human visual system to simulate the visual experience in the virtual world of an observer in the real world. They are thus capable of achieving for more realistic results then rasterisation based methods. Although many physically accurate algorithms have been proposed (for a good overview see [13]) these can take many seconds and even longer to authentically render only a modestly complex scene. Modern graphics hardware (GPUs), traditional parallel rendering, and visual perception techniques, such as selective rendering and Level of Detail methods [12], are playing an increasingly key role in significantly reducing these computational times, but still quality is often compromised in order to achieve interactive rates within the VEs. Sound is another key sense which has frequently being included in virtual environments, for example [16]. In addition to increasing the sense of "presence" in the virtual environment [20], recent work has shown that that the addition of audio could in fact enable the quality of rendering of high-fidelity graphics to be significantly reduced, with thus a substantial saving in computational time, without the viewer being aware of this quality reduction. Further senses which have been incorporated in VEs include haptics, for example [1] and motion, especially for flight and driving simulators [21].

2 Virtual Reality

Virtual Reality (VR) is a simulation in which computer graphics are used to create a realistic-looking world or imagined environment, in which people are the active participants. VR system has been used in treatments of patients with different disorders, i.e. social phobia, acrophobia, fear of public speaking, post traumatic stress disorder, flying phobia, spider phobia, treatment of eating disorders [2, 3, 4, 5, 6, 7, 10]. Situations which are created in a virtual environment need to be sufficiently similar to real world situations for successful patient exposure therapy. The significant advantage of VR is that it is more controlled and cost-effective and it allows therapists to create many and varied situations and environments for patients which are not life threatening for them. cite4 Therapists can also control how frightening individual objects and activities are. The virtual world thus provides a protected environment persuading the participants to be more willing to be treated by VR.

2.1 Characteristics of Virtual Reality

According to Sherman and Craig the defining features of virtual reality are: [9]

- It is a medium of communication
- It requires physical immersion
- It provides synthetic sensory stimulation
- It can mentally immerse the user
- It is interactive

Virtual reality as a medium offers interaction with models in three spatial dimensions and gives feedback from actions without noticeable pause. [10] These characteristics make virtual reality a useful tool for communication between therapist and participant. We have the ability to manipulate the sense of time and space and to change a degree of interactivity. The potential advantage for participants is that they can impact the narrative flow of the experience.

2.2 Virtual Reality in Cognitive Behavior Therapy

Virtual Reality-Enhanced Cognitive Behaviour Therapy offers many advantages over traditional treatment for an abundance of disorders. It can reduce the length of treatment, reduce relapse rates and is often more effective than traditional forms of therapy. VR therapy has been developed to overcome some of the difficulties inherent in the traditional treatment of phobias. It can provide stimuli for patients who have difficulty in imagining scenes or are too phobic to experience real situations. By providing a variety of stimuli such as 3D visual, binaural audio, vibratory, tactile, haptic and olfactory in an immersive and sometimes interactive manner, virtual reality enhances the therapeutic experience in a safe and controlled manner, especially in treatment that traditionally involves imagination. Virtual reality can be described as a technology or tool for influencing cognitive operations. The participant learns to consider different interpretations of a situation and he develops his own list of problem situations, which he discusses with the therapist and he makes the decision on how to proceed next. Cognitive behavior therapy (CBT) is an approach based on modifying distorted beliefs, attitudes and cognitive processes that maintain disordered behavior. [8] All of the existing VR systems in previous projects were focused on the individual human subject. Common indicators in these projects are that the patient is confronted with the feared stimuli and allows the anxiety to attenuate gradually. The most important thing to solve is the treatment of the individual's disorder. Not eating is a real problem whereas phobias are the products of anxiety. This problem needs to be solved by altering and learning it is not enough just to imagine some scene.

3 Virtual EgoCentric Holistic Environments

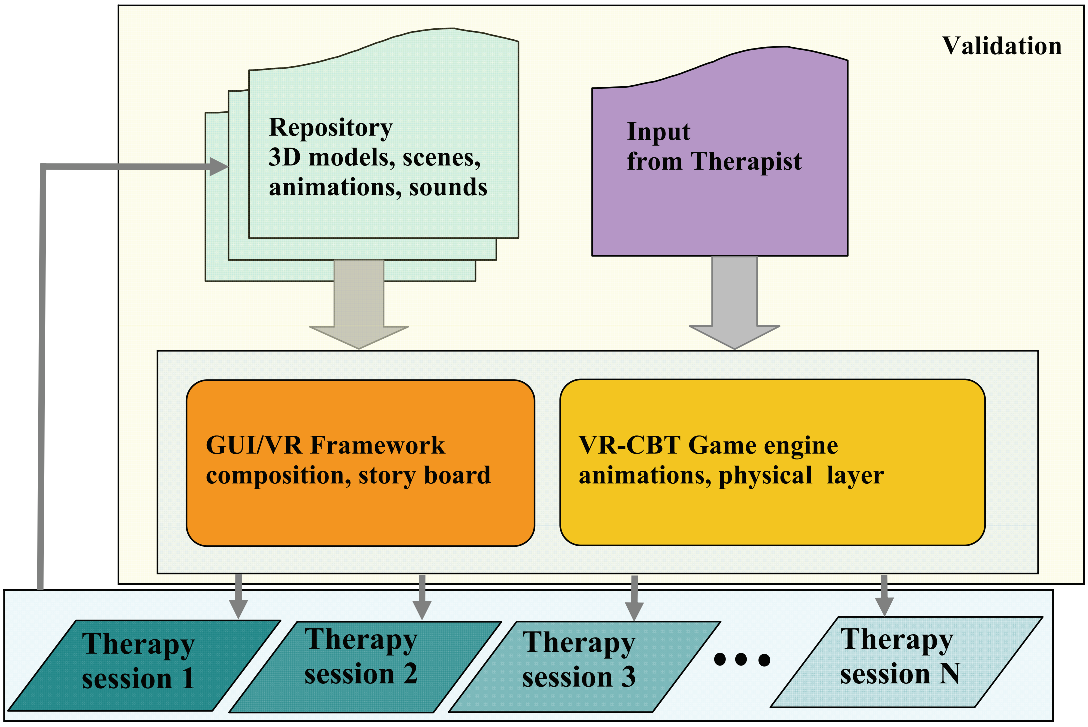

Figure 1: A development process of therapy sessions using VECeHe.

The Virtual EgoCentric Holistic Environments (VECeHE) is an alternative to the classical virtual environment used for CBT and will be built on various spectrum of information acquired by the therapist before the very first session; information that provides the framework for the therapist's understanding of a patient (see Figure 1). It has to include memorable information from childhood, because beginning in childhood people develop certain beliefs about themselves, other people and their words. Their most central beliefs are so fundamental and deep that they often do not articulate them, even to themselves. Another major aspect of the information package includes facts about patient's social life from the teenage years until present. It is important for the therapist to put himself in his patient's shoes, to develop empathy for what the patient is undergoing, to understand how the patient is feeling and to perceive the world through the patient's eyes. Cognitive therapy is based on the cognitive model, which hypothesizes that people's emotions and behaviors are influenced by their perception of events. Hypotheses are confirmed, disconfirmed or modified as new information is presented. The whole process is called conceptualization. During the conceptualization the therapist collects data and makes hypotheses about a patient, based on information the patient presents. After collecting all

necessary information and facts the therapist will then set up VECeHE and generate virtual environments suitable very precisely for that patient and treatment of his/her disorder. The VECeHE, with therapist's help, will teach the patient to identify, evaluate and modify his/her thoughts in order to produce relief of the symptom. The VECeHE is a combination of holistic and egocentric attributes with focus to particular needs for treatment of several cognitive behavior disorders. In psychology the term EgoCentricrism is defined as the characteristic of regarding oneself and one's own opinions or interests as most important. Our EgoCentric system is based on this definition with the focus to a patient and his/her needs. The system will be seamlessly adapted to patients needs in a controlled manner. The generated virtual environments will be based on the patient's health and psychological conditions and will be adequate to the phase of the treatment of his/her disorder.
The system is Holistic in the content of multimodal sources of perceptions. The system will be accompanied with several important attributes which will stimulate the human senses such as vision and hearing. All the different sources of stimulus will be then use together with the VR scenario to generate virtual environment with focus on particular disorder.

4 Virtual Reality for the Efficient Treatment of Infants with Feeding Difficulties

We are developing a VECeHE for individualized interactive therapy incorporating components of Cognitive Behaviour Therapy, which will be built on various information, such as the right environmental conditions for meal times, eating distractions, the right positioning during feeding, adaptive and social skills of the child, suggested by physicians and psychologists which will enable parent's successful experience with our application. It is necessary to make this environment easy to use for the participant and to let them think what to do in the current situation to build their self-confidence. People often develop certain beliefs about themselves, other people and their world, for example they believe that their baby doesn't eat enough or is not able to feed itself and they force-feed the baby. The basic hypotheses of cognitive therapy are that people's emotions and behaviors are influenced by their perception of events. It is important to teach the user to identify, evaluate and modify his/her own thoughts. We are developing a stand-alone application to conduct a pilot study and a series of experiments in conjunction with new parents to build the preliminary VR system (see Figure 2). All the experiments are discussed with clinical psychologists and pediatricians. We will compare how different levels of quality and multimodal aspects of a virtual reality system may influence the users. We will undertake initial comparisons for the VR solution, such as a solution with or without sound, different quality levels of animation, and several types of display techniques. On the basis of the pilot study we refine the VR system and fully investigate a novel, on-line Virtual Reality solution, which will be available for a wide range of users.

The main idea is to give the caregiver the ability to build his/her own scenario according to his/her problem. They should be able to choose a room to feed in for example, a kitchen or a living room, as a mealtime environment with the food and set up options relating to the child's problem. They should be able to simply select everything from the menu, so there would be no need to navigate around the environment, but they could also change the environment later by rearranging the furniture, adding some people and pets, and perhaps to be able to drag and

Figure 2: Cartoon avatar represents a 12 months old child.

drop some objects. We decided to have a child who would not yet be mobile and will not be breastfeeding. We consider that a significant object in the virtual world is a reclining adjustable chair with the help of a sensible pillow, because one of the prerequisites for successful feeding is appropriate positioning and body posture during feeding [32]. Many parents usually pay attention only to nutritional intake. They loose any fun and play during the feeding of their child, therefore the main focus of therapy is to make eating a pleasurable experience again [32]. Parents may not feel able to ask for help and may struggle on until they have a serious problem. Consequently, our project is aimed at prevention not treatment. We are aiming to help parents deal with the normal problems that many parents experience when feeding their children to prevent them from escalating into the sorts of problems that require clinical interventions. It is very important to give caregivers support to learn independently what to do in different situations and to attempt it with virtual babies and professional and skilled support, offered by intelligent decision support systems based on experts' knowledge.

5 Virtual Reality Therapeutic Intervention for Social Anxiety

A virtual reality exposure is based on the assumption that people feel "present" in the virtual environment [28]. The sense of presence is elicited by the user developing a mental representation of the virtual stimuli as one's own environment. Thus, it can occur when people feel deeply involved in the simulation then they can experience a variety of emotions [34]. Based on this fundamental phenomenon, we are developing a virtual therapeutic intervention for social anxiety with high-fidelity 3D graphics so as to increase the sense of presence. The aim of our initial study is to create a fear provoking virtual job interview that might elicit anxiety as a real job interview. The job interview situation is very simple to reconstruct for comparison in the real world, but it is an exceptional situation to have repeated exposure without limitations. We therefore designed a virtual office that was consisted of desks, chairs, a bookcase, a white board, and a female interviewer sitting on the chair (see Figure 3). In order to simulate natural attitudes of virtual human (VH) in real-time, the human models was scripted by Maya Embedded Language (MEL) to express positive, neutral, negative reactions, For instance, deadpanning, smiling, nodding, head-shaking, yawning, turning away, avoiding eye contact, interlocking fingers and interesting looking, folding arms and disinteresting looking. In addition, real interviewer's voices were implemented to the virtual human for delivering 10 job interview questions and a number of short sentences such as Thank you, Yes, I heard enough, Ok I am goanna go to the next question, and Yes good! Tell me more please. Here we want to outline

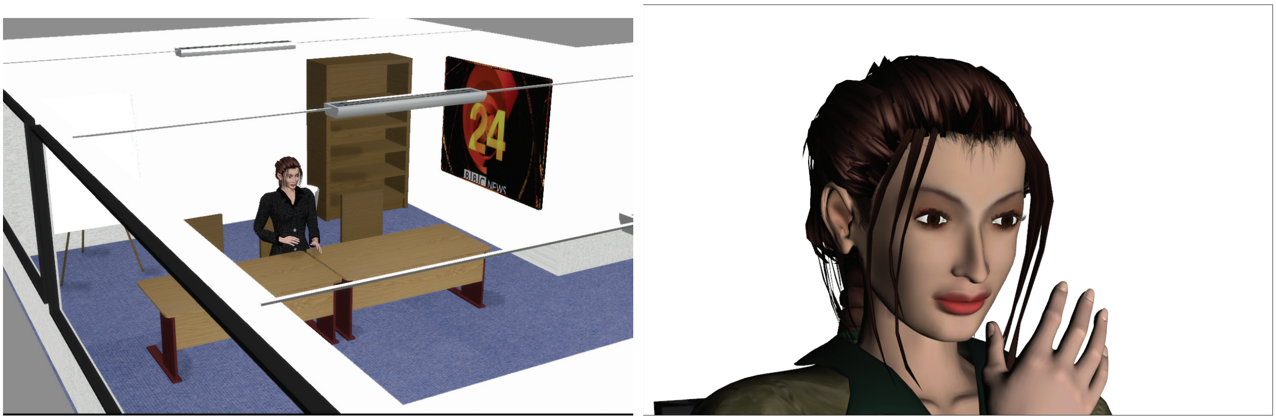

Figure 3: Virtual job interview scene. (left) a virtual office scene (right) closed up VH's face

our result of pilot study. Our pilot study was conducted with four volunteers (2 male and 2 female) who have not any history of mental illness. We measured subjects' gaze avoidances with a face tracking device (FacelabTM), and also asked several questions to obtain their feeling about the virtual job interview. When participants started being exposed to the virtual job interview, the VH interviewer performed in response neutrally with smiling or nodding from first to third job question. We also controlled the VH interviewer to simulate negative reactions (e.g. head-shaking, yawning, and turning head way) with a heavy sigh during the fourth to sixth question, and positive reactions (e.g. big smiling, frequently nodding, and interest looking) with expression of interesting chat "Yes Good! That sounds interesting" from seventh to last question. All participants successfully completed the virtual job interview exposure study without any difficulties, and demonstrated very obvious results on it. Firstly, we compared the subject's gaze movements depending on VH interviewer's three different reactions: negative, neutral and positive. The result showed similar tendencies for each subject that presented less visual contact avoidance at positive reactions, and more avoidance at negative reactions. Due to the limited sample size we could not conclude the difference with statistical significant, however, it were able to see that the gaze tracker seemed to be a potential measurement to identify subject's avoidance behaviours. The animation of VH model could be an essential factor to elicit user's anxiety during the virtual reality exposure session. Moreover, we could obtain very interesting comments from the subjects. Most subjects said that they did not expect too much about the virtual interview, however, after entering the virtual interview they were able to feel pressure or tense as much as real job interview. Especially, three subjects said they wanted to stop the interview when the VH interviewer yawned or turned the head away, and these kinds of reactions made them to be nervous or irritated. Overall from this pilot study, we could see the potentiality of virtual reality to use as a therapeutic tool for social anxiety care. Clearly we have to do more experimental study with large number of subjects to verify the relationship between the sense of fear and meaningful set of stimulus in other virtual conditions.

6 Virtual Reality for Treatment of Flying Ants Phobia

Insect phobias are very common it is believed that at least one person in 10 is affected at some time in their life. A severe phobia about them can be as disabling as any anxiety disorder. In some cases they become almost prisoners in their own homes for fear of common insects that the majority of people literally never notice [35]. Most people are at least wary, if not

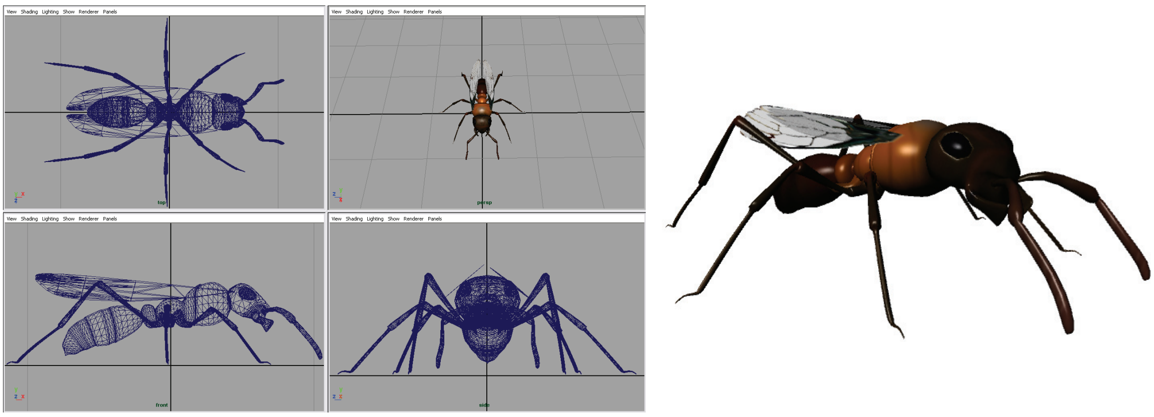

Figure 4: 3D model of a flying ant.

fearful, of certain insects (more correctly arthropods). This may be a reasonable fear based on knowledge or experience (bees, wasps, spiders, mosquitoes), an unreasonable but culturally understandable repulsion (cockroaches or flies), or a misplaced fear resulting from inadequate information (dragonflies, moths, crickets). A true insect phobia, on the other hand, is defined by the following criteria [35]:

1. A persistent irrational fear of and compelling desire to avoid insects, mites, spiders, or similar phobic objects.

2. Significant distress from the disturbance despite recognition by the individual that the fear is excessive or unreasonable.

3. Not due to another mental disorder such as schizophrenia or obsessive compulsive disorder (modified from [36]).

Traditional specialist treatment will be as for other phobias, and is largely determined by the therapist's individual preferences. Methods reported in the literature include supportive psychotherapy, desensitization [37], insight psychotherapy, combination of therapies (possibly including group therapy), drug therapy (anxiolytics), modeling [38], hypnotic regression and reframing [39], and implosive therapy [31].

Our approach for treatment of flying ants phobia combines the traditional approaches with novel VR technologies. We are using immersive VECeHE and head mounted display to maximize the level of presence. To create the VECeHE a Spheron 3D camera has been used to capture the real living environment (see Figure 5). A model of wild flying ant and its animations have been created and combined with VECeHE (see Figure 4). As the last step, the strategy of the treatment has been developed and consulted with cognitive behavior therapist.

Figure 5: Virtual environment for treatment of flying ants phobia (left) panoramic image of a kitchen captured by spheron 3D (right) part of a virtual kitchen used during the therapy session

7 Conclusion

An interesting question for every creator of virtual environments is how important the role of realism in immersion perception is. What kind of input and output devices do we need for good immersion? What are the key elements of virtual reality? In order to increase the reality in VR environments, several studies combined VR immersive devices to provide a better quality of VR environments. Although VR immersive systems can create different levels of presence, the levels may not directly lead to an enhanced treatment outcome. It is important to determine whether we need extremely realistic environments or we can also add some elements like having a cartoon-like look to the environment. Therefore we create both low and high quality models of babies, avatars, furniture and the rest of the objects in the virtual world. Another question is if full mental immersion is necessarily required for the application to be useful. Most current VR systems are primarily visual experiences, displayed either on a computer screen or through special stereoscopic displays, but some simulations include additional sensory information, for example, sound through speakers or headphones. A sense of how the world looks and sounds is very useful to improve the participant's immersion, so they should be chosen carefully.

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ART & MATHEMATICS – THE WEB-BASED PROJECT "SCIENAR"

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Abstract. Mathematics is the universal language of human abstract thought and it subtly pervades all forms of Art. Art and Mathematics have evolved in parallel, alongwith changes in our ways of conceiving, perceiving, experimenting and representing "reality". One can divide this evolution into four steps. The first refers to Classical Art; a second to the development of Perspective; a third one to non-Euclidean Geometry; a fourth to "Synthetic Geometry" and the deconstruction of rigid forms, with the development of fractals and topology alongwith modern and avant-garde forms of Art. In the era of Digital Art a skillful use of Web and Computer Technologies, alongwith Multimediality, offers new challenging ways to introduce Mathematics starting from a careful reading of artworks. We have therefore elaborated, proposed and begun experimenting an innovative "teaching and visualization project" mainly based on advanced techniques at the intersection of Mathematics, Computer Science and Digital Art. The project – called "SCIENAR" - is based the innovative possibilities that new media offer for a better Communication, Visualization, Divulgation and Didactics of Mathematics. It aims to creating three scientific scenarios concerning the theory of Relativity, the notion of Symmetries and the foundations of Geometry; starting from these it will be possible to produce artistic objects by means of a collaborative work between researchers, scientists and artists. "SCIENAR" will generate a fully innovative portal, whereby artists will find ready-to-use mathematical forms, including the more traditional Web Portal "MArs". Next steps will be aimed at fostering, stimulating and supporting the physical and the virtual mobility of artists and cultural players interested to use mathematical models for the production of artistic objects, improving links among single artists, cultural operators, scientists and experts in digital technologies.

Key words. Didactics of Mathematics, Art & Science, Visualization, Digital Technologies

Mathematics Subject Classification: 00A35, 00A99

1 Communication and Visualization of Theoretical Science in Our Century

As we have already started to discuss in 2006 in a recent paper of ours [1] (presented at the CMDE2006 Conference of Aveiro devoted to the "Communication of Mathematics in the Digital Era" and published in 2008) the task of Divulgating, Communicating and Visualizing the fundamental achievements of Mathematics (as well as of all other theoretical aspects of Science) is by no means a simple one. Mathematics is the language of all possibile forms of abstract human thought and - even if it pervades most of the products of human experience - still the majority of mankind considers Mathematics as a sort of “Science for Initiates”. A form of human thought eventually perceived by the “collective imagination” as a rather tedious discipline. Some relatively “modern” mathematical schools have in fact privileged abstraction rather than intuition in mathematical learning, further contributing to this widespread misunderstanding. Fortunately, new tendencies tend to revitalize Mathematics without renouncing to rigour but at the same time enhancing its more aesthetic and directly intuitive aspects (see also [2]). Teaching, divulgating, communicating and visualizing Mathematics is definitely a fascinating challenge, not only a stimulating exercise for “experts” but rather a necessity dictated by the new trends of our Society. As an example, we quoted in [1] the “Web Portal for Research” recently launched in Italy (www.ricercaitaliana.it) [3] with the explicit aim of reviewing some of the most intriguing national research initiatives. One of the “specials” hosted by this Web Portal refers to the Communication of Science [3a]: *“Scientists, nowadays more than before, are facing new challenges in the field of Communication. […] Communicating Science has nowadays become an absolute necessity […] The European Union considers the <<Dialog between Science and Society>> one of the fundamental steps towards the construction of a European space for research. […] The themes related with Communication of Science have therefore to become an essential part in the formation of young reserchers […].* And in another part [3b] of it one reads: *“the “Communicator of Science”, an intellectual character who might be still unexisting. However the new Society of Knowledge needs him more and more to tell herself her own fresh origin and her fast becoming. […] Communicating Science is becoming a more and more diffuse social need. […] We are therefore facing a two-fold necessity. Independent but convergent. Scientists have to communicate with non-experts, to share with them decisions that have a great impact on their own work. […] The world of Science is a complex system with several dimensions. […] The system of Mass Communication is evolving and articulating in a such a way not to require and privilege rigid specializations, but rather requiring and privileging flexibility. […] These consequences define the unalienable characters of the Dwarf who is obliged to climb the shoulders of Giants to tell the story of the Society of Knowledge: an intellectual figure who is not lost in the complexity of Science, of Society and of their growing compenetration; someone who, at the same time, is able to <<govern media>>, i.e. to master the fastly evolving and diversifying means of Mass Communication”.*

Mathematics (as well as all other theoretical scientific disciplines) is facing, at the turn of the Third Millennium, the two-fold need of renewing its basic teaching and of promoting a “task force” of specialists in Communication in its interior; central figures who, on the basis of a sufficiently broad and deep knowledge of the main fields of the discipline, as well as of its most promising further lines of development, can help in promoting the dissemination of its specific mathematical culture with the aid of rigorously constructed (as well as artistically valid) multimedial presentations that can attract the audience towards a deeper immersion into the discipline itself. A lot can be achieved in these directions by relying on a skillful use of all the new tools that the technologies of the Digital Era offer to our ways of teaching and communicating Mathematics. The potential offered by modern tools - as e.g. Digital Art and Web Technologies - is extremely well suited for the task of producing objects aimed at achieving a new and more modern form of Scientific Communication and Visualization in which one can conjugate high scientific quality and rigorous technical frames together with enjoyable realizations enriched by easily visualizable concepts and special effects. Thus appropriately softening all the difficulties that are related with the transmission of an exact scientific message that (as it often happens for extremely formalized theories as Mathematics is) might be abstract and rather far from common experience.

Recent experiences of ours refer to the field of Mathematical Physics, in which an especially important example is provided by Einstein's Theory of Special Relativity; in this specific domain we have realized a short video and multimedia aimed at divulgating and visualizing that fundamental and revolutionary Theory in an easy and short way [4]. The experience was in fact realized for the "Pirelli Relativity Challenge" [5] stimulated in fact by the World Year of Physics (WYP; see [6]), proclaimed by UNESCO in 2005 to celebrate those papers by Albert Einstein that in the "Annus Mirabilis" 1905 drastically revolutioned both the Physics and the Mathematics of XX Century. Visualizing the most recent discoveries of Physics is in fact fascinating and difficult at the same time. Difficult, since these Theories are often far from standard experience: Relativity, for example, has provided a deep conceptual revolution that eradicated the classical immutability of "Space" and "Time" as separate entities, replacing them with "Spacetime" (Space and Time loose an absolute meaning of their own to acquire a "relative" meaning which depends on observations and measures). Nevertheless, this Divulgation is also a fascinating challenge, since each one of us would like to tell and explain in simple words the fundamental story of the spacetime structure. We refer the reader to [4] and [7] (and references quoted therein) for further details.

Here we want instead to discuss in greater detail the impact that the new technologies of our era may have in related projects involving an innovative approach to Mathematics through the "main door" of Art. We strongly believe that new ways of making Scientific Divulgation and Communication - as well as approaching new didactical pathways based on the use of modern tools; as, e.g., discussed in [7] - should in fact profitably use the new media for the purposes mentioned, provided the "book paradigm" is suitably reconsidered and replaced by a suitably adapted "laboratory paradigm" (see also the preliminary discussion in [1]). Already existing "multimedia books" contain in fact images, text and videos, although they are usually structured as "unidirectional tales". In a laboratory, on the contrary, one forecasts and investigates "knowledge", allowing one (or more) mental models to be confronted with reality. Science is born out of this relationship. In a sense Science *is* this relationship. As already mentioned in [1], in our opinion a dialectic relationship between reality and the investigator is a founding feature of Divulgation, that has to guide the experience and allow the acquisition of mental models of reality, to be later tested in order to survive or to be refused. Scientific Divulgation is therefore something in-between a technical treatise and laboratory research, since notions that are acritically assumed cannot communicate the true essence of scientific experience (see also [8]).

Following indirectly a new pathway that was clearly traced by the Italian Mathematical Union (UMI) in [9] - a book devoted to "structurized teaching of Mathematics for external users" - we have recently launched a full innovative teaching project based on a consistent and skillful use of the interrelationships between "Mathematics & Art", that was already announced in several occasions (see [1], [8] and [10]). UMI declares in fact the following in [9]: *"Mathematics is universally recognized as a powerful language to describe the World, to construct models, to calculate and forecast. Because of this it is considered to be a very useful tool, often indispensabile, for a lot of disciplines. […] Also in various disciplines pertaining to Human Sciences, as e.g. the Science of Cultural Heritage and Archaeology. […] The teaching and understanding of Mathematics is therefore a relevant cultural and didactical problem, that each year concerns thousands of University Teachers and hundreds of thousands of Students […] To reach the type of knowledge indicated it is not necessary to insist too much on formalism, since it often obscures the meaning of mathematical objects and puts a brake on the development of modelling abilities. The specific competences […] are better exploited through […] the following abilities: * reading and interpretation of texts; * writing and, more generally, Communication; * organization, storage and retrieving of knowledge […] also by means of informatic devices"*; ([9], page xvi). The far-reaching name given to this extremely important piece of "mathematical teaching philosophy" is "*mattoncini*" (in Italian literally "*little bricks*"), meaning that any coherent teaching project should consist of sets of small basic pieces to be combined together according to specific needs.

We pointed exactly in this direction by means of a teaching experience realized in the Academic Years 2003-2004 and 2005-2006 at the University of Calabria, in the framework of a Ph.D Course in "Psychology of Programming and Artificial Intelligence" [11]. In full consonance with "mattoncini" a new course in "Mathematics and Art" has been successfully tested on a variegate audience, including artists. A written and rather innovative text [12] was produced and it will appear in electronic format on the Web (see also [13] and [14] for earlier reviews; [10],[15] and [16] for developments and didactical proposals; see also [7] for didactical developments based on the use of virtual agents). The didactical path we followed starts from Euclidean Geometry and ends up with Riemannian Geometry, peeling off the structures that enter our understanding of the Euclidean and sensibile World. The starting points are of course the definition of Geometry as "art of measurement" and the deductive methods of Euclidean Geometry (that can be treated in a critical and a-dogmatic manner, in line with the logical deductive approach followed by the ancient Greeks, that still to a great extent can be used today to describe, analyse and understand extended objects in space). Synthetic Geometry can be consequently introduced at different levels of abstraction and understanding (Set Theory, abstract and structured Spaces, Morphisms and Transformations between spaces, Topology, Erlangen's programme and Klein's work, in which Geometry is defined as the study of the invariant properties with respect to a characterizing Group of Transformations). Symmetry plays of course a dominant role in understanding Mathematics through Art. Other topics which can be conveniently extracted out of Art are: Curves and Surfaces; Euclidean Optics [17]; Projective Spaces; Perspective; Design; Tassellation of Surfaces (Picasso and Escher's work - see e.g. [18]); Fractals; Knots; Crystallographic Groups [19]; Curvature and non-Euclidean (i.e., Riemannian) Mathematics and Art ([18] and [20]).

2 A New Didactical Challenge: Mathematics & Art seen in their Parallel Development

All what we said up to now refers to the broader field of interest that can be collectively called the context of "Applications of Mathematics to Cultural Industry" [21]. As we shall shortly discuss below in this Section, Mathematics and the Art of Painting have developed along parallel tracks (see, e.g., [15] and [22]). A pathway followed also in all other forms of Visual Art (e.g. in Architecture; [23],[24]) as well as in the evolution of Music. In the present days we see Mathematics to play a renewed role in all forms of Art (visual, plastic and musical – see, e.g., [25], [26]), so that "essentially mathematical methods" are often used to generate Art and Music by means of computers and electronic devices ([27] and refs. quoted therein). As we already said in [1], Mathematics is not only an essential tool for Science and Technology, but also for Humanities, in particular for Art. And out of Art we may say that Mathematics gains one of its main reasons of developing and changing in time. Mathematics contributes to our way of conceiving and shaping the World we live in, while Art develops the means to harmonise, describe, represent aesthetically - or even to transcend and transfigure - the World of our sensations and perception.

The interplay of Mathematics with "Cultural Industry" [21] is therefore rather wide and it is hard to enumerate the many existing initiatives in this fascinating field, where conferences and/or installations at the crossroad between Mathematics, "Cultural Heritage" and Modern or Traditional Art are common (see, e.g., [28-36]). But this is not enough. We claim that a new way of teaching, communicating and visualizing Mathematics can and should be envisaged. A way which deeply requires the understanding of all the mentioned profound intertwining between Mathematics and Art. A way that rejects dogmatic ways of teaching Mathematics and only occasionally recognizing its relations with Art. We strongly suggest to follow a kind of "reverse path", i.e. a path in which Art is the central theme out of which the existence of mathematical structures is recognized, first as a part of the piece of Art and later grasped at a deeper level; Art should in a sense be considered as a mean to extract and understand structures, symmetries and broken symmetries out of their more or less evident appearance within the structure, symmetry or apparently broken symmetry of each single piece of Art (see, e.g., [37],[38]).

In order to obtain such a different and completely new perspective on Mathematics a clever use can and should be done of all the modern tools that new technologies provide us to represent, communicate and visualize Science: Digital Technologies, Digital Art, Computer Graphics, Computer Vision, Virtual and Augmented Reality, Multimedia and Web Technology, and so on. The explicit aim is to revitalize the understanding of the central role that Mathematics plays in everyday's life, starting not from far reaching and astonishing scientific results but from the emotional and aesthetic side of our consciousness and perception. Not only to contribute to a better spreading of Mathematics among non-specialists in the field, but also to promote a broader attention of large portions of students to the beautifulness of Mathematics and even to help specialists to create new pathways for introducing deep mathematical concepts in an easier, more intuitive, more sensible and more palatable way ([1] and [8]).

Mathematics is in fact less and less known by young people and one of the major problems seems to be the low appeal that Mathematics exerts on new generations. Against a Society that privileges images and motion, the teaching standard of Mathematics seems to be still too much attached to traditional schemes that have a scarce appeal on students. Because of this, we believe that new tools more broadly based on sound and images, as well as multimediality and new communication tools might help a lot to invert such a negative tendency. Exactly in this sense Art – in its multi-layered aspects - can play a central role in helping Mathematics to find new pathways. The idea we are following and its elementary steps have been presented in [1],[8],[10],[15],[16].

As far as the historical perspective is concerned, let us shortly recall from our previous papers of ours the following. As is well known, Mathematics has developed in a parallel and sometimes precursor way not only alongwith the scientifical thought but also alongwith our way of perceiving, describing and representing the sensibile world by means of Arts. Our cultural history shows in fact the closeness of the links between Mathematics – a means for discovering and describing reality – and Art, which aims to express or represent reality ([13],[14],[22]). The transition from Greek's Euclidean Geometry to the Geometry of Perspective in Renaissance, to non-Euclidean Geometry in the XVIII and XIX Centuries, up to the development of Geometry of "topological forms" in the XX Century, can and should be seen as a counterpart to the static paradigms of Arts and Architecture in the antiquities, to the conception of "beautiful painting" of Pier della Francesca, to the evolution of artistic shapes in the eighteenth Century (e.g., Divisionism, Expressionism, Impressionism) up to the complete destruction of symmetry in the modern, contemporary and avant-garde forms of Art (Cubism, Fractal Painting and so on – see, e.g., [18],[39]). The theory of proportions was at the basis of Greek Science and Geometry; in Renaissance the artist was a complete man: painter, sculptor, architect, mathematician and also scientist (we may quote Piero della Francesca, Dürer, Brunelleschi and Alberti). The idea of an exactly Euclidean world and the need for painters to represent faithfully the three-dimensional world in just two dimensions eventually gave birth to Projective Geometry while discussions about the validity of Euclid's V postulate pointed to develop Hyperbolic Geometry that, in the artistic field, paved the way to Impressionism. Artists in the

XVIII Century begun to represent what the eye actually sees rather than what the eye is pretended to see in a fully Euclidean world; the XIX and XX Century have finally seen the introduction of time as a fourth "sensibile" dimension alongside with height, lenght and depth (think of Einstein's Theory of General Relativity): motion and curvature become part of the World and not something which is embedded into the World. Objects are therefore endowed with a dynamics that is impossibile to represent in just two or three dimensions (even if artists like Balla, Boccioni and Duchamp tried to provide pictorial representations of movement; but think also of Picasso or Dalì, who designed a four dimensional hypercube that opens up in three-dimensional space). The new Mathematics of the XX Century is – finally - the Mathematics of Curvature, Discreteness, Fractals and Chaos, and modern ways of making Art reflect these new ideas (think of Picasso, Pollock and so on). We cannot finish this short discussion without mentioning the work of Escher, unchallenged inventor of impossibile objects and imaginary worlds, who was strongly influenced by the mathematical theories of Poincarè in creating his striking pieces of Art. The connection existing between Art and Mathematics is therefore universally recognized and needs no more examples. Several papers on the above subjects can be found in the collectanea [25], [28], [36].

3 The "Conventional" Web Portal "MArs"

The Ph.D Course and the Book [12] mentioned at the end of Section 1 (based on the historical parallels described in Section 2) are precious tools for deepening the basic understanding of Mathematics along the lines we have indicated. However a much broader project has to be developed, by changing completely the viewpoint and using Art as a first theme to develop a modern Communication, Visualization, Divulgation and Teaching of Mathematics, at all possibile levels and for all possibile applications. We have therefore projected a "conventional" Web Portal "MArs" dedicated to "Art & Mathematics", that is currently under development and implementation ([40],[41]) and will form part of the much broader project "SCIENAR" that we shall shortly discuss in the next Section. The leading idea is to allow the progressive understanding, at progressive levels of deeper and deeper mathematical reasoning and abstraction, in which Art comes first and touches the emotions, thus stimulating the need and the desire to penetrate more intimately into the structures which underly Art itself often without revealing themselves in an explicit way ([1] and [10]).

The Web Portal "MArs" is a central object in this project and it aims to present Art as a way to approach Mathematics, to enjoy the beautifulness of the structures existing in our vision and representation of the World and eventually to reconstruct the structures and theoretical tools that are necessary to understand and elaborate their true essence, as well as to show or stimulate interdisciplinary further applications to other fields in which Mathematics plays a key role. The Portal aims and intends to propose artistic objects and use them to discover their (usually hidden but sometimes evident) mathematical richness, by proposing also a reasoned selection of links to other (usually differently oriented but extremely important) websites (such as, e.g., the WebSite [42]) and also information about initiatives in the field of Art & Mathematics (Conferences, Exhibitions, Courses, and so on). The Portal intends also to exhibit a collection and a review of multimedia explicitly produced for the Visualization of Mathematics (and, more generally, of Science) as well as in the appropriate use of web-technologies aimed at implementing the interactions between Mathematics and Art.

In the future, especially through the related project "SCIENAR" shortly described in the next Section of this paper, the whole project will envisage an Electronic NewsLetter (in which both innovative and more traditional ways of communicating Mathematics through Art will find an appropriate forum), where Visual and Digital Art, Music, Architecture, Biology, Visualization, e-Learning – and other fields worth of being explored – will be seen in their interaction. The Portal will finally contain a less conventional space dedicated to "work in progress", such as: presentation of multimedia and/or innovative ideas; simulations and generative approaches to "Mathematical Art"; digital technologies to produce Art through Mathematics or to understand Mathematics through Art; new frontiers in Mathematics stimulated by Digital Art and Artificial Life, Virtual and Augmented Reality.

As we announced in [1], the final task of our project will be four-folded. It will: 1) broadly cover a wide range of interests; 2) open up a real dialectic between Mathematics and Humanities; 3) stimulate interdisciplinary attention to new and innovative products that exulate the traditional competence of single frameworks and, as such, cannot be easily classified under the traditional disciplinary classifications; 4) give adequate space to new tendencies and new problems generated by web-technology. This four-folded task will be reached by a four-folded set of explicit aims: A) to give a preferential attention to the most stimulating, innovative and interdisciplinary products (larger space to critical discussions about the infinitely many relations between Art, Architecture, Mathematics and Humanities); B) to create an interactive dialogue between different "poles", that is becoming of strict actuality and is made possibile by web-technologies and Digital Art; C) to strongly re-propose new ways of interaction between Art, Science and Technology, on the basis of the key role that "open minded" mathematicians can exert alongwith the role of "mathematically inclined" artists; D) to try to eventually re-create the fruitful "constructive and internal interaction" that existed between Mathematics and Art in Renaissance or in the Dutch painting of XVII Century; a liaison between the two disciplines that, as we said in [1], has been interrupted and replaced by "external" interactions, only at a marginal, secondary and less promising level of efficiency.

The innovative role of "MArs" will be enhanced by its centrality in the scientific project "SCIENAR" that shall be shortly presented and discussed in Section 4 below.

4 The European Project "SCIENAR" on Mathematics & Art

As we already mentioned above, the new approaches aimed at introducing Mathematics through Art are not completely finding their way to the "large public" and still the understanding that Mathematics and Art share a subtle interplay remains in the hands of a relatively small number of researchers. The purpose of the International Project that we have envisaged and we have called "SCIENAR" (achronymous for "*SCIEN*tific *SCIEN*aries in *AR*t") is therefore to make all aspects of this new approach reach both the artistic and mathematical communities, as well as the "global society", introducing this common understanding into a multicultural and international context. In order to achieve this objective the project intends to create three scientific scenarios, based on mathematical models, explicitly concerning: a) the Theory of Relativity; b) the role of Symmetries; c) the bases of Geometry. Starting from these scenarios it will be possible to produce artistic "mathematically motivated" objects, by means of a collaborative work between researchers, scientists and artists. Other significant purposes of the project are the following: a) to foster more stimulant and interesting methods in the research of new Digital Art style; b) to stimulate the creativity of artists by means of the innovative scientific frameworks related to the above scenarios; c) to make more visible and accessible to the public community the use of the state-of-the-art technology that forms part of the current International activities in "Performing Art". A heterogeneous partnership is thence called to work in several fields (Mathematics, Art as well as technological and multimedia sectors), to create in the aforementioned three scientific scenarios innovative artistic objects - to be later available on CD-Rom or DVD-Rom and/or on Internet - mainly in order to facilitate the access to people belonging to different cultures. The project will in fact provide a strong basis for an international cultural network, improving synergies and contacts already existing and creating new reference points and contacts, relying on the web portal "MArs" (discussed in Secion 3) to support both the network and the project results diffusion; the project will of course stimulate events like exhibitions and intercultural exchanges devoted to spread the results obtained and to further stimulate new ideas. Intercultural exchanges among artists, scientific experts and cultural operators has been in particular emphasized in 2008, that was the *"European Year of Intercultural Dialogue"*. The European Project "SCIENAR" has five UE Members as participants: besides the University of Calabria (Italy), that serves as Project Coordinator, the remaining partners are the Virtual Image Co. (Stockport, UK), the EMR – Electronic Media Reporting (The Netherlands), the ITC – Institute for Computers of Bucuresti (Rumania) and the Slovak University of Technology of Bratislava (Slovak Republic).

The three scenarios we have chosen among the many more possibile ones are well suited to provide good examples that allow to understand how Mathematics and Art have developed in a parallel Paths (as we already mentioned in Section 3). We are in particular planning actions along the following specific items: i) new applications to the field of Mathematics & Art of the *Mathematica* ® package; with the software *Mathematica* ® artists can in fact think at a higher level and by means of it they can create complex objects all at once. The functional programming capabilities of *Matematica* ® let artists easily build up complex algorithms from simple parts; [43] ii) further extensions of the results achieved within the project *Connections in Space* [44]; iii) further applications concerning *Einstein's Theory of Special and General Relativity*. In these applications, a "sensible" dimension of the last Century is considered on an equal footing with the three sensible dimensions of space: Einstein's Theory of General Relativity is at the top of this line of thought and it is not surprising that Einstein had influence on Picasso and Dalì (see [4]-[7]).

The Geometry scenario is first introduced as the "art of measurement" (see Sections 1 and 2) together with its logical-deductive structure; Synthetic Geometry is consequently introduced at different levels of abstraction and understanding and used to introduce the next Scenario, where it is shown how and why Symmetry plays a dominant role in understanding Art through Mathematics (or Mathematics through Art); e.g., [18],[20] or [38]. Concerning symmetries, the project intends then to treat the Art and Science of Symmetry Breaking, starting from the previous Scenario (see [20],[26],[37],[38]). The aim will be to inspire new works of Art by introducing the artists to the idea of broken symmetries, as applied in physical contexts; see, e.g., [32]. There are good aesthetic reasons to expect that a discussion of symmetry breaking might be valuable to the artists. Although some symmetry in an artwork might be appealing, too much is monotonous and uninteresting. Indeed, the most symmetrical artwork would be an infinite blank canvas. A musical example of the aesthetic value of a small amount of broken symmetry is the preference for the rhythms of a human drummer over the relentless perfect repetition of a drum machine. The Scenario of Relativity is strongly related with the recent experiences of ours recalled above [4], in which a paradigmatical example is provided by the short video and multimedia that we realized, aimed at visualizing Einstein's Theory of Special Relativity; [5],[6],[7]. Do not forget – in fact – that Relativity provides an emblematic field where Geometry and Symmetry play in fact a leading role in understanding the structure of the Physical World and in a suitable sense of great part of Modern Physics. As an emblematic example we quote – in the context of "SCIENAR" - the "Superstring Installation" presented in [32] and [33], that has recently evolved into a true artistic "Superstring

Performance" and will be further pursued in the project (see the link – http://www.unical.it/portale/portaltemplates/view/view.cfm?11094 - in Italian; an English version will soon be available online on the web-site of the project).

In the subsequent developments of the project, artists will be involved in order to give artistic and cultural value to scientific objects produced in the previous activities. Scenarios and produced objects have in fact to be tested from different points of view: artists have to provide feedback on scenarios in terms of usability and user-friendliness of interfaces, and on produced objects in terms of artistic value. Remote artists have to test the virtual scenarios published on the web to evaluate from home if they are easy to use and understandable.

As already mentioned above, the Project "SCIENAR" will be necessarily accompanied by a number of "public events" (intercultural exchanges and exhibitions) aimed at spreading the results and the produced artistic items. As well as by the constitution of an international network concerning Scientific Scenarios and Art, aimed at enhancing the exchange of information and expertise related to the topics of "SCIENAR", facilitating relations among Institutions of the sector: Universities and other centres of scientific research, the artistic world (centres working in innovative Art methods), technological sectors (centres working mainly in computer science and innovative technologies) and cultural worlds (cultural associations, museums, and so on). The visibility of this unique international network will be assured by an explicitly built-up web site, supporting the network as a part of the "MArs" project (see above in Section 3, where we discussed of e-Newsletter to provide a forum of discussion on the Web). "SCIENAR" plans to establish also a communication system supporting the interaction and the communication among partners, that will communicate by means of ad hoc thematic discussion forums set-up on the project web site where technical, scientific and informative material will be stored (some areas will be accessible only to registered users). Communication will be also assured using networking facilities or by participating to standard meetings. An efficient sharing of work among technological partners will be assured by setting-up a centralized code repository (shortly, "CVS"). The project "SCIENAR" intends finally to implement its communication and /or dissemination plan by means of several interrelated activities.

To summarize, the Project "SCIENAR" offers the following: 1) a concrete possibility to favour the creativeness of artists; 2) innovative environments for generating artistic items; 3) a contribute to research in the Scientific and Art areas; 4) a concrete contribute in the innovative Art sector, based on new technologies; 5) the possibility to spread scientific culture concerning Relativity, Symmetries and Geometry and their relations with Art; 6) the possibility to use produced artefacts for new and more intuitive educational paths, aimed to teach Mathematics and scientific concepts. Then, the project has a potentially strong social impact, since it represents culture, cultural objects and landscapes in a way that can be easily integrated in educational and entertainment sectors, stressing interconnections among countries from South to North, from West to East, broadly using new communication media.

On one hand "SCIENAR" has educational and recreational aims for the final user, on the other hand it provides an International reference point for researchers and cultural operators interested in the use of scientific scenarios based on mathematical models in Art field. Artists are now looking to new emerging information technologies for creating innovative artefacts. Activities and results will be known and divulgated at International level and then other scientific and cultural institutions will take advantage from these activities. At the same time, artists and cultural operators of several Countries, by using the produced artefacts, will give them a specific cultural value, expression of a given national culture. Furthermore, each national expression will be known, by co-operating at International level, in many other nations. In addition"SCIENAR" will also offers the opportunity to many International projects to develop and join a *"Pan-European"and International Network of Information on "Performing Art"* by using Scientific scenarios.

In order to achieve the specific aims (out of which the rest flows through) the project "SCIENAR" will also support actions of "virtual mobility" through the creation of a fully innovative Web Portal (of which "MArs", see Section 3, will be just a "conventional" portion). This innovative portal will be based on a collaborative approach, something that will allow to manage a collection of hypermedia documents and will exhibit galleries of multimedia objects, among which those produced within the project itself (as well as many more of different origin). The Portal associated with "SCIENAR" is thence intended to support the following: a) the consultation of galleries of multimedia objects; b) the collaboration and the exchange of opinion among artists; c) the cooperative work among Artists and Mathematicians; d) the production of new artistic objects: starting from mathematical models, so to allow the realization of several kinds of artistic and cultural representations. All these Portals will be based on "wiki technologies", with a large selection of links to other similar websites, as well as information about initiatives in the field of Art & Mathematics. It will have also the aim to gather information about needs of artists and cultural operators in order to provide them reference points and also to provide solution within the same project, as well as to take training needs concerning the use of digital technologies for producing artistic objects. Cultural operators and artists will thence be involved in order to give artistic and cultural value to mathematical objects available on the web, that in turn will offer a user interface very friendly and will allow to interact without little assistance. Then, also remote artists coming from different countries will have the possibility to express their opinion and their viewpoint about virtual objects published on the web and to effectively collaborate. Also scientists and artists interested to this new art form will cooperate in order to exchange ideas about artefacts and the related mathematical models. Intercultural exchanges will be very stressed by the Project "SCIENAR".

5 Further Perspectives

As previously discussed in Sections 3 and 4, the use of mathematical models in the artistic creation activity is today one of the most stimulating and interesting methods in the research of new Digital Art style. In fact, this allows researches and artists to create artificial worlds within them, in order to make innovative experimentations. Alongwith the Project "SCIENAR", its innovative web-portal and the included "conventional" web portal "MArs", a third action has then to be envisaged. Namely, an action based on the creation of a network of scientists and artists explicitly aimed at strengthening the artists and cultural players mobility, creating a platform for facilitating interchanges of ideas and experiences, within the European and International Cultural and Artistic Space and, in particular, to improve the possibility to have links (on the network and directly) among single artists and cultural operators, artists and scientists, artists and experts in digital technologies. This will form the core of future projects, that will have the following among the main aims: 1) to support virtual mobility and the collaboration among artists, among scientists and artists and among artists and technologies experts, as well as to allow artists to exhibit, to publish and to distribute their works in rapid and effective way in a very large area; 2) to support the creation of artistic objects (Music pieces, Digital Art objects, etc) and the exchange of opinions among artists about the produced objects; 3) to enhance the exchange of information and expertise related to the topic of Art and Mathematics facilitating relations among institutions working in the sector in order to obtain an efficient cooperation network of experts and a creative exchange; 4) to define good practices about virtual mobility and physical mobility involving cultural players and artists; 5) to establish permanent and effective relations among several kinds of Institutions (universities, cultural centres, etc.), to be continued also after the end of the project terms.

By supporting mobility and the collaboration among artists, among scientists and artists and among artists and technologies experts, the project “SCIENAR” and its further developments will facilitate the establishment of new intercultural exchanges and to promote in a very large area innovative artistic works. Activities of artists will be well known and also the market of Art will be improved. In addition, our project will provide an International reference point for researchers and cultural operators interested in the use of new technologies and Mathematics in “Performing Arts”. The establishment of permanent relations among several kinds of Institutions (universities, cultural centres, etc.) will allow to have lasting and sound benefits. In particular, the project “SCIENAR” and its further developments will allow to have synergies and improvement of artistic activities and consequentially of the market. This is very important because actually the International cultural sector is characterized by a many number of small and medium enterprises, different kinds of autonomous and subordinated work. Then, this sector needs in particular way the creation and the development of network links, coordination and spreading knowledge and information. “SCIENAR” has also a potentially strong social impact, since it represents cultural and artistic objects in a way that can be easily integrated and used in several sectors: Art and cultural sector (as innovative and creative artefacts produced by using new approaches based on mathematical models and new digital technologies), educational/training sector for experimenting new didactic methodologies to teach mathematical concepts (in fact, it is very interesting to start from the realized artistic artefacts for extracting mathematical and scientific concepts, as we discussed in Sections 1 and 2).

In the very end “SCIENAR” - taking into account that the International Community (and the European Union in particular, trough its extension and openings) - will therefore support at international level both the cultural players and artists’ mobility and the diffusion of cultural works and will contribute, in very strong way, to the European and International cultural cooperation; it aims thus to be a decisive factor for the spreading of knowledge, experience, and mutual breathing and for making well-known cultural diversity. In this way, the whole set of projects will foster the development of the quoted European and International Common Cultural Space. In this space, artists and cultural operators of several Countries, collaborate each other’s and, by using their realized artefacts, will give them a specific cultural value, expression of a given national culture. Furthermore, each national expression will be known, by co-operating at International level, in many other nations. Their activities and the achieved results will be known and divulgated at International level and then other cultural, scientific and technological institutions will take advantage from these activities. In addition “SCIENAR” will offer the opportunity to many International Scientific projects to create and join the aforementioned “Pan-European” and International Network of Information on “Performing Art” by using mathematically minded models To summarize, among the results anticipated on a long-term basis by “SCIENAR”, we can mention the following: 1) a good platform for supporting concretely virtual and physical mobility in cultural sector; 2) the harmonization and the spreading of know-how and successful examples on the use of new approaches in Performing Art; 3) the expansion of the network about Mathematics and Art aimed to improve the scientific and cultural cooperation among cultural worlds (cultural associations, museums, etc.), academic (mainly constituted by universities) and technologies laboratories (digital graphic centres, etc.); 4) the development of the already mentioned European and International Common Cultural Space.

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DYNAMICAL WEBARTS

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Abstract. webMATHEMATICA is a new web technology that allows the generation of dynamic web content with Mathematica. It integrates Mathematica with a web server. webMATHEMATICA harnesses the full range of Mathematica technology to build sophisticated web applications, especially in creating dynamical web art objects.
webMATHEMATICA provide immediate access to the technical computing software with very firm abilities especially in *MATHEMATICA* graphics from any web browser. It allows to incorporate also dynamical possibilities to creating graphics objects, so the graphics are live, interactive and responsive to user needs.
In this paper we will present the possibilities at the creation dynamical graphics objects in this way and the short guide - how to create this form of graphical objects. }

Key words and phrases. webMATHEMATICA, web art, dynamical graphical web objects

Mathematics Subject Classification. 65D17, 65D18 , 53A60

1. Introduction

webMathematica is the clear choice for adding interactive calculations to the web. This unique technology enables the user to create web sites that allow users to compute and visualize results directly from a web browser. Based on the world's leading technical computing software and Java Servlets, a proven server technology, *webMathematica* is fully compatible with Mathematica and state-of-the-art dynamic web systems.

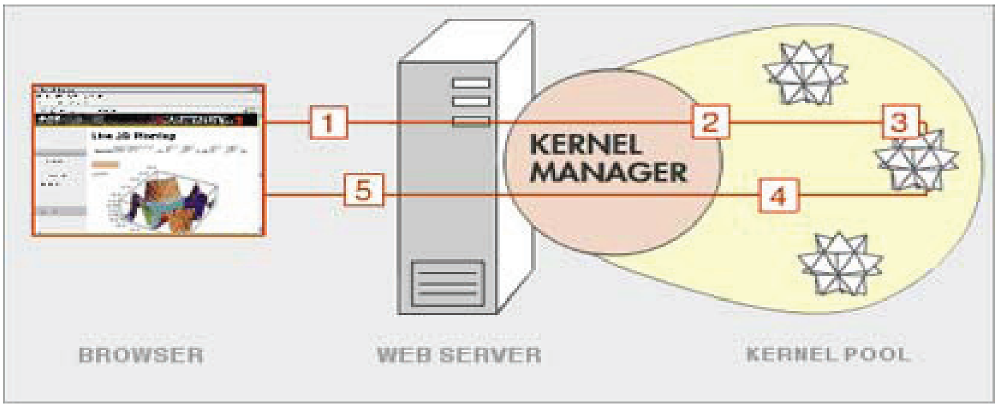
In this paper we will demonstrate an operation of *webMathematica.* We will present only only introductory examples and demonstration for presenting the possibilities of *webMathematica*. This introduction explain the reasons for using *Mathematica* in a web site, examines a few areas in which you might use *webMathematica*, briefly discusses the underlying *Mathematica Server Pages* technology, and outlines the requirements for running *webMathematica.*

WebMathematica technology is scalable, reliable, and easy to use. It allow the deployment of the building of calculators, algorithms, and problem-solvers over the web or intranets, custom web sites that provide specialized calculations to customers, delivery of interactive courseware over the web and publishing of interactive textbooks and book supplements on the web.

webMathematica allows a site to deliver HTML pages that are enhanced by the addition of *Mathematica* commands. When a request is made for one of these pages, the *Mathematica* commands are evaluated and the computed result is inserted into the page, see Example 1 at the end of this chapter. This is done with JavaServer Pages (JSP), a standard Java technology, making use of custom tags. After the initial setup, all that you need to write *webMathematica* applications is a basic knowledge of HTML and *Mathematica*. *webMathematica* is based on two standard Java technologies: Java Servlet and JSP. Servlets are special Java programs that run in a Java-enabled web server, which is typically called a "servlet container" (or sometimes a "servlet engine"). There are many different types of servlet containers that will run on many different operating systems and architectures. They can also be integrated into other web servers, such as the Apache web server.
Other software can readily be incorporated into *webMathematica* with MathLink technology. It is particularly easy to connect Java into Mathematica with J/Link, providing many exciting possibilities for *webMathematica* development. Teacher can call functionality in the server to examine HTTP headers, create and inspect cookies, or use JDBC for database connectivity.

webMathematica works seamlessly with the *Mathematica* application packages. They allow you to implement additional specialized functionality without months of development time. All standard *Mathematica* packages can be added to the web*Mathematica* jsp pages. Also scientist's own packages can be simply implemented to the created presentations. Included in *webMathematica* are professionally designed web page templates that can artists modify for their needs, thus saving design time. All standard web technologies, CSS styles allow creator to create more attractive and powerful sites.

An important part of *webMathematica* is the kernel manager, which calls *Mathematica* in a robust, efficient, and secure manner. The manager maintains pools of one or more *Mathematica* kernels; by maintaining more than one kernel, the manager can process more than one request at a time. Each pool takes care of launching and initializing its kernels. When a request is received for a computation, a kernel process is utilized to process the request and, upon completion, is returned to its pool. If any computation exceeds a preset amount of time, the kernel process is shut down and restarted. When the server is shut down, all of the kernel processes are also shut down. These features maximize the performance and stability of the server.



webMathematica is built on platform-independent standards such as HTML, Java, and Java Servlet technology. For example, Java Servlet technology is supported, either natively or through plug-in servlet containers, by all modern web servers-including Apache, Microsoft IIS,-as well as by application servers such as IBM WebSphere.

webMathematica is based on two standard Java technologies: Java Servlets and JavaServer Pages (JSP). **Servlets are special Java programs that run in a Java-enabled web server**, which is typically called a "servlet container" (or sometimes a "servlet engine"). There are many types of servlet containers that can run on many different operating systems and architectures. **There is no need to know technical details of servlet containers for teachers created jsp pages.**

webMathematica **allows a site to deliver HTML pages that are enhanced by the addition of *Mathematica* commands.** When a request is made for one of these pages, the *Mathematica* commands are evaluated and the computed result is inserted into the page. This is done with a standard Java technology, JSP, making use of custom tags.

webMathematica integrates effortlessly with standard web technologies. You choose, install, and configure the web server, Java, and servlet container of your choice on the supported platform of your choice. You can then add the *webMathematica* application, add *Mathematica*, and edit the *webMathematica* configuration files accordingly to fit your needs. This enables you to use proven web technology to manage your website.

2. Basic graphical examples of *webMathematica*

Any calculation done in *Mathematica* can be done using *webMathematica*, with two note worthy limitations. First, when using *webMathematica*, you will not have access to the full capabilities of *Mathematica*'s front end. Second, license restrictions may prohibit you from enabling certain calculations when you use *webMathematica*.

In this section we will show several simple examples of web*Mathematica* jsp pages. Many of these can be copied and used as the basic for your own work. You can test their functionality on appropriate web pages. The description given here will work through a collection of sample JSPs, each of which will demonstrate some details of feature. The sources for all these examples are fully working JPSs pages and reader can find them on our http://www.webmathematica.eu web site.

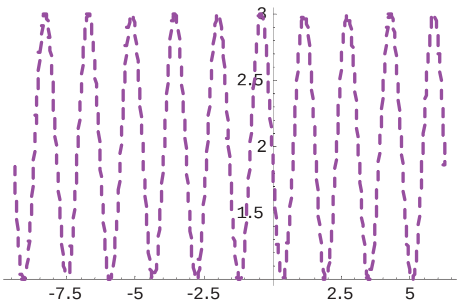
This section gives you some ideas for getting started to develop your own site and create Your own graphics on the webpages.

2.1 Page with various graphical outputs

This example generates a various plots. We will show the use of the MSP function **`MSPShow`**. **`MSPShow`** takes the Mathematica graphics object from the **`Plot`** (**`ListPlot`**, **`Plot3D`**...) command and generates a GIF image, which is stored on the server, returning an HTML **`<img>`** tag.

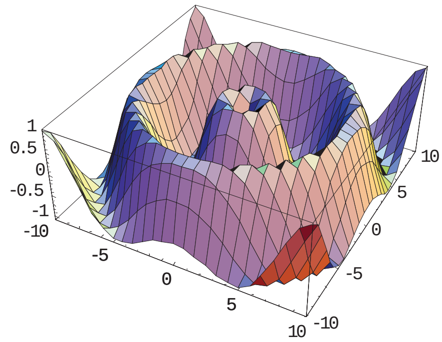
There are also several problems and questions with sending image to the browser, but in this simple example we will not discuss. on it. *Mathematica* can output graphics in a number of formats including GIF, animated GIF, and JPEG, SVG. Using the Java applet `LiveGraphics3D`, you can also create and manipulate three-dimensional graphics interactively in your web browser.
First we will show calculations done in *Mathematica.*

In[1]:=
```
Plot[Sin[4 x - 3] + 2, {x, -3 Pi, 2 Pi},
  PlotStyle → {Dashing[{0.02}], Thickness[0.01],
   Hue[0.8]}]
```



Out[1]= - Graphics -

In[2]:=
```
Plot3D[Sin[Sqrt[x^2 + y^2]], {x, -10, 10},
  {y, -10, 10}]
```

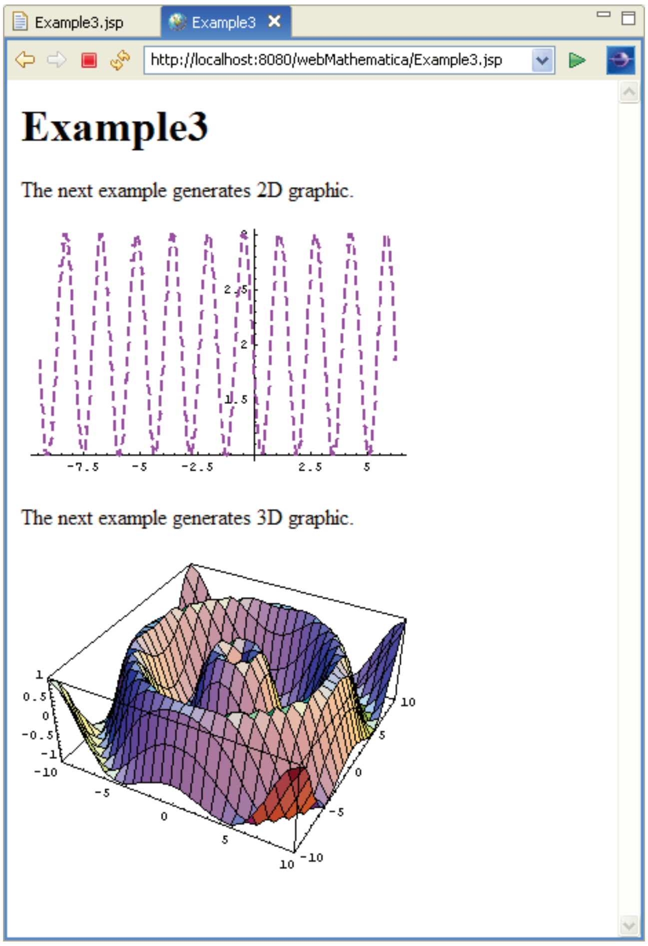


Out[2]= - SurfaceGraphics -

The same calculation can be provided by next jsp page.

The source code:

```
<?xml version="1.0"?>
<?xml-stylesheet type="text/xsl"
href="HTMLFiles/pmathml.xsl"?>
<!DOCTYPE html PUBLIC "-//W3C//DTD XHTML
1.1 plus MathML 2.0//EN"
        "HTMLFiles/xhtml-math11-f.dtd">

<%@ page language="java"  %>
<%@ taglib uri="/webMathematica-taglib"
prefix="msp" %>

<html>
<head>
<title> Example3 </title>
</head>

<body>
<h1> Example3 </h1>

<msp:allocateKernel>

<p> The next example generates 2D
graphic.</p>
<msp:evaluate>
       MSPShow[Plot[Sin[4x - 3] + 2, {x,
-3Pi, 2Pi},
             PlotStyle ->
{Dashing[{0.02}], Thickness[0.01],
Hue[0.8]}]]
</msp:evaluate>

<p> The next example generates 3D
graphic.</p>
<msp:evaluate>
       MSPShow[Plot3D[Sin[Sqrt[x^2 + y^2]], {x, -10, 10}, {y, -10, 10}]]
</msp:evaluate>

</msp:allocateKernel>
</body>
</html>
```



2.2 Manipulation on the web page with Graphical Outputs

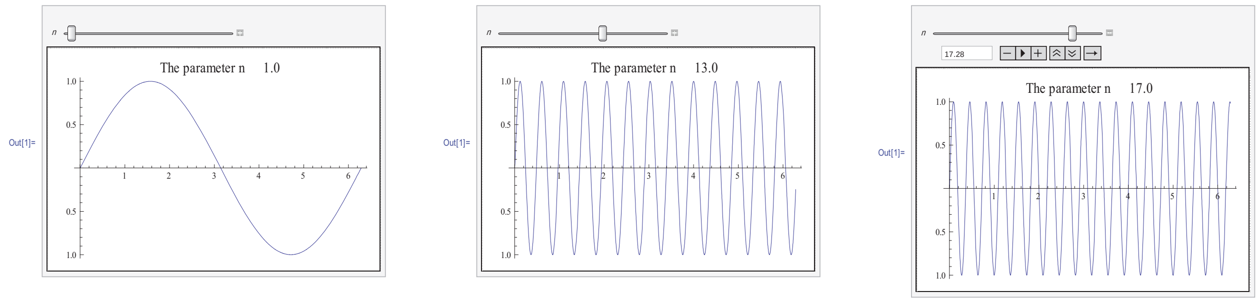
One of the key features of *Mathematica* 6 (*Mathematica* 7) was a simple way to construct user interfaces. Previously, user interface construction required specialist knowledge and expertise as well as complicated tools. The result was that many users were not able to create their own user interfaces, they would work with specialists or go without the extra facility of a user interface. However, the new ideas and technology based on Manipulate, allowed these users to create a wide range of interesting user interfaces without their needing to learn this special knowledge.

web*Mathematica* 3 introduced a web version of Manipulate technology, and we will show how does it works.

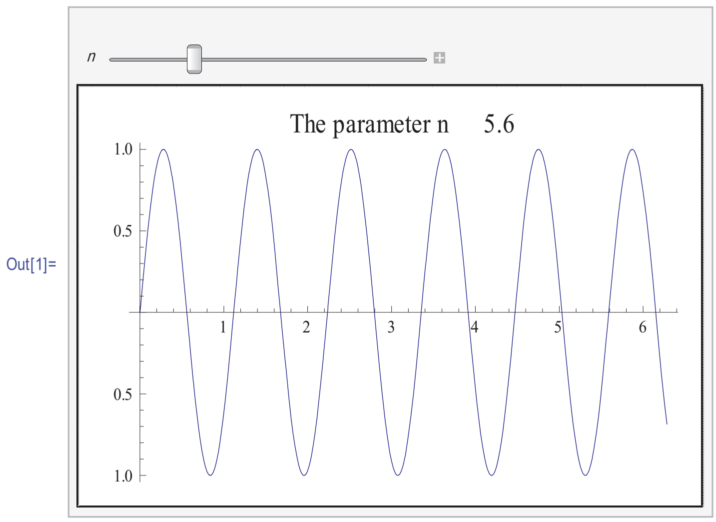
The core of an example web page that demonstrates the web interactive tools is special command MSPManipulate. The source code of one simple page follows. On this page we can realize the same calculation as in *Mathematica.* In *Mathematica* we simply write

```
In[1]:= Manipulate[
    Plot[Sin[n x], {x, 0, 2 π},
     PlotLabel → Style[StringForm["The parameter n = ‘1‘", PaddedForm[n, {2, 1}]], 16]],
    {n, 1, 20}
   ]
```

The output you get from evaluating a **Manipulate** command is an interactive object containing one or more controls (sliders, etc.) that you can use to vary the value of one or more parameters. The output is very much like a small applet or widget: it is not just a static result, it is a running program you can interact with.



The same calculation can be done by **MSPManipulate** via web page.



There are three key parts in this page. First, the **MSPManipulate‘** package is loaded with **Needs**, as shown below. This needs to be done in its own `evaluate` tag, and at the start of the page.

```
<msp:evaluate>
  Needs["MSPManipulate‘"]
</msp:evaluate>
```

Secondly, the special header is put down, with a call to **MSPManipulateHeader**.

This needs to refer to the variables `$$updateArgs`, and `$$manipulateNumber` exactly as shown (note that you cannot rename these). **`MSPManipulateHeader`** initializes the interactive features.

```
<msp:evaluate>
   MSPManipulateHeader[$$updateArgs, $$manipulateNumber]
</msp:evaluate>
```

Finally, you have to add the actual interactive code. This is done with a call to **`MSPManipulate`**; an example follows.

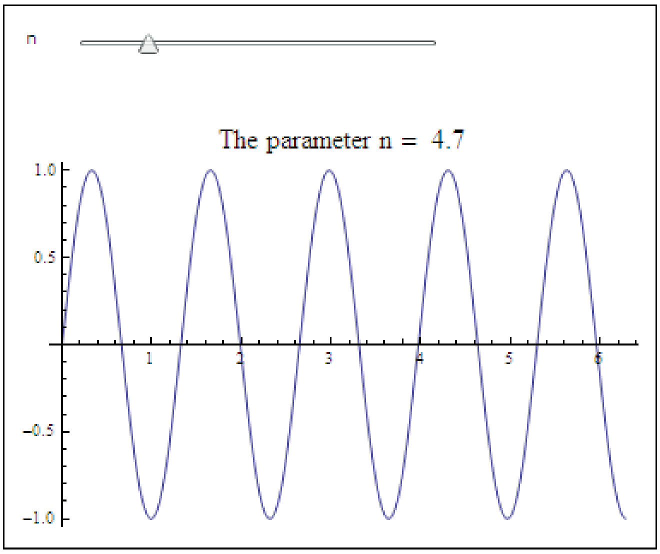
```
<msp:evaluate>
    MSPManipulate[Plot[Sin[x*a],{x,0,4 Pi},
            Filling   Axis,
            PlotRange   {{0,4 Pi},{-1.1,1.1}},Frame   frame],
      {a,0,5},{frame,{True,False}},AppearanceElements   All]
</msp:evaluate>
```

`MSPManipulate` supports a number of different interactive controls, which are all similar to those of **`Manipulate.`** With help of web*Mathematica* we can create a number of other examples of the use of its interactive web tools.

The source code for the whole page follows and the picture from web page You can find on the right-hand side.

The source code:

```
<%@ page language="java"  %>
<%@ taglib uri="/webMathematica-taglib" prefix="msp" %>

<msp:evaluate>
  Needs["MSPManipulate‘"]
</msp:evaluate>

<html> <head>
<title> Manipulate and Plot </title>
</head>

<msp:evaluate>
  MSPManipulateHeader[$$updateArgs,
$$manipulateNumber]
</msp:evaluate>

<body>
<h1> Manipulate and Plot </h1>

<msp:evaluate>
  MSPManipulate[
    Plot[ Cos[var+x], {x,0,2Pi}, Frame
-> frame],
      {var, 0,20}, {frame, {True,False}}]
</msp:evaluate>

</body>
</html>
```



The web version of **`Manipulate`** is based on Flash web technology. Flash is commonly used to add interactive features and effects to web sites and is quite well supported over a variety of different platforms and browsers. web*Mathematica* actually uses Flash 9, so any browser that does not support this will not be able to support web*Mathematica* interactive web tools.

2.3 Another graphical objects

We can add on the web page also another web controllers as a check box, radio box, popup menu, setter... In the next examples You can find the the source code and some picture to demonstration the ability of webMathematica.

6

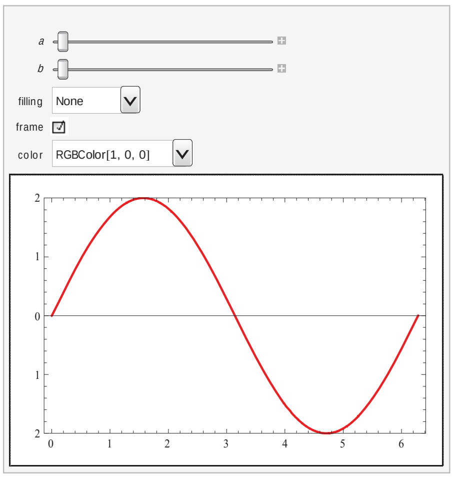
```
In[2]:= Manipulate[Plot[Sin[x * a], {x, 0, 4 Pi}, Filling → Axis,
          PlotRange → {{0, 4 Pi}, {-1.1, 1.1}}, Frame → frame],
        {a, 0, 5}, {frame, {True, False}}, AppearanceElements → All]
```

Out[1]=

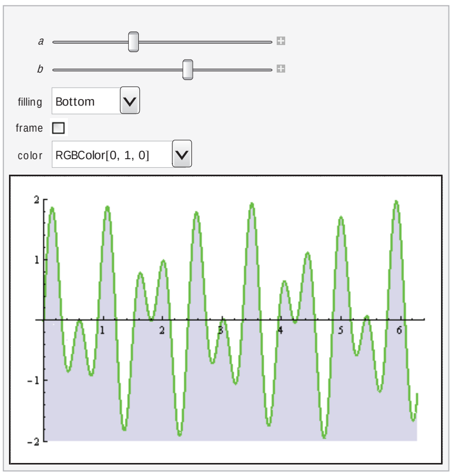
Out[2]=


```
In[3]:= Manipulate[Plot[Sin[a x] + Sin[b x], {x, 0, 2 Pi}, Frame → frame, Filling → filling,
          PlotStyle → {color, Thick}, PlotRange → 2], {a, 1, 20}, {b, 1, 20},
        {filling, {None, Axis, Top, Bottom, Automatic}, ControlType → PopupMenu},
        {frame, {True, False}},
        {color, {Red, Blue, Green, Orange}, ControlType → PopupMenu}, OutputSize → {800, 1000}]
```

Out[3]=



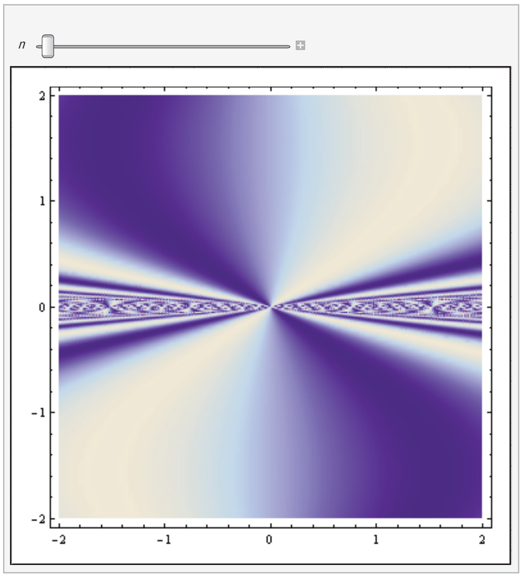
Out[3]=



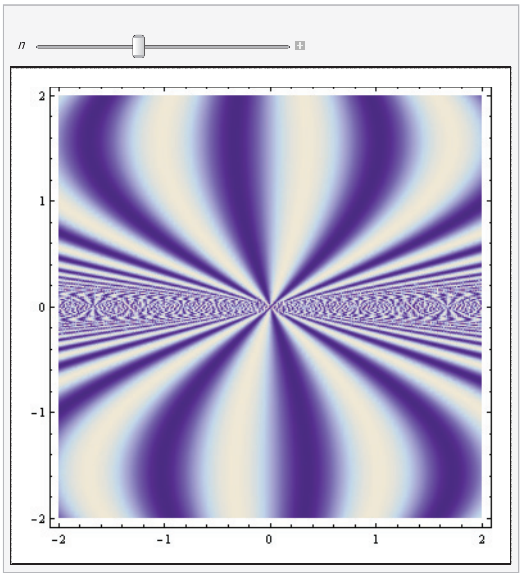
We can manipulate not only with 2D graphics, abut also wit other Mathematica graphical commands as DensityPlot, ParametricPlot, Plot3D... The same result we can receive with command **`MSPManipulate`** on our webpage.

In[4]:=
```
Manipulate[
 DensityPlot[Sin[n x / Sin[y]], {x, -2, 2}, {y, -2, 2}, PlotPoints → 100],
 {n, 1, 10}]
```

Out[4]=



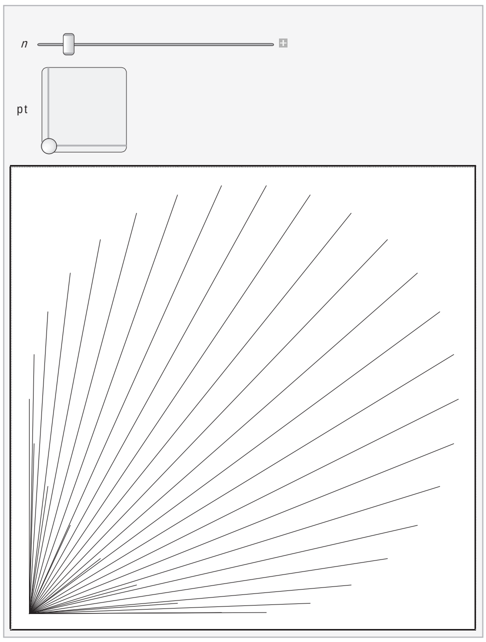
Out[4]=



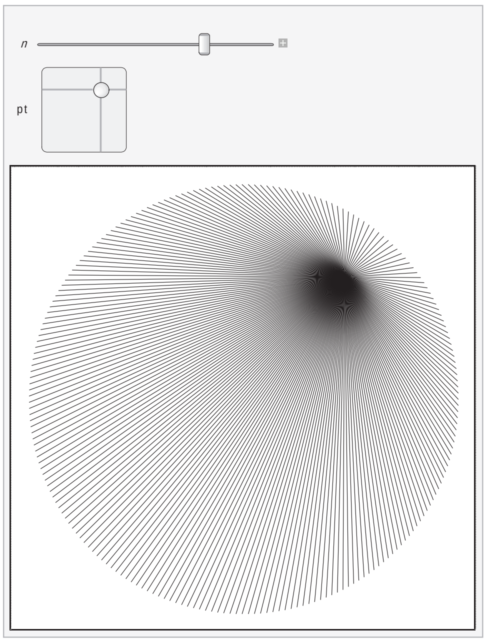
2D Slider allow to manipulate with position of 2D object

In[5]:=
```
Manipulate[
 Graphics[{Line[Table[{{Cos[t], Sin[t]}, pt}, {t, 2π/n, 2π, 2π/n}]]}, PlotRange → 1],
 {{n, 30}, 1, 300, 1}, {pt, {-1, -1}, {1, 1}}]
```

Out[5]=



Out[5]=

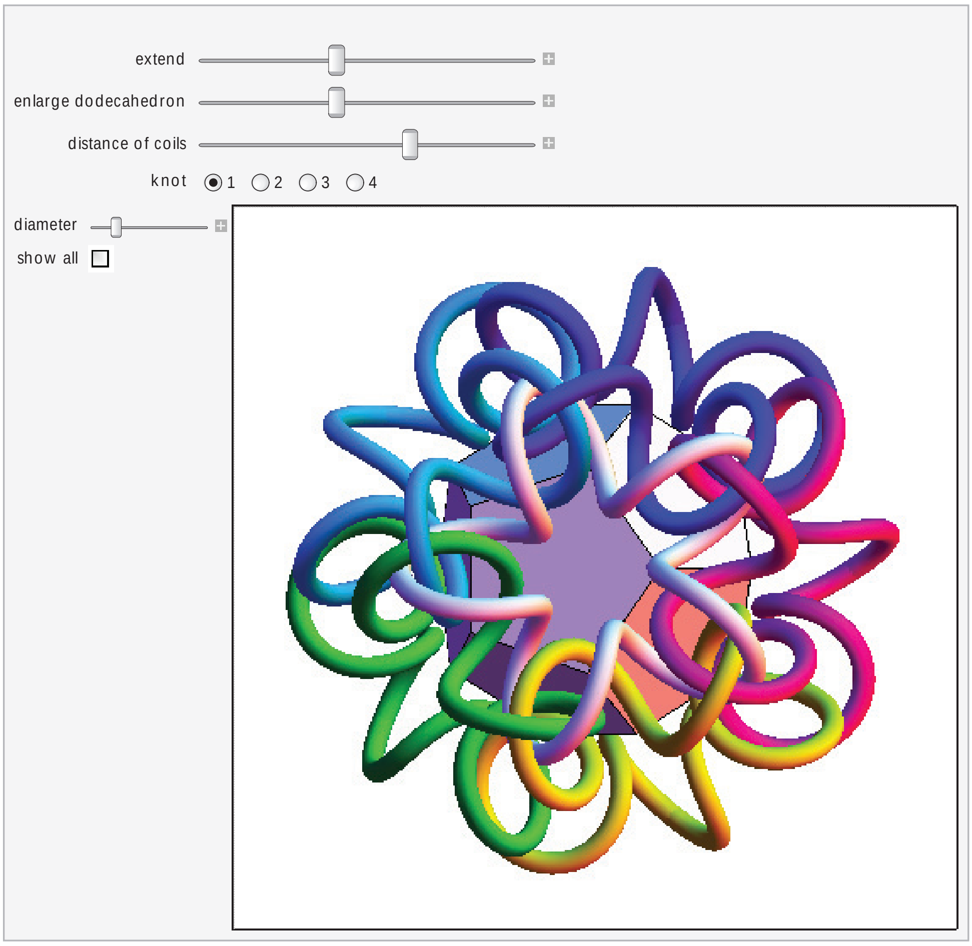


We can create also more complicate and more powerfull graphical object with multiple controles:

Entangled Torus Coils

```
In[6]:= Manipulate[
  par = ParametricPlot3D[{(ext + Cos[5 x]) Cos[knot x], (ext + Cos[5 x]) Sin[knot x], Sin[5 x]},
     {x, 0, 2 Pi}, PlotStyle → Tube[dia], PerformanceGoal → "Speed"][[1]];
  p1 = Translate[par, {0, 0, -dis}];
  p2 = Rotate[Rotate[p1, Pi, {0, 0, 1}], 2 ArcTan[0.618], {0, 1, 0}];
  p3 = Table[Rotate[{Hue[0.18 i], p2}, i 2 Pi / 5, {0, 0, 1}], {i, 5}];
  dod = Scale[Rotate[PolyhedronData["Dodecahedron", "Faces"], -ArcTan[1.618] - ArcTan[1 / 1.618^2],
     {0, 1, 0}], en {1, 1, 1}, {0, 0, 0}];
  bottom = Rotate[{p1, p3}, Pi, {1, 0, 0}];
  Graphics3D[{If[all, bottom, {}], p1, p3, dod}, Boxed → False, SphericalRegion → True,
   ViewPoint → {0, 0, -10}, ImageSize → {400, 400}, ViewAngle → Pi / 30], {{ext, 2, "extend"}, 0, 5},
  {{en, 2, "enlarge dodecahedron"}, 0, 5}, {{dis, 3.2, "distance of coils"}, 0, 5},
  {{knot, 1, "knot"}, 1, 4, 1, RadioButton},
  {{dia, 0.18, "diameter"}, 0.04, 1, ImageSize → Tiny, ControlPlacement → Left},
  {{all, False, "show all"}, {True, False}, ImageSize → Tiny, ControlPlacement → Left},
  TrackedSymbols → Manipulate]
```

Out[6]=

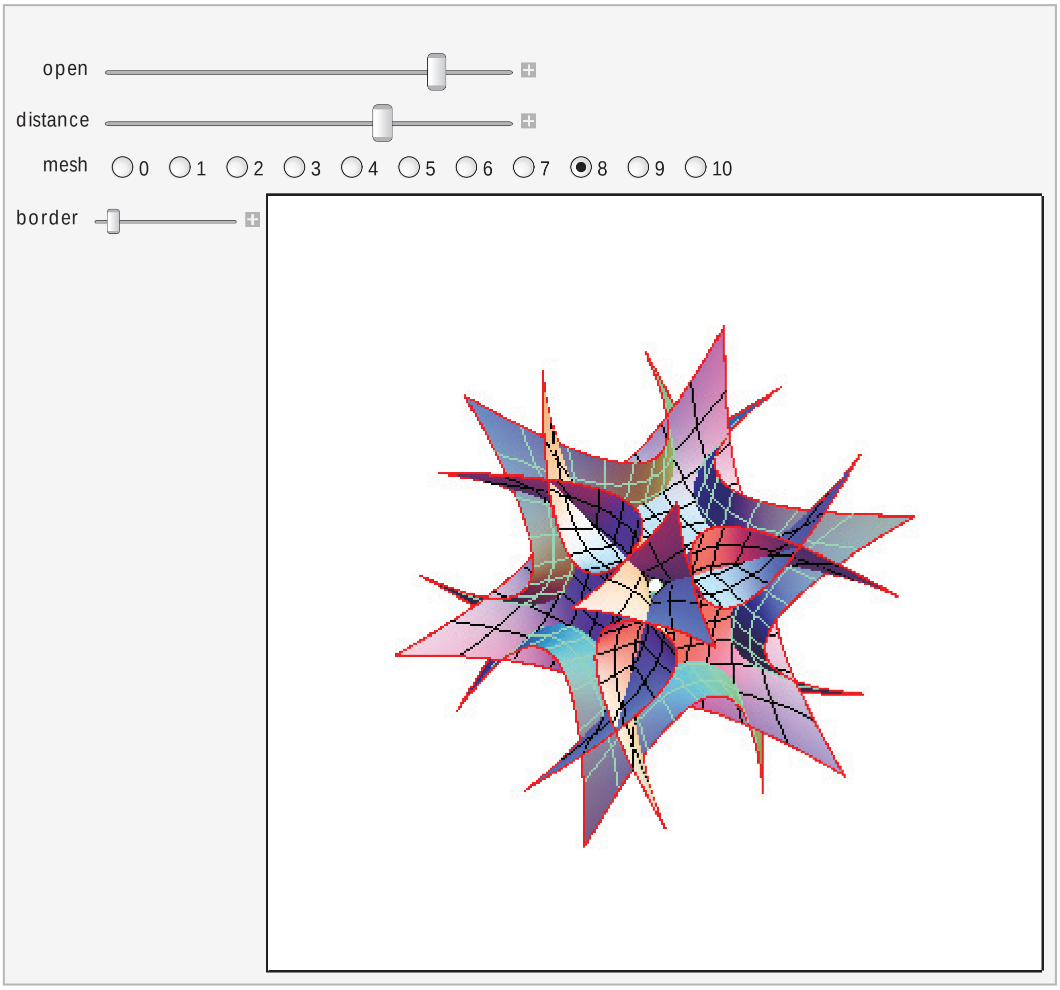


Open a Cube

Open a cube at the corners also to produce various graphic designs.

```
In[7]:= Manipulate[p1 = ParametricPlot3D[{x, y, dis + open x^2 y^2}, {x, -1, 1}, {y, -1, 1}, Mesh → m,
    BoundaryStyle → Directive[Thickness[bord], Red]][[1]];
    p2 = Table[Rotate[Rotate[p1, ArcTan[Sqrt[2]], {1, 1, 0}], i 2 Pi / 3, {0, 0, 1}], {i, 3}];
    p3 = Rotate[p2, Pi, {1, 1, 0}];
    Graphics3D[{p2, {RGBColor[.6, .86, .71], p3}}, SphericalRegion → True, Boxed → False,
      ViewAngle → Pi / 30, ViewPoint → {0, 0, 10},
    PlotRange → 3, ImageSize → {350, 350}],
    {{open, 1.3, "open"}, -2, 2},
    {{dis, 1, "distance"}, -3, 2},
    {{m, 0, "mesh"}, 0, 10, 1, RadioButton},
    {{bord, 0.003, "border"}, 0.001, 0.04, ImageSize → Tiny, ControlPlacement → Left},
    TrackedSymbols → {open, dis, bord, m}]
```

Out[7]=



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[5] http://www.webmathematica.wolfram.com
[6] http://www.wolfram.com
[7] http://webmathematica.mathematica.sk

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SUBLIMINAL MATHEMATICS IN ART

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Abstract: Three objects of art, the Pyramids of Ghizeh, the Michelangelo's Genesis and the Endless Column of Brancusi are surprisingly and unexpectedly connected by their subliminal mathematical message. A short presentation of our binocular vision leads to an open problem.

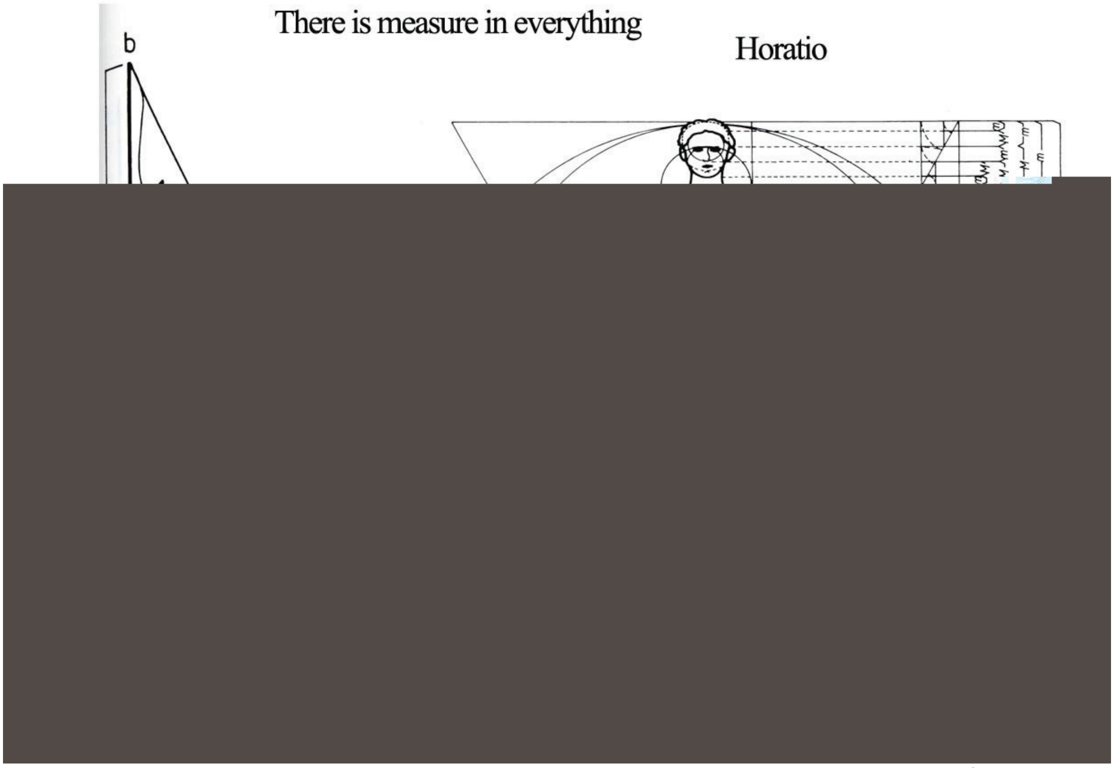
Key words: Light Cone, gates, 4-dimensional space, binocular vision

We don't need mathematics to breath but there is a lot of mathematics in the description of breathing act and we know that mathematics.
A cell doesn't need mathematics to divide itself but there is some mathematics which has as fundamental internal operation not the addition but the multiplication. We don't know it and probably that's why nature is so beautiful.
We use art as a tool of fulfilling our need of wonder and beauty. Generally we don't transpose in colors, sounds, words or stones brilliant ideas from mathematics.
But we can find subliminal mathematics in art. Some of them are preceding their time some of them are included, more or less voluntary, in the theme of the art object after their apparition.
Let us give some examples and for this purpose we have to get down from our mathematicians clouds and speak for all listeners.

1. We can see around us a space with three different types of movement possible: left-right, forward-backward, up and down. Clearly this is a three dimensional space. But we can locate an object in this space at a specified time. Motions can be represented as lengths spanning both space and time in a system of coordinates. In 1908 H. Minkowski claimed that if things can be rearranged in time then the universe can be considered as a four dimensional space. In this space he considered the length of a segment to be

$$l^2 = (ct)^2 - x^2 - y^2 - z^2$$

where x, y, z are spatial displacements along the corresponding axes, t is time and c is light velocity. The invariance of the space-time segment leads us to some hard to understand conclusions: space and time are deformable; the speed of light is a universal constant. More, things traveling with the speed of light in any direction have the length of the space-time segment equal to 0. So, we obtain a surface, namely the Minkowski Light Cone,



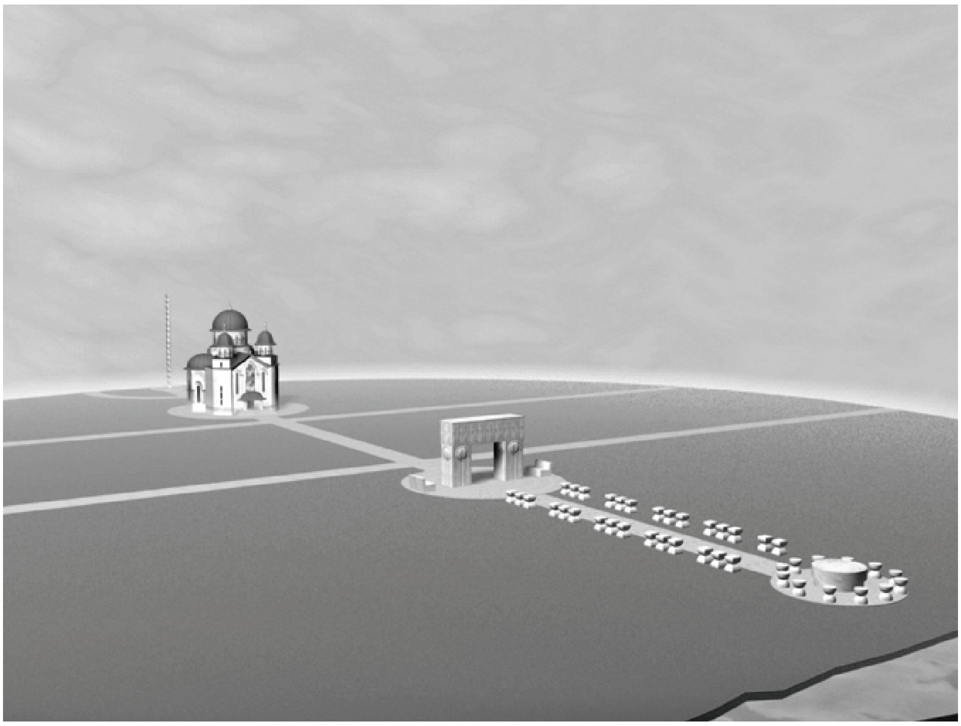
Courtesy of Sorana Georgescu Gorjan

Some consider that Brancusi, the sculptor coming from rural area of Romania did not know these geometric considerations and he carved his works using his profound artistic intuition. Others consider that his studies at the School of Arts and Crafts from Craiova and the National School of Fine Art from Bucharest learned him these basics.

However, we cannot but observe, following the pertinent analyze of Stefan Georgescu Gorjan, the engineer who offered Brancusi technical support for erecting the column in Targu Jiu, that “25 of Brancusi’s ovoid works respects the norm of golden section. From these, at 5 the ratio is 0,617 and 0,620 compared with this ideal ratio of 0,618. At 13 from the remained 17, this ratio is respected with a tolerance of 1% and only at 4, there is a deviation of 2-3% from the nominal value” [Am lucrat cu Brancusi, Stefan Georgescu Gorjan, Ed. Universalia Bucureşti 2004]

Above are presented a few of Brancusi's works that respect the golden rule of harmonious proportions. Perfect works of art belonging to the modern world of art, expressing the essence of things, contemporary with the most abstract trends and tendencies and with revolutionary evolutions in science are based of a rule as old as the world, that of harmonious proportions. In this context is more easily understood Brancusi's statement "everything that I have new is coming from the very old …".

Furthermore, we can extend the analysis to the Monumental ensemble of Targu Jiu, Romania. It is well known that this ensemble has been built by Brancusi for honoring the heroes who fought against the enemies during the first world war. It is a manifesto of Brancusi's credo, a gift to his native country, the first integration of a sculptural ensemble in a public space (in 1937) " integrating the idea of a road of heroes, long and difficult, starting from the river, and the Table of Silence, passing through the Gate of Kiss and reaching the Endless Column that "brings us to the infinite and beyond" [Hulten et al. Brancusi 1995, p.230].

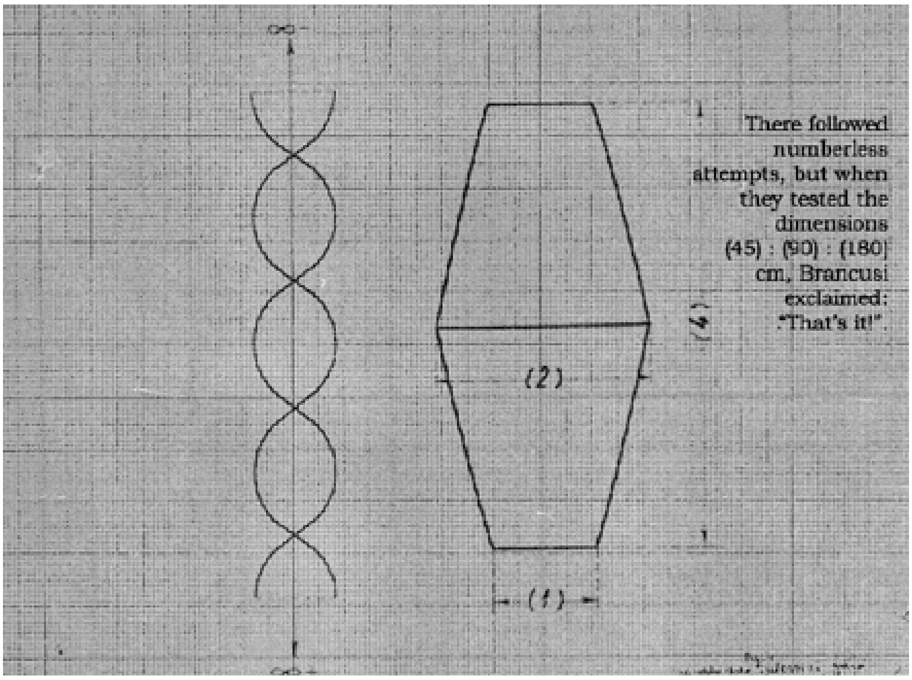


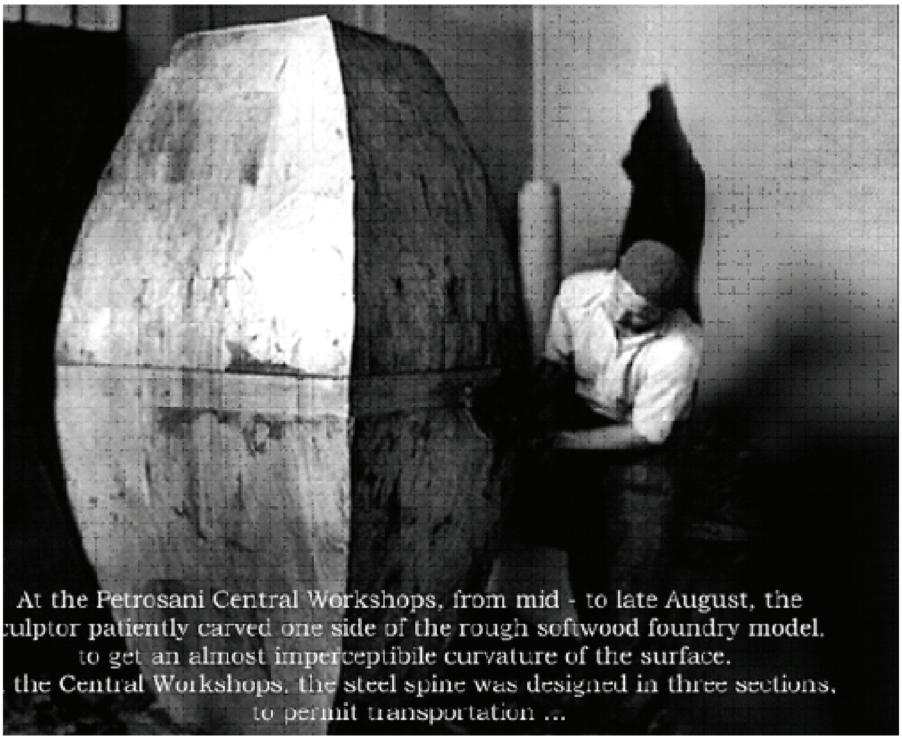
Courtesy of ITC

It is easily observed that the internal harmony of the Targu Jiu Ensemble is given by certain proportions of the volumes, in the successive repetition or to the distance between pieces of every part, to the grouping of the elements corresponding to key numbers, in their order, as well in the distance between them relative to their height. [Ion Mocioi]. In the name of discovering of a supposed numeric symbolism, it is believed that the secret behind Brancusi's forms relies on certain rapports and dimensions.

The works of Georgescu Gorjan related to the column, being the only who stood next to Brancusi in his artistic effort of creation of the Endless Column, are of important significance and again help us to go deeper in the analysis.

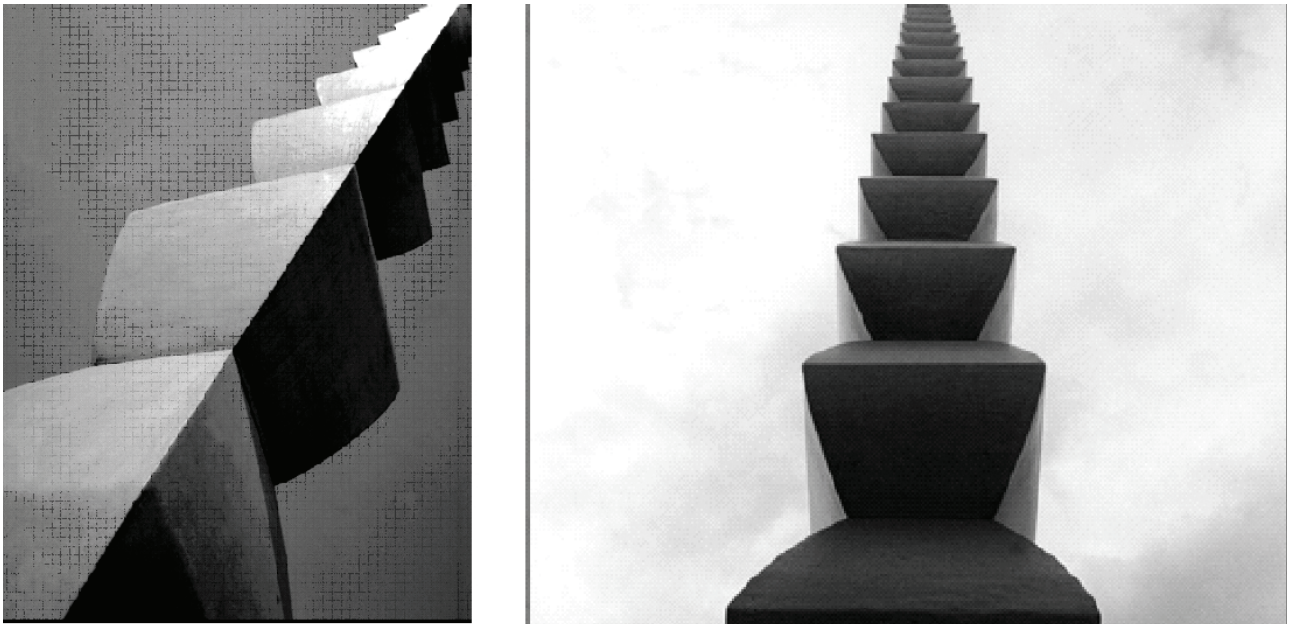
First of all, Brancusi's columns respect a factor of slenderness. Relative to their height, there are known columns with 3 module and two half modules, 6 modules and two half modules, 9 modules and two half modules. The Targu Jiu column has 15 modules and two half modules. Despite the fact that it is not known a column with 12 modules, we can consider that Brancusi followed this rule of harmony which lead him to a perfect slenderness of the columns. Relative to the dimensions of the module, again Stefan Georgescu Gorjan gave us useful information. The modules respect the law of harmonious proportions. The dimensions of the module are in proportions of 1:2:4, that means 45 cm: 90 cm:180 cm.





Courtesy of Sorana Georgescu Gorjan

It is then better to see Brancusi's work as a continous balance between rational and shadowed zone. It makes appeal to the <<mother-ideas>>, to the goethean conception of primary plant, to the archetype.
Are their forms geometric or biotic then?
Jack Burnham divides methodically the modern sculpture in geometric and vitalist and says the following about Brancusi:
<<The history of modern sculpture from Rodin to Moore is the history of an absurd conflict between the two principles: vitalism and geometry-fight brought sometimes inside the artist's conscience. This was the case of Brancusi>> and Burham mentions a number of Brancusi's works being <<in the same time geometric and biotic>>.
Referring to this, another critic says: <<There is a geometry which does not kill life, but on the contrary is integrating it. It is the geometry of decorative art of Romanians. In this way, the geometry of Romanian folklore is organic as the objects created by Brancusi.

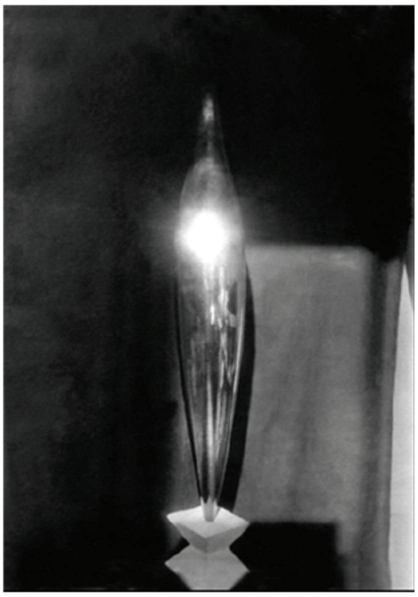
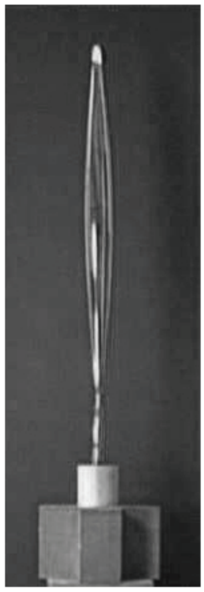


Courtesy of ITC

The slender Endless Column from Targu Jiu pointing to the sky

2. The second for us relevant dilemma in our quest to discover interferences with science is the one related to the realism and abstractionism of his art.
Talking about abstract art, it is impossible not to observe that Brancusi's art enlarges and modernizes the frames of Art itself going to its object. If, speaking in a general manner, it is true that "science, in its cognitive approach has the tendency of eliminating the subject and subjectivity considering as factors of perturbation and of incertitude, while art in its relation object-subject not only intends not to eliminate the subject, but on the contrary, fights to enforce it, because, its motivation stays in discovering the subject, knowing it in its relation with the object". When we speak about Brancusi, we observe that his art has a greater aim, that of knowing artistically the World-which includes Man, when he reoccupies its natural place. In this way, the spirit of Art can be very nearly to the spirit of Science despite the different ways it uses to reach this goal.

If we regard the new style of modern scientific thinking, imposed by the revolutionary conquests of Physics of the beginning of the 20th century we have the revelation that they are in synchronism with Brancusi's search for essential. The genius science of nature researchers tried to put in evidence other essences of the world, not less generators of "unforeseeable" and paradox. In this way, despite the fact that the nature and its corps give us the impression of stability, of static, their interior is dynamic, unstable, agitated, something like a swarming of the born matter, a forever becoming. The apparent real, called by philosophers "sensible" falling under senses incidence, is born from the atoms, from the quantum and is naturally for the two worlds to be in "correspondence" [Max Born, La Physique atomique, En roumain, Ed Ştiinţifică, Bucarest 1973, p.151].
Being infinitely big, our world is born from the quantic, from essential, and here is a process marked by a quantum leap.
From here we can say, using the analogy, that brancusian's work being „essential" (a nuclear condensed visual representation of our world)-presents a direct correspondence with real. If the Miraculous Bird (Maiastra) continues to be a natural bird, little by little, it is melting (by a qualitative jump) into its (simple) symbol, with "The Bird in Space"

The Birds evolution from form to idea.

"The Maiastra-writes Serge Fauchereau-it is reduced to the essential; there are no more feathers, feet, wings and open beak…[…] the bird of Brancusi evolves toward a gradually greater abstraction, to the point when it is no more than flight and song".
Following his quest for essential during his entire life, in his series of birds, we notice Brancusi's words "…It is not a bird that I want to represent: I want to present its quality in itself, the flight, the soaring. I doubt that I will do it…. Meanwhile the days, the years pass. And I am still looking for a definitive form."
Similarly, not being directly accessible to the sensible, the quantum world pf physics cannot be perceived instantly. We can arrive by a complementary manner; that means adding to the first

impressions (observations) the next, recorded under different angle. In this way appears the exhaustive image of the micro object.
The analogy with Brancusi's way of discovering the essential and the truth in sculpture it is evident.

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SURFACE MEASURES

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Abstract. Paper is an attempt to enable confrontation of the concepts of measures and measurement strategies in fine art and in mathematics. Short analysis of the surface measure definitions and its application within the two environments is presented, together with first efforts of finding their common features and differences.

Key words. Measure definitions, mathematics and art, modelling of surfaces

Finding a measure of any kind of unknown, we get the feeling that we have reached the first level of understanding the matter.

1 Mathematics and Fine Arts

Mathematical theorem is essentially an exactly formulated truth, which is generalisation of a certain precisely repeating phenomena. It is a description of observed behaviour feature of an object, presented in abstract form, and explaining simultaneously also the circumstances, under which the studied phenomena appear. These circumstances are assumptions of the preposition. Mathematical hypothesis is a coded message, understandable only to experts, but it is subdued to strict laws of logics and its truth must be exactly verified by means of logical rules. It can be well understandable also to a layman, when the main idea of hypothesis is explained and illustrated, and its truth can be visualised on a particular example, a geometric model preferably.

Geometric interpretation helps to show the human face of mathematics. It serves as platform for science and art matching, as space for overlapping of artistic and mathematical abstraction. Anyhow, their common intersection is not only usage of geometric forms in fine arts. Into a much greater extent it can be found on the abstract level, in searching for a description of natural relations, principles and rules of our Universe.

Artist is also expressing and describing observed reality and truth by creating his masterpiece, and using this artistic approach he presents them in abstract form too, but "in his way". His observations are reflected in his mind and intimate emotional world, receiving thus remarkable emotional mettle and generating the most remarkable power of the created piece of art.

Fine art and mathematical abstract worlds are two parallels dealing with understanding of our world and its laws, each from its own point of view and by its own means of expressions. Maths and Art represent two "on the edge" attitudes of describing the world's truth. They seem to be the most imaginative, but in spite if this common feature, also the most descent achievements of human mind. Anyhow, their naturally different character in attitude and work processes is more unifying then separating them, as these two worlds are meeting on a higher abstract level, not clearly apparent on the first sight. Mathematical preposition and fine art work are two different representations of the observed reality, two abstractions appearing seemingly on the opposite poles of human mind, two different illustrations of truth and effort to comprise it and present in an abstract way. These two representations of "world truths" should perfectly complement each other.

2 Definitions of Measures

Measuring is one of the pretty natural human activities, when attempting to evaluate whatever we might come across, objects we are meeting, activities we are performing and their consequences and results, our relations and their manifestation, but at the first place, space, we are obliged to live or work in. Finding a measure of any kind of these unknown things, we get the feeling that we have reached the first level of understanding the matter.

Concept of measure is a well defined phenomenon in mathematics and measure theory represents a complex field of mathematical theory on a highly abstract level, with verified geometric models illustrating on particular examples and calculations of different measures as length, area, or volume of geometric figures.

What exactly is a measure in mathematics?

Let X be a non-empty set and M a non-empty system of its subsets. M is called σ-algebra, if it is closed with respect to all complements and countable unions, i.e.

$$A \in M \Rightarrow X \setminus A \in M, \quad A_n \in M \Rightarrow \bigcup_{i=1}^{\infty} A_n \in M .$$

If X is a topological space, then elements of σ-algebra B generated by system of all opened sets in X are called Borel sets.

Let M be a σ-algebra on a set X. Function $\mu : M \to [0,+\infty]$ is called (non-negative) measure, if it is σ-additive, i.e.

$$\mu\left(\bigcup_{i=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

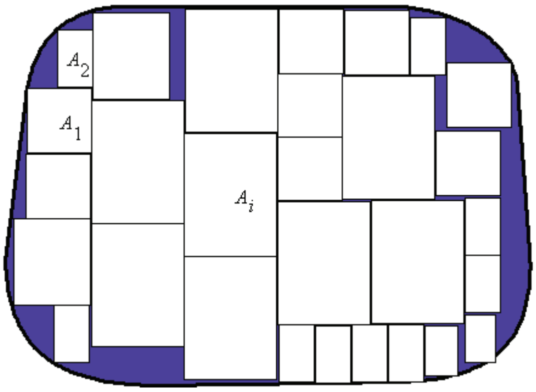
for any arbitrary system of mutually disjoint sets, while $\{A_n\}_{n=1}^{\infty} \subset M$ and $\mu(\Box) = 0$. Elements of M are called measurable sets, while triple (X, M, μ) is called space with a measure.

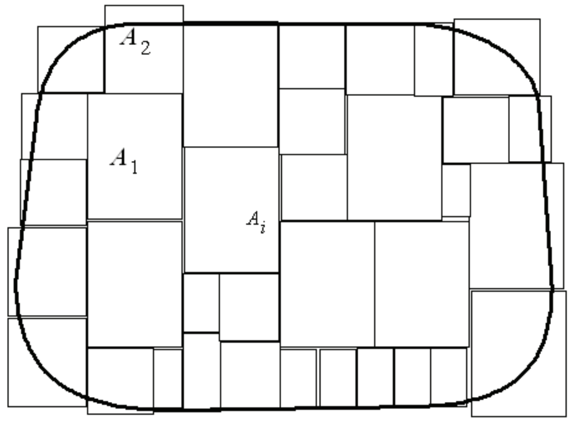
Let $A \subset E_n$, and let $S = \{I_1, ..., I_m\}$ is a system of closed intervals, in which no two different intervals have common interior points. Then we define $|A| = 0$, if set A does not contain interval, and if it contains some interval, then we define

$$|A|_* = \sup \sum_{k=1}^{m} |I_k|, \quad \bigcup_{k=1}^{m} I_k \subset A, \quad |A|^* = \inf \sum_{k=1}^{m} |I_k|, \quad A \subset \bigcup_{k=1}^{m} I_k .$$

For any bounded set A the following relation is true: $|A|_* \le |A|^*$.

Number $|A|_*$ is the inner Jordan measure of the set A, number $|A|^*$ is the outer Jordan measure of the set A. Bounded set A is measurable in the Jordan sense, if $|A|_* = |A|^* = |A|$.





Inner Jordan measure of set A - $|A|_*$ Outer Jordan measure of set A - $|A|^*$

Number $|A| \neq 0$ is called the Jordan measure of set, it is represented as

$$|A| = \int_A \chi_A(x)dx = \int\int_A \ldots \int 1.dx_1 \ldots dx_n \,,$$

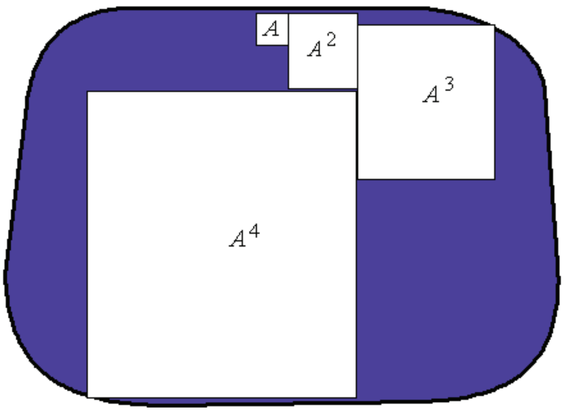
where $\chi_A(x)$ is a characteristic function such, that

$$\chi_A(x) = 1 \text{ for } x \in A,\ \chi_A(x) = 0 \text{ for } x \in E_n - A.$$

If (X, ρ) is a metric space and p is a positive number, then for any $A \subset X$ holds

$$\mu_p^*(A) = \sup_{\varepsilon>0} \inf \left\{ \sum_{n=1}^{\infty} \left(d\left(A_n\right) \right)^p ; A = \bigcup_{n=1}^{\infty} A_n, d\left(A_n\right) < \varepsilon, n = 1, 2, \ldots \right\}.$$

Outer measure μ_p^* is called the Hausdorff p-dimensional measure.



Hausdorff p-dimensional measure of set A

Measure is a certain limit that can be reached in-between two different measurements based on similar, anyhow not completely equal principles. Limit, in which inner and outer measures meet and overlap finding an agreement reflected in the real value, must not exist necessarily in all cases.

Measure of an empty space is zero. Only the measurement units, objects filling the space, determine its non-zero measure. Consequently, these objects themselves, i.e. a kind of items representing our expectations, define the final measure value. The preconditions – defined assumptions influence a sort of the measure quality or type. Mathematics states these clearly by determining assumptions, initial conditions related to the specific desired type of measure, which are generally agreed and accepted and they serve as certain guide for related measurements. We can speak about Jordan, Hausdorff, Borel, Radon, Dirac, Carathéodory or Lebesgue measures, about regular, complete, finite, σ-finite, arithmetic, and many other measures, which differ in the measuring approach and the measure quality for which they are applied.

Let (X, M, μ) be a space with measure and let simultaneously X be a topological space. Measure μ is called

- Borel measure, if M contains all Borel sets, so $B \subset M$
- regular, if it is a Borel set and for all $A \in M$ holds
  $\mu(A) = \inf\{\mu(U); A \subset U, U \text{ opened}\}$
  $\mu(A) = \sup\{\mu(K); K \subset A, K \text{ compact}\}$
- (non-negative) Radon, if it is regular and for any compact $K \subset X$ holds $\mu(K) < \infty$
- complete, if for any $A \in M$ holds: if $\mu(A) = 0$ and $B \subset A$, then $B \in M$
- finite (or bounded), if $\mu(X) < \infty$
- σ-finite, if there exists $A_n \in M$ such, that $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$.

1. Let X be and arbitrary set and $M = \exp(X)$ is system of all subsets in X.
   Let $\mu(A) = |A|$, which is the number of elements of a finite set A, otherwise $\mu(A) = \infty$.
   Then μ is the arithmetic measure of X.
2. Let $x \in X$, $M = \exp(X)$ and setting $\mu(A) = 1$ if $x \in A$, $\mu(A) = 0$ if $x \notin A$.
   Then μ is the Dirac measure located in point x.
3. Let $X = \mathbf{R}^n$. Then there exists a unique total Radon measure, translationally invariant on X and such, that for $A = (a_1, b_1) \text{ x } (a_2, b_2) \text{ x } \dots \text{ x } (a_n, b_n)$ holds
   $$\mu(A) = \prod_{i=1}^{n} (b_i - a_i).$$
   This measure is the Lebesgue measure in $\mathbf{R}^n$. Set A $\subset \mathbf{R}^n$ is measurable in the Lebesgue sense iff it cannot be written in the form $B \cup N$, where B is the Borel set and N is subset of some Borel set with zero measure. Further holds $B \subset\neq M \subset\neq \exp(\mathbf{R}^n)$.
   For any countable set $A \subset \mathbf{R}^n$ holds $A \in M$, $\mu(A) = 0$.
4. Let $\varphi: \mathbf{R} \to \mathbf{R}$ be a non-decreasing function. Let $a < b$, setting
   $$\mu([a,b)) = \lim_{x \to b-} \varphi(x) - \lim_{x \to a+} \varphi(x).$$
   Function μ can be then extended to a complete Radon measure, and it is called the Lebesgue-Stieltjes measure. Specially for $\varphi(x) = x$ we receive the Lebesgue measure.
5. Let X be a infinitely dimensional separable Hilbert space and let μ be a translationally invariant measure on X and such, that $\mu(B_1) < \infty$, where B_1 be a unit ball in X.
   Then it must hold the following equality, $\mu(B_1) = 0$.
6. Let (X, M, μ) be a space with measure and let $\emptyset \neq A \in M$. Let us set $M_A = \{B \in M; B \subset A\}$ and let μ_A be the restriction of measure μ on M_A.

Then (A, M_A, μ_A) is again a space with measure. If μ is a Borel (or regular, Radon, σ-finite, complete) measure, then the same property holds also for the measure μ_A .

7. Function $\mu*$: $\exp(X) \rightarrow [0, \infty]$ is called outer measure, if: $\mu* = 0$ and $\mu*(A) \le \sum_{n=1}^{\infty} \mu*(A_n)$ *for* $A \subset \bigcup_{n=1}^{\infty} A_n$.

For an arbitrary outer measure $\mu*$ we can define set

$$M = \{A \subset X; \mu*(T) = \mu*(T \cap A) + \mu*(T / A) \text{pre } \forall T \subset A\}$$

Then it holds that M is a σ-algebra and restriction $\mu*$ on M is a complete measure called the Carathéodory measure.

In the same way there can be constructed also the Lebesgue measure, if for A $\subset \mathbf{R}^n$ we set

$$\mu*(A) = \inf\left\{\sum_{k=1}^{\infty} \mu(I_k); A \subset \bigcup I_k\right\},$$

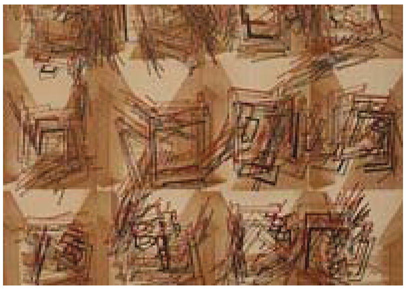
where I_k are sets in the form $(a_1^k, b_1^k) \times \cdots \times (a_n^k, b_n^k)$ and $\mu(I_k) = \prod_{i=1}^{n} (b_i^k - a_i^k)$. Function $\mu*$ is in that case called the Lebesgue outer measure.

Do similar laws exist also in art? Is the imagination space of an artist completely unlimited? Are there certain obstacles, defined by authors themselves, imposed maybe intuitively due to culture development of mankind, that are influencing creative work on the masterpiece, in the sense of its pre-defined inner desirable measure? Are the different measures reflected in different artistic styles that are approaching different aims and looking for different qualities?

Artists – painters might apply a principle similar to mathematical definition of a measure, when filling the empty 2 dimensional space represented by a piece of canvas, in order to change it into a painting. According to his/her own strategy he/she starts to distribute separate objects in there, creating thus a unique composition reflecting his/her own inner criteria posed on a surface, i.e. painting measure. Authors emotional world and creative potential influence quality of his/her measurements, it has a strong impact on the whole composition, as it determines the inner measure of the created piece of art. Outer measure of a painting is perceived by the observers. It comes out of the 2D space of the painting and it is independent on the viewer in certain sense. Although this outer measure can be generally perceived differently on the individual level, it is defined and influenced by well formed rules of our culture heritage applicable in fine arts, painting, or sculpture and design, which are developing together with the mankind knowledge and culture.

Ester Šimerová Martinčeková

Svetlana Ilavská



Lýdia Jergušová-Vydarená

It is very interesting and inspiring to compare, how a fine artist would react to an abstract formalised mathematical proposition from the measure theory coded in mathematical symbolism, and how mathematician could understand his artistic illustration of the abstract mathematical imaginations in the fine art shaping the intuitive vision of artist.

We know very little about general connections between different abilities of human brain to perceive pieces of art and to understand mathematical formulas. Is the first one not an illustrative emotional visualisation and model of the latter one? Are they interfering and cooperating, or neglecting each other passing in parallels? How can we make these two abstract abilities to respect or influence each other, and utilise thus the synergetic effect of imaginative human abilities resulting from this cooperation? Do there exist certain general rules, which human beings apply when measuring space by dividing it into smaller parts that actually live in there, populating it make the space alive and are the carriers of interrelation with the exterior viewer? Are all these measurement processes intuitive, inborn or natural, or are they only side effects of our material obsession and desire to evaluate all our possessions?

Exhibition "Surface measures", which was born in cooperation of the artistic association VEKTORYart and Aplimat 2009 international conference on applied mathematics, is one of the first attempts to find answers to some of these interesting questions.

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STATISTICAL HYPOTHESIS TESTING IN EXCEL

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Abstract. The paper deals with statistical hypothesis testing in Excel. At the Faculty of Education at Trnava University a huge amount of students needs the basic knowledge of mathematical statistics to test hypothesis in their seminar or bachelor works and dissertations. The authors describe easily operated excel sheets which enable students to test their hypothesis. An e-learning course which the authors designed to integrate the sheets and videos presenting how to operate them is characterized, too.

Key words. Statistical hypothesis testing, Excel, ICT, e-learning.

Mathematics Subject Classification: 97U50

1 Introduction

The importance of modern information and communication technologies has rapidly grown in recent years. These technologies have penetrated into all spheres of human activities. Nowadays, it is quite easy to operate a huge amount of data using modern technologies, for example Excel which contains a lot of functions from different categories. Some calculations in mathematical statistics, which lasted for a long time in past because of complicated formulas, can be made in few seconds using modern spreadsheets. That is why the methods of teaching statistical hypothesis testing have changed significantly. It is not necessary to spend hours by routine calculations. On the other hand, it is necessary to obtain special abilities to operate the spreadsheets.

The faculty of Education, Trnava University, prepares students who will work as teachers at nursery, primary and secondary schools or as educators in social institutions. These students not only have to learn how to teach, but they also have to learn how to design experiments, how to collect data from these experiments, operate and analyse them in an efficient way. In many cases these experiments are part of seminar or bachelor works and dissertations. For the students it is not important to understand the statistical methods in details, neither do the students need to master all statistical functions in Excel. The students just need to know how to choose the most appropriate method to operate their data from experiment and to test their hypothesis. That is why we have prepared Excel sheets for statistical tests that are operated easily and we teach our students how to use them at the subject *Basics of statistics*. It is important to notice that our students are not specialists neither at the field of ICT nor operating computers. They are ordinary users. However, they are taught how to work effectively with ICT in the subject *ICT in education*.

2 Statistical hypothesis testing in Excel

As we mentioned above, there is a lot of students in our faculty who have to test hypothesis of their experiments. These students are not specialists in mathematical statistics, they just need to choose an appropriate statistical test and implement it in Excel as simply as possible. That is why it is sufficient to teach them how to operate the excel sheets which we created for them and how to choose the appropriate sheet to solve their problem.

The majority of problems that our students solve when they need to test their hypothesis can be sorted into the following groups.

1. The students make an experiment on a sample of people and they need to compare the selected sample with the whole population or a part of population.
2. The students make an experiment on a sample of two groups. The first group, which serves as a control group, is taught by a classical method. The second group, which serves as an experimental group, is taught by an experimental method. The students need to compare the knowledge or abilities of selected groups before the experiment and after the experiment.
3. The students make an experiment on a sample of people and they need to compare the knowledge or abilities of these people before and after the experiment.
4. The students make an experiment on a sample of people under two different conditions and they need to compare the results.
5. The students make an experiment on a sample of people and they need to formulate conclusions for the whole population or a part of population.
6. The students make a questionaire on a sample of people and they need to formulate conclusions for the whole population or a part of population.
7. The students make a questionaire on a sample of people and they need to compare the answers of two different groups, for example to compare the answers of men with the answers of women.
8. The students make a questionaire on a sample of people and they need to know whether there is some correlation between answers on two different questions.
9. The students explore a correlation between two variables and they need to determine a confidence interval for correlation coefficient.

For each group we designed excel sheets that help students to test their hypothesis. We tried to prepare the sheets as simply as possible, so in majority of cases students need just to choose the appropriate sheet, put their own values and add or remove some cells. The students usually do not have to master the excel functions that are necessary to solve the problem. The following table shows which tests we used to solve the problem from the groups above.

Group no.	Statistical Tests
1	t-test
2	D'Agostino's normality test, F-test and t-test when the population is assumed to be normally distributed; the Mann-Whitney U test for assessing whether two samples of observations come from the same distribution; the Kolmogorov–Smirnov test
3 and 4	t-test for related samples
5, 6 and 9	computing confidence interval
7 and 8	tests for discrete variables

Table 1

3 E-learning course

We decided not only to create excel sheets for different statistical tests, but our aim was to integrate these sheets into a complex and homogenous material. We have already designed five e-learning courses from different parts of mathematics and integrated them into the learning management system *EKP*^{TM} of our university. More information about the LMS can be found in [7]. We have been successfully using the LMS and e-learning courses in teaching process at our department since 2003 and our experience with it is strongly positive, as well as experience, reactions and results of our students. So we have decided to design e-learning course *Statistical Hypothesis Testing* and integrated it into the LMS. The course is available for public at the address http://elearning.truni.sk/, where you can use *guest* as both login and password. The course keeps both *AICC* and *SCORM* standards, so it can be used also in LMS of other university. We have also prepared off-line version of the course, which you can obtain from the authors of the paper.

As the course is designed for students who are not specialists neither in mathematics nor statistics, we adapted the content of the course. The course, which consists of ten modules, contains the minimum of theory and a lot of solved problems closely related to the experiments of our students. As the students are not specialists in operating ICT, we have also created 16 videos which show how to work with the sheets from the course.

In this academic year more than 100 students are studying the course. The aim of the course is not to substitute the teacher, but to serve as an effective study material in combination with classical lectures. However, using the course enable to reduce the number of contact hours from 24 to 8 without negative impact on the knowlegde of students (see [5] or [6]).

The preparation of an e-learning course is much more difficult than the preparation of a classical textbook. We tried to utilize elements which can be used in electronic materials, but they cannot be used in printed materials, such as videos or interactivity. Moreover, we tried to respect the knowledge from cognitive psychology about the way how people learn from electronic materials (see for example [1], [2], and [3]). It is ideal if methods used in the course lead into integration of texts and graphics into existing structure of knowlegde in the long-term memory of students. To reach it, we focus the attention of students on important information in the course, we tried to use effectively the limited capacity of working memory, integrate information from visual channel into structures in the long-term memory, use methods that enable to access the knowlegde and abilities from the long-term memory when necessary, provide support for student with less developped metacognitive skills, etc. Unfortunately, we were not able to add sound into the course. It is also important that students need to understand that it is useless to read the content of the course in a passive way. To master the content of the course, the students should apply the obtained knowledge in the examples from the course. So the students have to work with excel sheets when they are studying the course.

4 Conclusion

Modern information and communication technologies can make human activities more efficient, especially in that case when it is necessary to operate a huge amount of information. In the paper we described a method how to save a lot of hours spent by routine calculations using functions in Excel to test statistical hypothesis, which represents a concrete example of efficient utilization of ICT.

It is necessary to use ICT in educational process, as they can make the process more effective and interesting to students. We believe that the faculties which prepare future mathematics teachers will cooperate in preparation of e-learning courses and materials and that whole mathematics curriculum will be covered by these materials. The example of such a co-operation is between departments of mathematics at Trnava University and Matej Bel University in Banská Bystrica. Both departments use learning management system to educate their students. Moreover, they have prepared tens of e-learning courses from the field of mathematics. More information can be found in [4].

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ACTIVE STUDENTS PARTICIPATION IN THE PROCESS OF STATISTICS TEACHING

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Abstract. The author deals in this paper with factors of motivation and active student's participation in the process of the teaching of the basic course of statistics at the Institute of managerial systems in Poprad – Branch of the Faculty of Economics, M. Bel University, Banská Bystrica. Active participation of students was realized during the summer semester of the academic year 2007/2008 and in the winter semester of the academic year 2008/2009. The active student's participation consist of the active preparing, presenting and lecturing some chosen simple parts at the lessons of Statistics. These student's presentations were corrected and modified by teacher some days before and during their presentations at the lessons, as well. In the paper the preliminary results are presented, which were obtained from the analysis of the questionnaires answered by students. The aim of the questionnaire was to identify and find some weak and/or strong aspects of this method of education.

Key words. statistics, methodology of education, teaching of statistics

Mathematics Subject Classification: Primary 62-01, Secondary 97C80

1 Introduction

A lecture of Prof. S. Knypstra from the University of Groningen, The Netherlands, which was communicated at the 10th International Scientific Conference „Applications of Mathematics and Statistics in Economy“ in 2007 [1] and the author's private discussion with him, as well, was the principal impulse for starting an active participation of students in the process of statistics teaching. An interview with Prof. R. Haňka, a member of the Scientific Board at the University of Economics, Prague, Czech Republic, working now at the Wolfson College, University of Cambridge, Great Britain, published in journal Trend [2], was a supporting impulse for starting this form of teaching for the author of this paper. In the interview Professor Haňka says: "The main role of a university is to teach students thinking. Not to teach facts. Here (in Czech and Slovak Republic) people still consider university to be a next step after a grammar or secondary school". He continues: "I tell to my students the source where they can find the subject of study and they will study it themselves. We do not lecture for teaching. We lecture to motivate students for their further individual studies. Students and lecturers discuss interesting problems drinking coffee, solve together tasks and problems they have met during their individual study. They discuss together, solve tasks and debate on problems stimulating and creatively.”

The aim of this paper is to present the author’s personal „know-how“, experiences and observations he gained applying this method of teaching in his teaching practice and share them with colleagues lecturing statistics. A new method of statistics teaching based on communication published in [1], a little modified on the author’s own conditions, was introduced in his practice at the Institute of Managerial Systems in Poprad, The Faculty of Economics, Matej Bel University, Banská Bystrica, Slovakia. It started in the summer semester of the academic year 2007/08 (4th term) and still is in course in the winter semester of the academic year 2008/09 (3rd term). Preliminary results, teacher’s experiences and students looking at it, opinions and positions are presented in this paper.

2 Active participation of students in the process of statistics teaching in practice

Institute of Managerial Systems in Poprad is a department of the Faculty of Economics, Matej Bel University, Banská Bystrica, where students study fully accredited study of economics in the study programme 3.3.16 Corporate Economics and Management in two levels: bachelor study program Business Economics and Management and master study program Economics and Management of Small and Medium Enterprises, both programs in presented (so called daily) and external form. In the third and fourth term students study subject Statistics, which was split into two parts: Statistics 1 in the third term (winter semester) and Statistics 2 in the fourth term (summer semester) of the second year of study. Detail information about these subjects (structure, contents, methods of lecturing, etc.) can be found in the paper of Kaščáková et al. [3].

Some parts of statistics, which were studied by students themselves and then actively presented by them during the lectures of statistics were, for example, these ones:

- Bayes´ theorem (after understanding the basic terms of probability),
- skewnes and kurtosis of distribution (after understanding data grouping, means and moments),
- measures of concentration and Gini coefficient of concentration (after understanding relative frequencies and cumulative relative frequencies used in data grouping),
- covariance (after understanding basic terms and measures of variability),
- Moivre-Laplace local and integral theorem (after understanding discrete and continuous variable distributions),
- large-sample statistical test for the population proportion (after understanding of basic notions of statistical testing and the test for the population mean),
- Spearman rank correlation coefficient (after understanding Pearson correlation coefficient), etc.

The process of active participation of students was realized in the following way: The teacher encouraged a small group of two or three students every week to study one of the proposed themes. Then teacher and those students organized a small working meeting where they discussed all practical aspects of their future presentation, e.g. contents, recommended literature, another sources of information about the study problem, form of presentation, possibilities to use new computer technologies during presentation, etc. On the first working meeting the time for the next working meeting was established, which was realized several days (usually three or two) before presentation on the lecture. During this second working meeting students informed teacher about their problems which they had met during their self-study and had to overwhelm in practical preparing the presentation, about some problems with right understanding or misunderstanding concerned the theory of statistics, examples, etc. Then students presented their contribution on this working meeting for the teacher. Every student presented his part separately, but his colleagues could help him during presentation. At the end of the presentation the teacher expressed his opinion on the presentation and advised them to improve or correct some parts of the presentation.

Students used the Power Point presentation mostly as a form of presentation, some of them presented their parts written in Word editor and some of them used chalk and blackboard. The last form of presentation was usually used when they calculated examples to demonstrate statistical theory in practice on problems from real economic life. The lecture in the classroom began with lecturing the statistical problem by teacher. In a certain part of his lecture his invited the group of chosen students to him and then they presented their own parts of presentation. During the students´ presentation teacher sometimes explained difficult parts more precisely and deeper or corrected some mistakes, misunderstandings or discrepancies, if any. The teacher was also involved into the students presentations by his questions addressed to the students or to the audience. At the end of the lesson teacher summarized the facts which were presented by him and by students.

3 Preliminary results

The aim of the author of this paper and the teacher of statistics why to introduce this form of teaching can be briefly summarized in the following:

- to educate students to use and work with scientific sources and literature effectively,
- to educate students to learn new facts individually, to obtain knowledge by self-study,
- to teach students to present their own thoughts, opinions in front of audience,
- to eliminate or minimize fear and doubts of students from public presentation, to gain self-importance and to encourage students for discussion, argumentation, to argue out their ideas, to be able answer questions of audience,
- to teach students to express their opinions and ideas using modern methods and forms of presentations using new techniques of communication and computer technologies,
- to help students to understand, that to obtain new information just by passive listening the lectures is not the best form of education, but, on the other hand, it is more joy-full to gain new information putting his own hands on it, to feel certain satisfaction for his own results.

The next aim of this paper is to evaluate the preliminary results of this educational experiment. It is important and valuable to identify weak and strong aspects of this method, to strengthen and support strong ones and to improve or correct weaknesses, as well. That is why we realized research consisted of spreading a questionnaire among students with the aim to know opinions of students on problems, difficulties, which they had met during preparing their presentations with. Not only those ones who actively presented their contributions on lessons took part on this research, but also students, who took part on lessons just as an audience in the classroom. Participation on the research was voluntary and anonymous. In the summer semester (the 4th term) of the academic year 2007/08 59 students were studying in the presented form of study (excluding students with individual plan of study), in the winter semester (the 3rd term) of the academic year 2008/09 this amounted 46 students. On the research 56 students together (53 % of all students) took part, while 25 students actively participated on the lecturing statistics by their own presentations, 31 students did not take part in the presentations.

The questionnaire consisted of seven questions, while the first question was devoted for separating students into two parts: firstly for whose ones, who actively presented their contributions and secondly for whose ones who did not take active part in education process. The next four questions were devoted to the students who actively took part in the teaching and the last two questions were devoted for all students without any differentiation. Questions of the questionnaire were constructed in the closed form with possibility to chose several answers from 4 up to 15 items. One item (the last one) of these questions was open, where students could express by their own words their individual opinions, suggestions, proposals and feelings concerning the arbitrary aspect of this form of statistics teaching. In all questions except the first one respondent could chose not only one answer but several ones of them. Because of the research was realized only on the small sample of students and the experiment with new form of teaching was realized in a very short time interval, just during two semesters, the results given here should be considered just as preliminary results and conclusions.

3.1 Students actively participated in the process of teaching – the results of questionnaire

The first queston for students actively participated in the process of teaching was devoted to the forms and ways of preparing presentations by students. Students clearly declared that they preferred the method of learning from printed books, text-books, some specialized printed study materials, which they obtained, bought or borrowed from libraries, friends, students, bookshops, etc. (34 % of all answers), while the second place was assigned to the sources found on internet (26 % answers).

In the second question, which was concerned to the problems which students met during preparing their presentations, the answers were distributed more uniformly than in previous case. The most frequent answers were these ones: lack of free time for preparing the presentation (25 %), absence of suitable study literature and other sources for studying (25 %), weak cooperation among students in their groups during the period of preparing presentation (19 %). It is interesting, that the students who as the most important problem considered the impossibility to understand new facts and information on statistics, they all stated the problems lay in the fact, that they did not have solid basic knowledge of statistics (or mathematics) which had been communicated during previous lectures.

The next question was focused on identifying positive aspects of this form of self-study system of education. The highest percentage of answers (31 %) was devoted to fact which reflected the pragmatic position of students: they considered as the most important advantage of this form of teaching the fact that they obtained extra bonus in the form of positive points, which were included into the final evaluation of examination. This answer were followed by these ones: I was forced to study new subject matter by myself (25 %), I understood better and deeper those subject matters which I had to study by myself (24 %).

The last question was focused on negative aspects connected with self-study and presentations. Fear, doubts and low self-importance was considered as the most frequent obstacle (38 %) connected with public presentations of students´ opinions among their students-colleagues. This fact was one of them which the author of this experiment wanted to eliminate from students positions.

3.2 Students actively participated and not-participated in the process of teaching – the results of questionnaire

The sixth question was devoted to the both groups of students, i.e. for actively participated and for activerly not-participated, and it asked: „Express your personal opinion on presented form of statistics teaching. Indicate those items which do you agree with.“. Student could indicate as many items as he/she wanted. The answers were analyzed separately for both groups of students. Graphic presentation of the answers are presented in Figure 1.

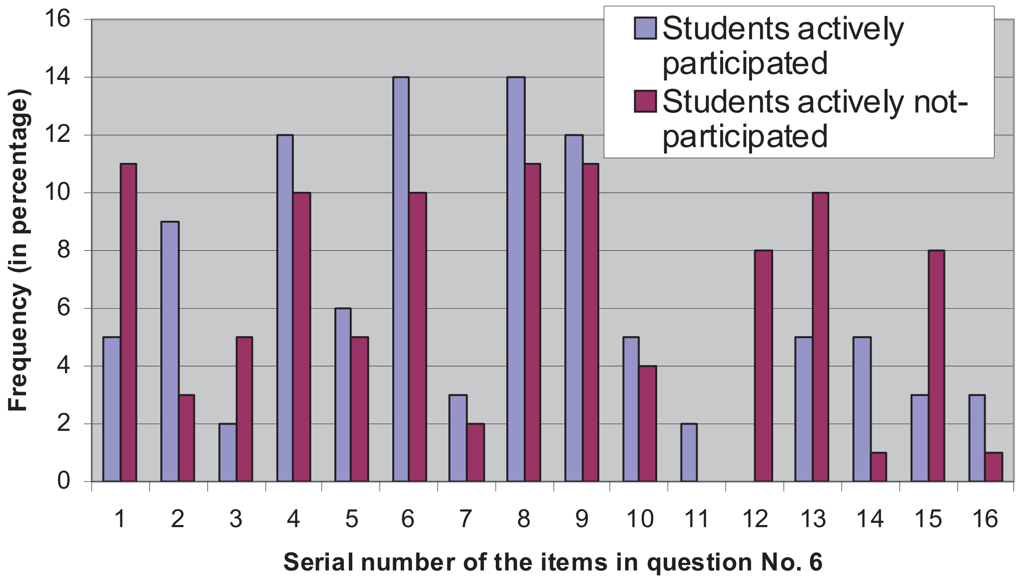

Fig. 1. Graphic presentatnion of the frequency distribution (in percentage) of the items in the question of No. 6 of the questionnaire. Frequency distribution is presented separately for two groups of students. Details in the text.

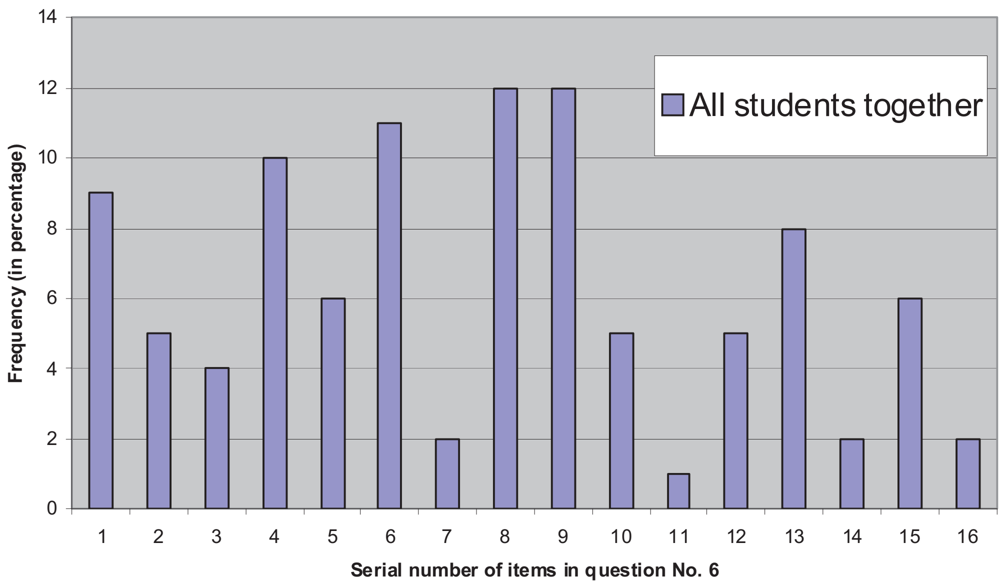

Fig. 2. Graphic presentatnion of the frequency distribution (in percentage) of the items in the question of No. 6 of the questionnaire. Frequency distribution is presented together for both groups of students. Details in the text.

Using the same way of presentation as in Figure 1 the results for all students – actively participated and actively not-participated – are presented in Figure 2. Here are the items which students could indicate as their answers on the question No. 6:

- Preparing the open presentation is enormous time-consuming problem for students.
- This way of study was for students, who actively participated in teaching statistics, very encouraging and a good impulse for further self-study.
- The students, who did not take part actively in teaching, had to devote much more time for studying presented subject matter, because they did not understand the presentation presented by their colleagues.
- Student has serious problems and difficulties to understand new subject matter adequately because of its severity and difficult language used in books and text-books.
- There are serious problems to find suitable literature, in which the topics are clearly explained.
- Students taking part in teaching did not know clearly to explain their thoughts, ideas; they did not know to present their ideas in proper way.
- I understood the presented parts of statistics by students better than from teacher.
- I understood the presented parts of statistics by students worse than from teacher.
- I had to study the presented parts of statistics by students again by myself.
- This form of teaching was an impluse for my absence at the statistics lectures.
- This form of teaching was a way by which the lectures of statistics had become better and more interesting.
- I consider this form of teaching as a way how teacher wants to avoid of his work and lecturing.
- I consider this form of teaching as a way how teacher wants to shift his responsibility for teaching on students.
- My opinion was changing in this way in the course of semester: from negative one at the beginning to positive one at the end of semester and this way of teaching is suitable for me now.
- My opinion was changing in this way in the course of semester: from positive one at the beginning to negative one at the end of semester and this way of teaching is not suitable for me now.
- Write your opinion on this way of teaching which you could not find in the items above.

It can be clearly seen from Figure 1, that the frequency of occurrences of individual items in the question No. 6 is not uniform. However, we can say without exact statistical analysis, that students who actively participated in teaching preferred items with serial numbers 2, 4, 6, 8 and 14, while higher preferences were assigned to the items with serial numbers 1, 3, 12, 13 and 15 by students actively not-participated in teaching. We can conclude from this, that for the group of students referring on lectures this form of teaching and learning encouraged in their self-study activities, on one side, but they met serious problems with literature, time for study, with understanding the subject matter properly and with team work in groups of students, on the other side.

The students who did not take part in teaching saw great problems in time factor and they understood this way of teaching as a form by which the teacher wanted to avoid his responsibility and to shift it from him to students. On the other hand, the author can say, that this form of teaching is for teacher much more difficult and brings him much more problems (e.g. from the point of view of time) in comparison with current lecturing.

We can say (see Figure 2) the most important problems connected with this form of teaching are the following ones: enormous time loading of students (item 1), problems and difficulties to understand new topics of the subject matter (4), problems with proper presentation of topics by students (6), serious difficulties to understand presented subject matter by students in comparison with those presented by teacher (8) and way of studying by oneself. (9).

The seventh question was focused on statistics teaching in the future. Students could express their proposals, suggestions, and at last, whether they wanted to be continued in this form of teaching or not. They could indicate some of these items:

- I prefer to be continued in this form of teaching without serious changes and improvements (1).
- I prefer to be continued in this form of teaching but after some corrections and improvements (2).
- I don't want to be continued in this form of teaching, I prefer to hear lectures on statistics as it was in previous form of teaching without participating students in teaching process (3).
- I do not agree with this form of teaching. I suggest to make serious changes in the form of teaching of statistcs, but not in this form (4).
- I do not know, it is not important for me (5).

The results obtained from this question we analyzed separately for two groups of students and they are presented in Figure 3. It can be clearly seen that for majority of students actively not-participated in teaching is this form of teaching not suitable way and they prefer previous form of teaching lecturing by teacher only (item 3 – 53 %). Students, who took active part in teaching, did not denied this form of teaching so strictly. Fiftyseven percent of these students (items 1 and 2) agree with this form of teaching without any changes or with small changes and modifications, but they propose to maintain this form of teaching. Twentynine percent of these students want to return to the previous way of statistics teaching.

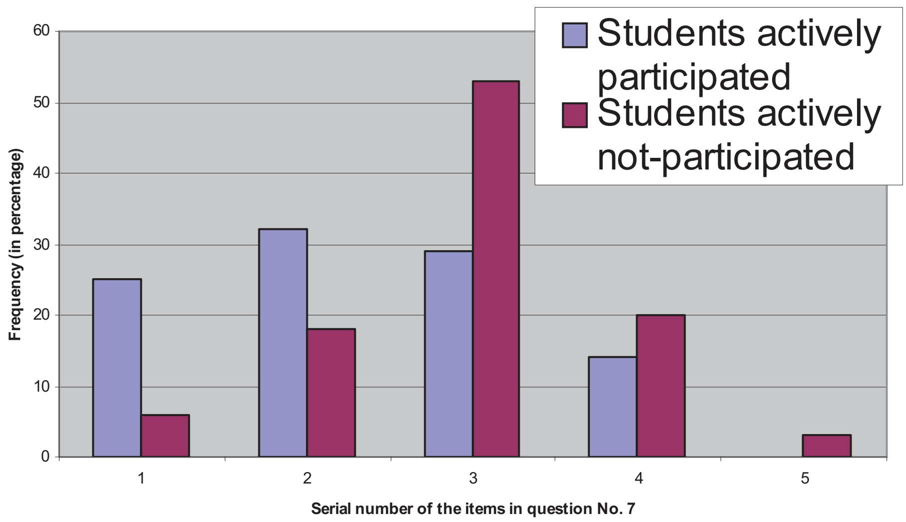

Fig. 3. Graphic presentatnion of the frequency distribution (in percentage) of the items in the question of No.7 of the questionnaire. Frequency distribution is presented separately for two groups of students. Details in the text.

4 Conclusions

The aim of this contribution was to introduce a little inovative form of statistics teaching into practice. One of the notion of the author of this paper was to help students in their study of statistics (according to the author mainly in the „simply" parts of statistics), because his intention was like this: if students hear explanations of some statistic problems by their students-colleagues, they will accept these facts more easily, simply and better, in general. On the other side, this way of teaching may be helpful not only for students who hear the lectures, but also for the students who will present their parts of subject matter. They can go deeper to the problem on which they are concerned, they have possibilities to consult difficult parts with teacher, they can gain some advantages in explanation the problem by their own words, etc. It has been proved many times and it is general known from the theory and practice of education, that a man will remember and save in his brain the fact better, more easily and for longer time, if he practiced it personally, made it by his hands, calculated it by himself, observed it himself, etc. than a man, who just heard or read about it. From the point of psychological aspect the author assumed, that students would accept new facts better from their colleagues than from teacher – they should have no obstacles to ask questions their colleagues without any fear, they should not feel any distance and barrier among them also from the point of view of age or position, and so on.

The preliminary results of this experiment do not indicate for satisfaction of all author's expectations of the statistics teaching. We can draw the following conclusions from the obtained preliminary results that can be formulated in the following way:

- Majority of students prefers current way of statistics teaching, when teacher lectures and communicates the new problems, facts and situations and students inactively hear and accept all what teacher presents.
- New way of statistics teaching was not denied definitively by all students. The author intuitively feels that this way of teaching is suitable mostly for students who have good results not only in statistics, but also in mathematics and other similar exact subjects.
- One of the factors which can negatively influence this preliminary results may be the fact the students got used to study not continuously, but just some days or week before exams.

The relation between results of students they gained in examinations in statistics or in other subject and their opinion on this way of teaching was not in the scope of this research. This relation may be a part of research in the future.

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MATHEMATICAL ASSISTANT ON WEB

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Abstract. In this paper we describe an online application which allows to solve selected typical problems using a simple and friendly Internet interface. The access to this application is free and the solution contains also intermediate steps necessary to obtain the final answer.

Key words and phrases. Computer algebra, Maxima, calculus, online.

Mathematics Subject Classification. Primary 97U70, 97U50; Secondary 97-04, 26A06.

1 Introduction

This paper describes from user's point of view an open source online application Mathematical Assistant on Web (MAW) which can be used for performing mathematical computations in the Internet browser. The idea to provide free Internet access to a computer algebra system (CAS) is not new. There are several projects which allow to use CAS like Maxima, Magma, Axiom and others for free and over the Internet without installing anything on local computer, see e.g. [2–4, 11]. There is also a promising computer algebra system Sage [5] which uses Internet browser as native workplace for mathematical computations. An advanced user can run almost arbitrary computation available in the corresponding CAS via these services. Another approach which enables online computations can be found on the server WMI2 [9]. This service tries to emulate symbolic calculator on web page.

Besides these solutions you can find several specialized one-purpose calculators for evaluating integrals, derivatives, determinants, ranks of matrices and much more, see e.g. [1,8,10,12]. These services allow to use CAS with no or minimal knowledge of commands of computer algebra system behind the scene. The user simply enters his or her data into a form on an Internet page and enjoys the answer. Doing computations with these tools is as simple as buying books on online e-shop. These tools could show not only the final answer to the selected problem, but the commercial service [1] shows also all important intermediate steps which are necessary to obtain the solution[1].

MAW is an online service which aims to sum up the best of existing solutions. This "best of" is taken from point of view of a weak student at technical university, which has poor mathematical skills and mathematics does not belong to his favorite topics, his primary interest is outside of mathematics. A typical user of MAW could be student of economics, landscape engineering or forestry, optionally a distant student. Having such a student in mind, we wish

- to enable online computations with minimal or no knowledge of computer algebra systems[2],
- to show all the steps in the worked problem,
- to cover as many typical problems solved in basic courses in mathematics on technical universities as possible.

The user of MAW simply enters his/her function (or whatever applicable) into predefined form and gets the solution which includes intermediate steps. Thus the user gets the answer which is in most cases not far from the complete worked problem, as it could be done by teacher or clever student. The language of MAW is Czech and English.

A secondary aim of MAW is to provide a tool which helps teacher to prepare problems for classes or for exams in a simple and comfortable environment.

2 The tools included in MAW

Tools available in MAW are divided into several group: Precalculus, Calculus, Integral calculus, Differential equations and Equations and inequalities.

2.1 Precalculus

In precalculus MAW provides tools for drawing functions, establishing natural domain of function, Lagrange polynomial of interpolation and least squares method for a line. The last two topics are self explanatory and hence we describe in more details the first two.

In the tool for drawing functions in one variable an analytical formula for function in one variable and boundaries for the picture are expected on input. After submitting the problem to the server, MAW parses automatically the function and tests, if it can be written in the form

$$y = Af(ax + b) + B, \tag{1}$$

where x is variable, a and A are nonzero constants, b and B are real constants and $f(\cdot)$ is one of basic elementary functions. If the user function can be written in form (1), the answer includes the graph together with the graph of the corresponding basic elementary function

[1] steps for some computations are available for free

[2] The computer algebra system which is currently used in MAW is Maxima 5.13.0, `http://maxima.sourceforge.net`, using Lisp GNU Common Lisp (GCL) GCL 2.6.8.

and description of the process which allows to transform one graph into another by scaling and shifting. User can also build and animation which shows this transformation in details as animated gif picture or as PDF file with animations in JavaScript. If the attempt to write user function in form (1) fails, the answer contains the graph of the user function only (with corresponding error message). The graphs are created by famous program `GNUplot` and the conversion between Maxima expressions into `GNUplot` expressions is done by program `formconv`. This makes MAW to be (as far as we know) the unique tool which draws graphs of odd roots also for negative argument[3].

In the tool for finding domain of functions in one or two variables MAW parses the user expressions, finds all denominators, even roots, logarithms and asin and acos functions. From here MAW builds the inequalities which restrict domain, solve these inequalities numerically on the interval given by user and presents these results together with intersection of all partial results – the domain of the function. Remark that this is the only point in MAW, where numerical calculations are used instead of symbolic computations.

2.2 Calculus

In calculus MAW provides forms for investigating functions from the first and second derivatives, for evaluation of derivatives[4], Taylor polynomial and local extrema for function of two variables.

The tool for investigating functions is the oldest in MAW, its first simple version has been created in November 2007. For a general function it evaluates the first and second derivatives, zeros of the function and its derivatives and shows a graph of the function. Moreover, if the function on input is rational function such that all zeros of denominator have the same multiplicity, MAW shows in details how to find and simplify the derivative and finds vertical asymptotes and asymptotes at infinity.

The calculator for local minima and maxima in two variables offers three predefined actions. The first action takes a function and a stationary point on input and evaluates the first two derivatives, Hessian at stationary point and uses this Hessian to recognize the type of the stationary point. The second action is similar, but allows more stationary points on input and the output is summarized in a table and is less detailed. The last action assumes a function of two variables on input and attempts to find stationary points using solve command of Maxima CAS.

2.3 Integral calculus

The most used application of MAW is the integral assistant which is the first tool in the group for integral calculus. In this application user enters a function which has to be integrated and submits the form. After this, some automatic tests on the input are executed. These tests reveal, if the function is suitable for integration by parts, for substitution, for direct application of integration formulas, for division into partial fractions, for some method of undetermined coefficients and so on. After these tests we provide some hints for the next step and the user

[3]This feature has been kindly added to formconv by G. Bakos on the request of MAW authors.

[4]This computation is also available at [1] for free, but we use slightly different notation and more natural output.

can freely choose between various integration methods. In this manner the user evaluates the integral as on the paper, but with minimal hazard of numerical error and he/she can concentrate himself/herself to trying various methods of integration and investigating, how the results after applications of various methods look like.

Another tools in this group can be used to evaluate area between two curves (the answer includes a picture of this region and MAW can be used to find intersections of two curves as well as to draw a solid of revolution about x-axis), to estimate the definite integral by trapezoidal rule (including a simple picture, if the function is nonnegative and the number of trapezoids is small) and to evaluate double integral (including Cartesian and polar coordinates, both orders of integration and picture of the domain of integration).

2.4 Differential equations

The collection of tools for differential equations includes first and second order differential equations solvers and tool for autonomous systems.

The first order differential equation solver can be used to solve the equation with respect to the derivative y' and to solve equations with separated variables, linear equations and homogeneous equations, including the steps in the corresponding method. The type of the equation is recognized automatically and if the equation is linear and can be also solved via separation of constants, both methods are shown. The integrals are evaluated automatically and contain links to integral assistant, which allows to try comfortably various integration techniques.

Another tool in the section for differential equations can be used to solve homogeneous and nonhomogeneous second order linear differential equation with constant coefficients. The user has two option for nonhomogeneous equation: variation of constants and method of undetermined coefficients[5]. The form of the particular solution in the second case is guessed automatically.

The tool for investigating autonomous systems expects a planar autonomous system and a stationary point on input and outputs the Jacobi matrix, characteristic polynomial and characteristic values at the stationary point. Stationary points can be computed using Maxima, if necessary.

2.5 Equations and inequalities

The last group of tools available in MAW involves tools for solving nonlinear equations and inequalities. Equations can be solved using method of bisection or by iteration method and a simple picture is drawn to visualize the function and successive approximations for solution.

Inequalities are solved numerically using essentially the same method as in the tool for drawing domains.

[5]MAW supports equations with quasipolynomial on the right hand, where the degrees of polynomial parts are smaller than 5.

3 MAW home installation

The primary installation for MAW is in virtual server at the Faculty of Forestry and Wood Technology at Mendel University of Forestry and Agriculture Brno. The server is hosted on a powerful machine and this makes the server fast despite its relatively weak parameters (128 MB RAM, Dual-Core AMD Opteron(tm) Processor 8218, 2.6 GHz, HDD 4 GB). MAW runs on Debian Linux, in a typical day it handles more than 1000 requests per day and since its establishing has been accessed from more than 100 countries. A special care prevents bad mathematical expressions on input and overloading of the server by long computations. This technical background has been described in [7].

The MAW project has been initiated on October 2007 and most of the programming work was done during the spring and summer 2008 with kind support of the grant FRVŠ 99/2008.

All current MAW sources are available on Internet at `sourceforge.org` [6].

Acknowledgment

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