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# APPLICATION OF THE CHAIN SEQUENCES TO LINEAR DIFFERENCE EQUATIONS 

ABDERRAMÁN MARRERO Jesús, (E)


#### Abstract

The chain sequences are here introduced in the study of some basic oscillatory and spectral properties of the solutions of self-adjoint operator associated to second order linear homogeneous difference equations. Applying these results on parameterized difference equations, approximated conditions are obtained for the values of the $\epsilon$-parameter where the absolutely continuous spectrum appears. As an illustration, the $\epsilon$ parameter is taken as the energy in current examples of discrete Schrödinger operators.


Key words and phrases. Absolutely continuous spectrum; chain sequences; oscillation theory; parameterized difference equations; Schrödinger operators; self-adjoint difference equations.
Mathematics Subject Classification. Primary 15A90, 39A11, 40A05; Secondary 47B39.

## 1 Introduction

The behaviour of some qualitative properties of the solutions of the difference equations, in particular for the second order linear difference equation, is a subject of current interest $[3,6,7]$. Not only is a line in growth, but it has practical applications in some branches of science as discrete dynamical systems and mathematical physics. Without loss of generality, the linear second order difference equation with variable coefficients can be write as, with $n \in Z_{n_{0}}^{+}=\{n \in$ $\left.Z^{+}, n_{0} \in Z^{+} \cup\{0\}: n>n_{0}\right\}:$

$$
\begin{equation*}
x_{n+1}=b_{n} x_{n}-a_{n} x_{n-1} . \tag{1}
\end{equation*}
$$

A solution of (1) is defined as $X=\left\{x_{n}\right\}_{n=n_{0}}^{\infty}$, and each term $x_{n}$ is denoted as the component solution at step $n$. Another equivalent representation is given by means of its $2 \times 2$ companion
matrix formulation [2],

$$
\binom{x_{n}}{x_{n+1}}=\left[\begin{array}{cc}
0 & 1  \tag{2}\\
-a_{n} & b_{n}
\end{array}\right]\binom{x_{n-1}}{x_{n}}
$$

The chain sequences $[4,11,12]$ have been introduced in the study of the spectral properties of the solutions from (1) or (2). The aim here is to apply the chain sequences to give basic results on the oscillatory properties. An approximation about the location of its absolutely continuous spectrum is also provided. For this task, in Section 2 a closed representation of the solutions in form of a product is given. It permits us to introduce the chain sequences for the obtainment of some basic oscillatory properties of the canonical self-adjoint equation associated to (1). In Section 3 the results are applied to parameterized difference equations. Thus an inequality is given for the coefficients of Harper as well as Fibonacci equation. These equations are current examples of difference Schrödinger operators. This simple inequality gives an approximation for the parameter values where the spectrum of the operator is absolutely continuous. In the Section 4 some brief conclusions and possible lines of future work are outlined.

## 2 Chain sequences and oscillatory properties

### 2.1 The associated self-adjoint equation

We assume the following change in the component solutions of (1), with $n \in Z_{n_{0}+1}^{+}$and the same initial conditions $y_{n_{0}}=x_{n_{0}} ; y_{n_{0}+1}=x_{n_{0}+1}$,

$$
\begin{equation*}
x_{n}=y_{n}\left(\prod_{i=0}^{\left[\frac{n-1-\left(n_{0}+1\right)}{2}\right]} a_{n_{0}+1+2 i+r}\right) . \tag{3}
\end{equation*}
$$

In the top of product, $[u]$ is the greater integer not exceeding $u$. The standard convention for products is used. That is to say, if $[u]$ is a negative integer then the product takes the value 1. The integer parameter $r=\left(n-n_{0}\right)$ mod2 can take two values, 0 or 1 . Note as this change is a variable scaling of the original solution from (1), with dependence on the step $n$. Then, the transformed difference equation results,

$$
\begin{equation*}
y_{n+1}=b_{n}^{*} y_{n}-y_{n-1} . \tag{4}
\end{equation*}
$$

with $n \in Z_{n_{0}}^{+}$, and

$$
\begin{equation*}
b_{n}^{*}=b_{n} \frac{\prod_{i=0}^{\left[\frac{n-1-\left(n_{0}+1\right)}{2}\right]} a_{n_{0}+1+2 i+r}}{\prod_{i=0}^{\left[\frac{n-\left(n_{0}+1\right)}{2}\right]} a_{n_{0}+1+2 i+r}} \tag{5}
\end{equation*}
$$

The equation (4) is defined as the (canonical) self-adjoint difference equation associated to (1). Its equivalent companion matrix formulation is,

$$
\binom{y_{n}}{y_{n+1}}=\left[\begin{array}{cc}
0 & 1  \tag{6}\\
-1 & b_{n}^{*}
\end{array}\right]\binom{y_{n-1}}{y_{n}} .
$$

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### 2.2 Closed representation in form of a product

If a new transformation of the Riccati type, $z_{n}=\frac{y_{n+1}}{y_{n}}$ is introduced in (4), then we have,

$$
\begin{equation*}
z_{n-1}=\frac{1}{b_{n}^{*}-z_{n}} \tag{7}
\end{equation*}
$$

Let now $\left\{g_{k}^{\left(n_{0}+1\right)}\right\}_{k=0}^{\infty}$ be a sequence defined as,

$$
\begin{equation*}
g_{k}^{\left(n_{0}+1\right)}=1-\frac{z_{n_{0}+k+1}}{b_{n_{0}+k+1}^{*}} . \tag{8}
\end{equation*}
$$

From (7) and (8) a new sequence $\left\{\alpha_{k}^{\left(n_{0}+1\right)}\right\}_{k=1}^{\infty}$ is obtained, whose terms are,

$$
\begin{equation*}
\alpha_{k}^{\left(n_{0}+1\right)}=\left(1-g_{k-1}^{\left(n_{0}+1\right)}\right) g_{k}^{\left(n_{0}+1\right)}=\frac{1}{b_{n_{0}+k+1}^{*} b_{n_{0}+k}^{*}}=\frac{a_{n_{0}+k+1}}{b_{n_{0}+k+1} b_{n_{0}+k}} . \tag{9}
\end{equation*}
$$

The following closed expression for the solutions will be the nexus between the chain sequences and oscillatory properties of the solution of Eq. (4).

Proposition 2.1 Every solution $y_{n}$, with $n \in Z_{n_{0}+1}^{+}$, of equation (4) admits the following closed representation in terms of the sequence $\left\{g_{k}^{\left(n_{0}+1\right)}\right\}$,

$$
\begin{equation*}
y_{n}=y_{n_{0}+r}\left(\prod_{i=0}^{\left[\frac{n-1-\left(n_{0}+1\right)}{2}\right]} \frac{1-g_{2 i+r}^{\left(n_{0}+1\right)}}{g_{2 i+r}^{\left(n_{0}+1\right)}}\right) . \tag{10}
\end{equation*}
$$

Proof. The expression can be construct inductively. Another equivalent proof follows if we observe, for the definition of $g_{k}^{\left(n_{0}+1\right)}$ and the equations (7) and (9), as one relation between consecutive component solutions can be obtained $y_{n}=\frac{1}{b_{n}^{*} g_{n}^{(0)}} y_{n-1}=\frac{1-g_{n-1}^{(0)}}{g_{n-1}^{(0)}} y_{n-2}$. If this relation is iterated down to either $y_{n_{0}}$ or $y_{n_{0}+1}$, then the result yields.

For an explicit computation, this representation is not better than the recursive method itself. Nevertheless, it is useful for the analysis of qualitative properties of the solutions. As a consequence of Proposition 2.1, we have the following

Corollary 2.2 Every solution $x_{n}$, with $n \in Z_{n_{0}+1}^{+}$, of the difference equation (1) admits the following closed representation

$$
\begin{equation*}
x_{n}=x_{n_{0}+r}\left(\prod_{i=0}^{\left[\frac{n-1-\left(n_{0}+1\right)}{2}\right]} \frac{a_{n_{0}+1+2 i+r}\left(1-g_{2 i+r}^{\left(n_{0}+1\right)}\right)}{g_{2 i+r}^{\left(n_{0}+1\right)}}\right) \tag{11}
\end{equation*}
$$

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### 2.3 Basic oscillatory properties

Continued fractions are closely related to second order linear homogeneous difference equations [5]. A systematic development of chain sequences in the analysis of the continued fractions can be found in [12]. The chain sequences are here applied on the oscillatory properties of the solutions from (4) and (1). This tool provides us the obtainment of some results without the use of continued fractions.

### 2.3.1 Chain sequences

The definition of chain sequences given in [4] is here detailed for clarity.
Definition 2.3 [4] A sequence $\left\{\alpha_{n}\right\}_{n=1}^{\infty}$ is called a (positive) chain sequence if exists another sequence $\left\{g_{n}\right\}_{n=0}^{\infty}$ such that:
i) $0 \leq g_{0}<1 \quad ; \quad 0<g_{n}<1$.
ii) $\alpha_{n}=\left(1-g_{n-1}\right) g_{n}$.

The sequence $\left\{g_{n}\right\}$ is called the parameter sequence for the chain sequence $\left\{\alpha_{n}\right\}$, and $g_{0}$ is called an initial parameter. A parameter sequence $\left\{g_{n}\right\}$ is called minimal parameter sequence for the chain sequence $\left\{\alpha_{n}\right\}$ if $g_{0}=0$. In general, the parameter sequence of a particular chain sequence is not unique.

A sequence $\left\{\alpha_{n}^{\left(n_{0}+1\right)}\right\}_{n=1}^{\infty}$ is a chain sequence if $\left\{\alpha_{n}\right\}_{n=n_{0}+2}^{\infty}$ is also a chain sequence. Analogous with its parameter sequence $\left\{g_{n}^{\left(n_{0}+1\right)}\right\}_{n=0}^{\infty}$. Note as $g_{k}=g_{k}^{(0)}$. It is of interest to know the behaviour of chain sequences that have all its terms $\alpha_{n} \geq \frac{1}{4}$. The following corollary obtained in [4] will be useful later.

Corollary 2.4 [4] Let $\left\{\alpha_{n}\right\}_{n=1}^{\infty}$ be a chain sequence, if $\alpha_{n} \geq \frac{1}{4}$ for $n \geq N$, then lim $_{n \rightarrow \infty} \alpha_{n}=\frac{1}{4}$. Hence, if $\beta_{n} \geq \beta>\frac{1}{4}$ for $n \geq N$, then $\left\{\beta_{n}\right\}_{n=1}^{\infty}$ is not a chain sequence.

### 2.3.2 Oscillatory properties

The definition for the oscillatory solution is also given for clarity.
Definition 2.5 [5] A nontrivial solution $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ of Eq. (4) is said to be a oscillatory solution (around zero) if for every positive integer $N \geq n_{0}$, there exists $n \geq N$ such that the product of two consecutive component solutions verifies $y_{n} y_{n+1} \leq 0$. Otherwise, the solution $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ is said to be a non oscillatory solution.

A first basic result on the oscillatory properties of the solutions from (4) is now introduced.
Theorem 2.6 Every solution $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ of (4) is a non oscillatory or strictly alternating solution if and only if the sequence $\left\{g_{k}^{\left(n_{0}+1\right)}\right\}_{k=0}^{\infty}$ from (8) is a parameter sequence of the chain sequence $\left\{\alpha_{k}^{\left(n_{0}+1\right)}\right\}_{k=1}^{\infty}$.

Proof. Suppose that $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ of Eq. (4) is a non oscillatory or strictly alternating solution. In both situations, from (10), all terms $\frac{1-g_{k}^{\left(n_{0}+1\right)}}{g_{k}^{\left(n_{0}+1\right)}}$ in the product must be positive. This fact is possible if and only if $0<g_{k}^{\left(n_{0}+1\right)}<1, \forall k \geq 0$. Thus, $\left\{g_{k}^{\left(n_{0}+1\right)}\right\}_{k=0}^{\infty}$ from Eq.(8) is a parameter sequence for the chain sequence $\left\{\alpha_{k}^{\left(n_{0}+1\right)}\right\}_{k=1}^{\infty}$. Then, we invoking the Sturm separation theorem [5], because (4) is a self-adjoint second order difference equation, to claim that every solution is a non oscillatory, or strictly alternating, solution.

Thus if the sequence $\left\{\alpha_{k}^{\left(n_{0}+1\right)}\right\}_{k=1}^{\infty}$ defined in Eq. (9) is not a chain sequence for any $n_{0} \geq 0$, then every solution $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ from (4) is an oscillatory (non alternating) solution. As a first consequence of Theorem 2.6, other result is given about the sign of the $b_{k}$ coefficients from (4) in the non oscillatory and strictly alternating situations.

Corollary 2.7 If the solution $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ of the self-adjoint difference equation (4) is a non oscillatory (strictly alternating) solution, then the sequence $\left\{b_{n_{0}+1+k}^{*}\right\}_{k=0}^{\infty}$ is a positive (negative) sequence.

Proof. Let $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ be a strictly alternating solution of Eq. (4). The sequence $\left\{b_{n_{0}+1+k}^{*}\right\}_{k=0}^{\infty}$ can not be a positive sequence [5], and from Theorem 2.6, the sequence must be negative. Indeed, if for $j \geq 0, b_{n_{0}+1+j}^{*}>0$ exists and $0<g_{j}^{\left(n_{0}+1\right)}<1$, then $\operatorname{signum}\left(y_{\left(n_{0}+j+1\right)}\right)=$ $\operatorname{signum}\left(y_{\left(n_{0}+j+2\right)}\right)$, which is a contradiction. Thus, $\left\{b_{n_{0}+1+k}^{*}\right\}_{k=0}^{\infty}$ is a negative sequence. In an analogous way, it is easy to show that if $Y=\left\{y_{n}\right\}_{n=n_{0}}^{\infty}$ of equation (4) is a non oscillatory solution, from Theorem 2.6 the sequence $\left\{b_{n_{0}+1+k}^{*}\right\}_{k=0}^{\infty}$ must be a positive sequence. When $b_{n_{0}+1+j}^{*}=0$, the sequence $\left\{\alpha_{k}^{\left(n_{0}+1\right)}\right\}_{k=1}^{\infty}$ is not a chain sequence and Theorem 2.6 does not hold.

If a component solution is null, say $y_{n_{0}}=0$, then the Theorem 2.6 remains right. We can take $y_{n_{0}+1}$ and $y_{n_{0}+2}$ as initial values. Besides $g_{0}^{\left(n_{0}+1\right)}=0$, and if the sequence $\left\{\alpha_{k}^{\left(n_{0}+1\right)}\right\}_{k=1}^{\infty}$ is a chain sequence, then the sequence $\left\{g_{k}^{\left(n_{0}+1\right)}\right\}_{k=0}^{\infty}$ is its minimal parameter sequence.

Now consider the oscillatory properties of the equation (1) through its solutions in the form of Eq. (11). Due to Theorem 2.6 the sequence of coefficients $\left\{a_{n}\right\}_{n=n_{0}}^{\infty}$ plays a dual character, according to if the sequence $\left\{\alpha_{n}^{\left(n_{0}+1\right)}\right\}_{n=1}^{\infty}$ is, or is not, a chain sequence. Its analysis will be intricate because, from equation (9), the sequence $\left\{\alpha_{n}\right\}$ also depends of the sequences $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$. Nevertheless, when (1) is a self-adjoint equation, with $\left\{a_{n}\right\}_{n=n_{0}}^{\infty}$ a positive sequence, the Theorem 2.6 stays right for the solutions of (1).

The chain sequences whose terms are greater than $\frac{1}{4}$ have been characterized in [11]. From this, the Theorem 2.6, and the Corollary 2.4, another corollary is obtained, which introduces a simple quantitative condition, similar to others known results on oscillation theory [5], and useful for the applications of the next Section.
Corollary 2.8 Let $\left\{a_{n}\right\}_{n=n_{0}+1}^{\infty}$ be a positive sequence, if $\alpha_{k}^{\left(n_{0}+1\right)}=\frac{a_{n_{0}+k+1}}{b_{n_{0}+k+1} b_{0}+k}>\frac{1}{4}$ for $k \geq 1$, and $\lim _{k \rightarrow \infty}\left(\alpha_{k}^{\left(n_{0}+1\right)}-\frac{1}{4}\right)>0$, then every solution $X=\left\{x_{n}\right\}_{n=n_{0}}^{\infty}$ of Eq. (1) is a oscillatory, non alternating, solution.

## 3 Application to parameterized self-adjoint difference equations

Suppose that the real coefficients $a_{n}(\epsilon)$ and $b_{n}(\epsilon)$ can present a parametric dependence. When the sequence $\left\{a_{n}(\epsilon)\right\}_{n=n_{0}+1}^{\infty}$ is positive, associated to Eq. (4), or to Eq. (1), exists a self-adjoint difference operator which has the parameter $\epsilon$ as main parameter. This can be corresponding to a major physical property of the system under study. An example is the energy in difference Schrödinger operators. Could be of interest to find the component functions of Eq. (1) or (4), as functions of the parameter $\epsilon$. Also, it is possible to verify the range of parameter values where this operator presents an absolutely continuous spectrum, and so on. The case of linear dependence in $\epsilon$ has important applications, as the orthogonal polynomials or the difference Schrödinger operators. The achievement of similar conditions seems admissible if the results are checked in other cases of dependency on $\epsilon$.

### 3.1 Absolutely continuous spectrum in difference Schrödinger operators

Difference Schrödinger operators are formulated with self-adjoint second order difference equations related to particular partial differential Schrödinger equations. Difference Schrödinger operators are obtained through of some physical and accurate approximations. The classical examples are Harper [8, 10] and Fibonacci [9] equations. The original differential equations were transformed in particular cases of equation (4). Then Corollary 2.8 can be applied on these discrete Schrödinger operators, which permits us to obtain an approximation, in some cases an exact result, on the location of values of $\epsilon$ where the generalized zeros [7] could be dense and the operator can display absolutely continuous spectrum. When applying the Corollary 2.8 on Eq. (4), the following inequality holds, with $n \in Z_{0}^{+}$,

$$
\begin{equation*}
\frac{1}{b_{n+1}^{*} b_{n}^{*}}>\frac{1}{4} \tag{12}
\end{equation*}
$$

Harper equation [8, 10] is for the energies $\epsilon$, that also depends on other parameters, $\lambda, \theta, \nu$. Here, the coefficients are $a_{n}=1$ and $b_{n}=b_{n}^{*}=(\epsilon-2 \lambda \cos (2 \pi \theta n+\nu))$. Fibonacci [9] equation has the coefficients $a_{n}=1$ and $b_{n}=b_{n}^{*}=\epsilon-c_{n}$. The coefficient $c_{n}$ only takes two constant values $A$ and $B$. But these values are introduced with a quasi-periodic sequence, the Fibonacci sequence.

Harper equation, with $\theta \in Z$, is an equation with constant coefficients, and one band of continuous spectrum $\sigma=[-2+2 \lambda \cos (\nu), 2+2 \lambda \cos (\nu)]$. Its Lebesgue measure is $L_{\sigma}=4$. If $\theta=\frac{1}{2}$ the sequence $b_{n}^{*}$ is a 2 -periodic sequence. In both cases the inequality (12) limits the range of values of $\epsilon$ where absolutely continuous spectra can appear. The limit points correspond exactly with the outer boundaries of the continuous spectrum, see Figure 1. For these cases, this shows the main role of chain sequences in Eq. (4). The application of chain sequences in orthogonal polynomials can be seen in [4].

When $\left\{b_{n}^{*}\right\}$ is a q-periodic sequence, with $q>2$, the spectrum have q bands. $\theta=\frac{p}{q} \in Q$. A smooth condition is to maintain the inequality (12) for the complete period:


Figure 1: Harper Equation. Exact continuous spectra (black lines) and borders (black points) of the location with the approximated condition. $\theta=\frac{1}{q}$, with $q$ from 1 to $4, \lambda=1$, and $\nu=0$.

$$
\begin{equation*}
\prod_{i=1}^{q}\left(\frac{b_{i}^{*}}{2}\right)^{2}<1 \tag{13}
\end{equation*}
$$

In Harper equation when $\theta$ is irrational, and in Fibonacci equation when $A \neq B$, the sequence $\left\{b_{n}^{*}\right\}$ is quasi-periodic, $q \rightarrow \infty$. The spectrum of Harper equation is a Cantor set [10] with Lebesgue measure $L_{\sigma}=4(1-|\lambda|) \geq 0$, with $0 \leq|\lambda| \leq 1$. Also condition (12) holds in the Fibonacci equation, but it does not work in Harper. It are not defined for $\epsilon$ values where the coefficients $b_{n}^{*}=(\epsilon-2 \lambda \cos (2 \pi \theta n+\nu))=0$.

## 4 Conclusions and future work

The basic properties of the chain sequences are adequate for a simple categorization of the oscillatory solutions from equation (4). Also, they have been suitable to initiate an exhaustive study of oscillatory conditions of the solutions from equation (1). A thorough application of the chain sequences properties could give rise to an improvement of these results. Chain sequences $\left\{\alpha_{n}^{\left(n_{0}+1\right)}\right\}$ with terms $\alpha_{j}^{\left(n_{0}+1\right)}>\frac{1}{4}$ are not included in (13), for example some periodic sequences. This produces in general that the outer bands of the absolutely continuous spectrum are not covered in this approximation, see Figure 1. An improvement of the condition is necessary in these situations. Finally, another line of research for a more exact location of the absolutely continuous spectrum is based in the bounded condition $\left|\operatorname{trace}\left(\mathbf{T}_{n}\right)\right| \leq 2$, for the trace map,
$[1,9]$, of the fundamental matrix $\mathbf{T}_{n}$ from the equivalent companion matrix representations (2) or (6).

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# NON-COMPLETE QUASI-TOPOLOGICAL GROUPS <br> BATÍKOVÁ Barbora, (CZ) 


#### Abstract

We introduce examples of (non-complete) $\mathrm{T}_{1-}$ and $\mathrm{T}_{2}$-quasi-topological groups that do not have group completion (for four types of completeness).


Key words and phrases. weak complete, semi-complete, weak Cauchy, semi-Cauchy.
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## 1 Notations and definitions

The main references are books [5], [6] for topology and [2], [8] for topological structures on groups.

Quasi-topological groups are groups with topology where multiplication is separately continuous (not necessarily continuous) and inversion is continuous.

First recall some definitions of topological structures on groups:
Definition 1.1 Let $X$ be a group with a topology $\tau$. We say that $X$ is a semi-topological group if the mapping $m$ (multiplication) is separately continuous, which means that it is continuous for the right and left variable.

We say that $X$ is a quasi-topological group if it is a semi-topological group and the inversion mapping $i: X \rightarrow X: x \mapsto x^{-1}$ is continuous.

We recall from [5] that a semi-uniformity on a set $X$ is a filter of subsets of $X \times X$ having a base of symmetric sets and the intersection of which contains the diagonal. Every semiuniformity $\mathcal{U}$ on $X$ induces a closure on $X$, which need not be topological, namely $\bar{A}=\bigcap_{U \in \mathcal{U}} U[A]$. We will use those semi-uniformities inducing topological closures.

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The couple $(X, \mathcal{U})$ is then called at-semi-uniform space, where the letter $t$ stands for topological.

There exist several semi-uniformities that induce the topology of a quasi-topological group, but we will examine only the two-sided semi-uniformities, which are used in the definition of complete quasi-topological groups. The description is the same as for topological groups, see e.g. [8]:

Definition 1.2 The semi-uniformity $\mathcal{R}$ consisting of sets
$R_{U}=\left\{(x, y) \in X \times X ; y x^{-1} \in U\right\}, U \in \mathcal{U}_{e}$, where $\mathcal{U}_{e}$ is a neighbourhood filter of the unit $e$ in the topology of a quasi-topological group $X$, is called the right semi-uniformity on the group $X$.

The semi-uniformity $\mathcal{L}$ consisting of sets $L_{U}=\left\{(x, y) \in X \times X ; x^{-1} y \in U\right\}, U \in \mathcal{U}_{e}$, is called the left semi-uniformity on the group $X$.

Supremum $\mathcal{L} \vee \mathcal{R}$, resp. infimum $\mathcal{L} \wedge \mathcal{R}$, is clearly also a semi-uniformity on $X$, see e.g. [5], and it is called the two-sided (or upper), resp. Roelke (or lower), semi-uniformity on $X$.

Each of the described semi-uniformities is topological (or t-) semi-uniformity. They have one more property (which is satified generally in uniform but not in semi-uniform spaces):
Definition 1.3 A t-semi-uniformity on a set $X$ is called point open if it has such a base $\mathcal{B}$ that $B[x]$ is an open neighborhood of $x$, for all $x \in X, B \in \mathcal{B}$.

The definitions of Cauchy-like properties and completeness come from [3]:
Definition 1.4 Let $(X, \mathcal{U})$ be a $t$-semi-uniform space.

1. A filter $\mathfrak{f}$ on a space $X$ is called a classic Cauchy filter on $(X, \mathcal{U})$ if for every $U \in \mathcal{U}$ there exists $F \in \mathfrak{f}$ such that $F \times F \subset U$.
2. A filter $\mathfrak{f}$ on a space $X$ is called a Cauchy filter on $(X, \mathcal{U})$ if for every $U \in \mathcal{U}$ there exists $F \in \mathfrak{f}$ such that $U[x] \in \mathfrak{f}$ for every $x \in F$.
3. A filter $\mathfrak{f}$ on a space $X$ is called a weak Cauchy filter on $(X, \mathcal{U})$ if for every $U \in \mathcal{U}$ and every $F \in \mathfrak{f}$ there exists $x \in F$ such that $U[x] \in \mathfrak{f}$.
4. A filter $\mathfrak{f}$ on a space $X$ is called a semi-Cauchy filter on $(X, \mathcal{U})$ if for every $U \in \mathcal{U}$ there exists $x \in X$ such that $U[x] \in \mathfrak{f}$.

Definition 1.5 A t-semi-uniform space is called (classically, weakly, semi-) complete if every (classic, weak, semi-) Cauchy filter converges.

Recall that in topological groups all Cauchy-like properties coincide.
Definition 1.6 A quasi-topological group is called (classically, weakly, semi-) complete if it is (classically, weakly, semi-) complete in its two-sided semi-uniformity.

Definition 1.7 Let $X$ be a quasi-topological group. A (classically, weakly, semi-) complete quasi-topological group that contains $X$ as a dense subgroup is a group (classic, weak, semi-) completion of the group $X$.

## 2 Weak and semi-completion of quasi-topological groups

Every (Hausdorff) topological group is embedded in a (Hausdorff) topological group that is complete in its two-sided uniformity. In quasi-topological groups the situation is different.

We show examples of $T_{1}$ - and $T_{2}$-quasi-topological groups that cannot be embedded into a weakly complete or semi-complete quasi-topological group, and, thus, they are neither weak complete nor semi-complete.

The following example introduces a $\mathrm{T}_{1}$-quasi-topological group that has not a weak (and thus not a semi-) completion (in t-semi-uniform spaces), and hence it has not a group weak (and thus not a semi-) completion:

Example 2.1 Let $Z$ be a quasi-topological group of integers
and take the coarsest $\mathrm{T}_{1}$-topology on $Z$.
The filter $\mathfrak{f}=\{F \subset Z ;|Z \backslash F|<\omega \wedge\{a, b\} \subset F\}$ is a weak Cauchy filter on the two-sided semi-uniformity on $Z$, for every couple $a, b \in Z, a \neq b$.
In fact, let $U$ be a neighbourhood of the neutral element $0, F \in \mathfrak{f}$.
The sets $a-U, b-U, F$ have finite complements, and, thus, a non-empty intersection. Take an element $z$ from the intersection. Now $z \in F$ and $a, b \in z+U$. Then $U[z]=z+U \in \mathfrak{f}$, and $\mathfrak{f}$ is weak Cauchy.

If the space $Z$ has a weak completion $Y$, then the filter $\mathfrak{f}$ is weak Cauchy on $Y$, see [3], Proposition 1.

As $Y$ is weakly complete, the filter $\mathfrak{f}$ converges to a point $y \in Y$. That means that every neighborhood of $y$ belongs to the filter $\mathfrak{f}$, and thus it contains the points $a, b$. Now $y \in \overline{\{a\}}^{Y}$, which is equivalent to $a \in \overline{\{y\}}^{Y}$, because the space $Y$ must be symmetric. From $a \in \overline{\{y\}}^{Y}$ and $y \in \overline{\{b\}}^{Y}$ it follows that $a \in \overline{\{b\}}^{Y}$. But that is a contradiction to $\mathrm{T}_{1}$ property of $Z$.

We recall two useful assertions from [4]. The first one does not hold in semi-uniform spaces in general.

Lemma 2.2 In point open t-semi-uniform spaces convergent filters are Cauchy.

Proof. If $(X, \mathcal{U})$ be a point open t-semi-uniform space.
Take $\mathfrak{f}$ a convergent filter. Let $U$ be a symmetric set from the semi-uformity $\mathcal{U}$.
Take $x$ a limit of $\mathfrak{f}$. Then $U[x] \in \mathfrak{f}$. As the space is point open $U[x]$ is a neighbourhood of each its point. Now $x \in U[y]$ for each $y \in F=U[x]$ because $U$ is a symmetric set. And $U[y] \in \mathfrak{f}$ because it is a neighbourhood of $x$.

Proposition 2.3 If a point open t-semi-uniform space $X$ has a point open semi-completion, resp. a point open weak completion, then every
semi-Cauchy, resp. weak Cauchy, filter on $X$ must be Cauchy on $X$.

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Proof. Let $Y$ be a pointwise open semi-completion of $X, \mathfrak{f}$ be a semi-Cauchy filter on $X$. Then $\mathfrak{f}$ generates a filter $\mathfrak{f}^{\prime}$ that is semi-Cauchy on $Y$ and thus convergent on $Y$. Then $\mathfrak{f}^{\prime}$ is Cauchy on $Y$ and thus Cauchy on $X$.

The proof for the weak completion is the same.
The next examples show Hausdorff quasi-topological groups that do not have group weak (and thus semi-) completion.

We will use Proposition 2.3.
First we recall special quasi-topological groups from [1] and [7]. In [1], par. 3, some orbits are shown to be quasi-topological groups:

Example 2.4 Let $(G,+)$ be an Abelian group, $Z$ a Hausdorff topological space.
If $\phi: G \times Z \rightarrow Z$ is an action (i.e., $\phi(0, z)=z, \phi(a, \phi(b, z))=\phi(a+b, z)$ for every $a, b \in G, z \in Z$, and $\phi(a,-): Z \rightarrow Z$ is a continuous mapping for every $a \in G)$, then every orbit $\left(O\left(z_{0}\right)=\left\{\phi\left(a, z_{0}\right) ; a \in G\right\} \subset Z\right)$ forms a semi-topological group.

If $G$ is an Abelian group of order 2, the orbits will be quasi-topological groups.
Special orbits are introduced in [7], Theorem 4:
Example 2.5 (Korovin's orbit.)
Take an Abelian group $(G,+)$ and a Hausdorff topological space $X$ such that $\left|G^{\omega}\right|=|G| \geq$ $|X| \cdot \omega$. In Example 2.4 let $Z=X^{G}, \phi$ be the shift of $G \times Z$ to $Z, \phi(a, f)(b)=f(a+b)$.

Denote by $\mathcal{A}$ the set of all finite subsets of $G, \mathcal{F}(\mathcal{A})$ the set of all mappings of $A$ to $X$ for $A \in \mathcal{A}, \mathcal{F}=\bigcup\{\mathcal{F}(\mathcal{A}) ; \mathcal{A} \in \mathcal{A}\}$.

Denote by $A(f)$ such a set from $\mathcal{A}$ that $f \in \mathcal{F}(\mathcal{A}())$, for $f \in \mathcal{F}$.
Denote $\tau=|G|=|\mathcal{A}|=|\mathcal{F}|$.
Take $\left\{f_{\alpha} ; \alpha \in \tau\right\}$ a well-ordering of the set $\mathcal{F}$.
By transfinite induction we pick up $g_{\alpha}: \alpha \in \tau$, such that that the family $\left\{g_{\alpha}+A\left(f_{\alpha}\right): \alpha \in \tau\right\}$ is disjoint:
Suppose that for an ordinal $\beta \in \tau$ we have a subset $\left\{g_{\alpha}: \alpha \in \beta\right\}$ of $G$ such that the family $\left\{g_{\alpha}+A\left(f_{\alpha}\right): \alpha \in \beta\right\}$ is disjoint.

Take an element $g_{\beta} \in G \backslash H$, where $H$ is a minimal subgroup of $G$ containing the set $\left\{A\left(f_{\alpha}\right): \alpha \leq \beta\right\} \cup\left\{g_{\alpha}: \alpha<\beta\right\}$.
$G \backslash H \neq \emptyset$, because $|H| \leq|\beta| \cdot \omega<\tau=|G|$.
The family $\left\{g_{\alpha}+A\left(f_{\alpha}\right): \alpha \leq \beta\right\}$ is disjoint.
Then take a mapping $f \in X^{G}$ such that restrictions $\left.f\right|_{g_{\alpha}+A\left(f_{\alpha}\right)}=\left.f_{\alpha} \circ l_{g_{\alpha}}^{-1}\right|_{g_{\alpha}+A\left(f_{\alpha}\right)}$ for any $\alpha \in \tau$.

The subspace $Y=\left\{f_{g}=f \circ l_{g} ; g \in G\right\}$ of the space $X^{G}$ is a semi-topological group, its elements are functions $f_{g}, g \in G$, where $f_{g}(a)=f(g+a)$, for $a \in G$, with the group operation $f_{g}+f_{h}=f_{g+h}$. The neutral element is $f_{0}=f$, inverse elements are $-f_{g}=f_{-g}$. Subbasic neighbourhoods of the neutral element $f=f_{0}$ are the sets $U_{c}^{f}=\left\{f_{g} ; f_{g}(c)=f(g+c) \in U\right\}$, where $U$ are neighbourhoods of $f(c)$ in $X$.

Subbasic neighbourhoods of an element $f_{g}$ are the sets $U_{c}^{f_{g}}=\left\{f_{h} ; f_{h}(c)=f(h+c) \in U\right\}=$ $\left\{f_{g+k} ; f_{g+k}(c)=f(g+k+c)=f_{k}(g+c) \in U\right\}=f_{g}+U_{g+c}^{f}$, where $U$ are neighbourhoods of $f_{g}(c)=f(g+c)$ in $X$.

If we take for $G$ a group of order 2 then $Y$ is a quasi-topological group. (Here $l_{g}^{-1}=l_{g}$.)
We recall a useful Lemma 2 from [4]:
Lemma 2.6 Take $G, X, f$ from Example 2.5. For any finite sets $\left\{x_{0}, \ldots, x_{m}\right\} \subset X$ and $\left\{d_{0}, \ldots, d_{m}\right\} \subset G$, where $d_{i} \neq d_{j}$ for $i \neq j, i, j=0, \ldots, m$, there exists a $g_{\alpha} \in G$ (picked by the transfinite induction in Example 2.5) such that $f_{g_{\alpha}}\left(d_{i}\right)=f\left(g_{\alpha}+d_{i}\right)=x_{i}, i=0,1, \ldots, m$. Moreover $g_{\alpha}+d_{i} \neq 0$ for all $i=0, \ldots, m$.

Proof. For the finite mapping $h$ assigning $x_{i}$ to $d_{i}, i=0, \ldots, m$, there must exist an $\alpha \in \tau$ that $h=f_{\alpha} \in \mathcal{F}$ and $A\left(f_{\alpha}\right)=\left\{d_{0}, \ldots, d_{m}\right\}$.

From $\left.f\right|_{g_{\alpha}+A\left(f_{\alpha}\right)}=\left.f_{\alpha} \circ l_{g_{\alpha}}^{-1}\right|_{g_{\alpha}+A\left(f_{\alpha}\right)}$ it follows that $f_{g_{\alpha}}\left(d_{i}\right)=f\left(g_{\alpha}+d_{i}\right)=x_{i}$.
Clearly we can suppose that $\alpha \neq 0$.
Thanks to the construction of the family $\left\{g_{\alpha}+A\left(f_{\alpha}\right)\right\}$ in the Korovin's orbit we have $g_{\alpha}$ that is not in the minimal subgroup generated by $A\left(f_{\alpha}\right)$.

Thus $g_{\alpha}, d_{1}+g_{\alpha}, \ldots, d_{m}+g_{\alpha} \neq 0$.
The Korovin's orbit of the group $G$ of order 2 is a Hausdorff quasi-topological group that has not a group weak (and semi-) completion:

Example 2.7 Take the Korovin's orbit $Y$ described in Example 2.5. Let $G$ be a group of order 2 so that $Y$ would be quasi-topological.

Denote by $\mathfrak{f}$ the filter generated by all the sets $F_{c_{0}, \ldots, c_{n}}=\left\{f_{g} ; f_{g}\left(c_{i}\right)=f\left(c_{i}\right), i=0, \ldots, n\right\}$, where $c_{0}, \ldots, c_{n} \neq 0$. We have: $F_{c_{0}, \ldots, c_{n}} \cap F_{d_{0}, \ldots, d_{m}}=F_{c_{0}, \ldots, c_{n}, d_{0}, \ldots, d_{m}} \in \mathfrak{f}$ and $f$ is in $\bigcap \mathfrak{f}$. Thus $\mathfrak{f}$ is a filter on $Y$.

First we show that $\mathfrak{f}$ is weak Cauchy.
Take an arbitrary set $F_{c_{0}, \ldots, c_{n}} \in \mathfrak{f}$ and a basic neighbourhood $U_{d_{0}, \ldots, d_{m}}^{f}=\left\{f_{g} ; f_{g}\left(d_{i}\right) \in U_{i}, i=\right.$ $0, \ldots, m\}$ of $f_{0}=f$, where $U_{i}$ are neighbourhoods of $f\left(d_{i}\right)$ in $X, d_{i} \in G, i=0, \ldots, m$.

If $d_{i} \neq 0$ for all $i=0, \ldots, m$ then $U_{d_{0}, \ldots, d_{m}}^{f} \supset F_{d_{0}, \ldots, d_{m}} \in \mathfrak{f}$. We have $f_{0}=f \in F_{c_{0}, \ldots, c_{n}}$ and $f_{0}+U_{d_{0}, \ldots, d_{m}}^{f}=U_{d_{0}, \ldots, d_{m}}^{f} \in \mathfrak{f}$.

Take now $d_{0}=0$. By the previous Lemma we find such an $\alpha \neq 0$ that $f_{g_{\alpha}}\left(c_{i}\right)=f\left(c_{i}\right), i=$ $0, \ldots, n$ (which means that $f_{g_{\alpha}} \in F_{c_{0}, \ldots, c_{n}}$ ) and $f_{g_{\alpha}}\left(d_{i}\right)=f\left(d_{i}\right), i=0, \ldots, m$ and $g_{\alpha}, d_{1}+g_{\alpha}, \ldots, d_{m}+$ $g_{\alpha} \neq 0$.

The set $F_{g_{\alpha}, d_{1}+g_{\alpha}, \ldots, d_{m}+g_{\alpha}}=\left\{f_{g} ; f_{g}\left(d_{i}+g_{\alpha}\right)=f\left(d_{i}+g_{\alpha}\right)=f\left(d_{i}\right)=f_{g_{\alpha}}\left(d_{i}+g_{\alpha}\right), i=0, \ldots, m\right\} \in$ $f$.

The last set is a subset of a neighbourhood $U_{d_{0}+g_{\alpha}, \ldots, d_{m}+g_{\alpha}}^{f_{g \alpha}}=U_{d_{0}, \ldots, d_{m}}^{f}+f_{g_{\alpha}}$ of $f_{g_{\alpha}}$, which finally must belong to the filter $\mathfrak{f}$.
Filter $\mathfrak{f}$ is weak Cauchy.

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The filter $\mathfrak{f}$ is not Cauchy.
The function $f=f_{0}$ is in $\bigcap \mathfrak{f}$ and if $\mathfrak{f}$ were Cauchy it would converge to $f$. It does not because by the previous lemma for any point $x \in X \backslash V$, where $V$ is a neighbourhood of $f(0)$ in $X$, and any $F_{c_{0}, \ldots, c_{n}}$, where $c_{0}, \ldots, c_{n} \neq 0$, we find an $\alpha<\tau$ such that $f\left(c_{i}+g_{\alpha}\right)=f\left(c_{i}\right), i=0, \ldots, n$, and $f\left(g_{\alpha}\right)=x$.

Thus the neighbourhood $V^{f}=\left\{f_{d} ; f_{d}(0) \in V\right\}$ of $f$ is not in the filter $\mathfrak{f}$ because for every $F_{c_{0}, \ldots, c_{n}}$ from the base of the filter $\mathfrak{f}$ there is a function $f_{g_{\alpha}} \in F_{c_{0}, \ldots, c_{n}} \backslash V^{f}$.

## 3 Classic completion and completion of Hausdorff quasi-topological groups

Now we introduce an example of a Hausdorff quasi-topological group that does not have a Hausdorff group classic completion:

Example 3.1 Take the Abelian group $\mathbb{R} \times \mathbb{R}$.
Denote
$e=(0 ; 0)$,
$a_{n}=\left(\frac{1}{p_{n}}, \frac{1}{p_{n}}\right) \in \mathbb{R} \times \mathbb{R}, n \in \mathbb{N}$,
where $\mathbb{N}$ is the set of positive integers and $\left\{p_{n}\right\}_{n \in \mathbb{N}}$ is a (increasing) sequence of prime numbers.
Denote $B_{\varepsilon}=\{x \in \mathbb{R} \times \mathbb{R} ; d(x, e)<\varepsilon\}$, where $d$ is the Euclidean metric on $\mathbb{R} \times \mathbb{R}$.
For all points $x \in \mathbb{R} \times \mathbb{R}$ define sets $\mathcal{B}_{x}$ :
for $x=e \in \mathbb{R} \times \mathbb{R}$ let $\mathcal{B}_{x}=\left\{B_{\varepsilon} \backslash\left\{a_{n},-a_{n}\right\}_{n \in \mathbb{N}}\right\}_{\varepsilon>0}$,
for $x \in \mathbb{R} \times \mathbb{R}$ let $\mathcal{B}_{x}=\left\{G_{e}+x\right\}_{G_{e} \in \mathcal{B}_{e}}$.
First we show that the system $\left\{\mathcal{B}_{x}\right\}_{x \in X}$ generates a topology on $\mathbb{R} \times \mathbb{R}$.
It suffices to show that if $y \in G_{x} \in \mathcal{B}_{x}$ then there is a $G_{y} \in \mathcal{B}_{y}$ that $G_{y} \subset G_{x}$, see Proposition 1.2.3 in [6].

In fact, if $y \in G_{x}=G \backslash\left\{a_{n}+x,-a_{n}+x\right\}_{n \in \mathbb{N}}, y \neq x$, where $G$ is a standard open neighborhood of $x$ in $\mathbb{R} \times \mathbb{R}$, then there is a standard open neighborhood $H$ of $y$ that $H \subset G$.
As $\left\{a_{n}+x\right\}_{n \in \mathbb{N}}$ and $\left\{-a_{n}+x\right\}_{n \in \mathbb{N}}$ converge to $x$ in the standard topology, which is Hausdorff, we can assume that $H$ contains only finitely many $a_{n}+x,-a_{n}+x$, and thus we can take $H$ without points $a_{n}+x,-a_{n}+x$. Now $H \subset G_{x}$.

Take $X=\mathbb{R} \times \mathbb{R}$ with the the topology $\tau$ generated by the neighborhood system $\left\{\mathcal{B}_{x}\right\}_{x \in X}$. Clearly, it is a Hausdorff quasi-topological group.

Take the filter $\mathfrak{f}$ generated by the set $\left\{\left\{a_{n} ; n \geq n_{0}\right\}, n_{0} \in \mathbb{N}\right\}$.
We show that $\mathfrak{f}$ is classic Cauchy on $X$ (with the right (denoted by $\mathcal{R}$ ), left or two-sided semi-uniformities, which are the same on Abelian groups).

Take $U \in \mathcal{B}_{e}, R_{U}=\{(x, y) \in X \times X ; x-y \in U\} \in \mathcal{R}$.
We find a set $F \in \mathfrak{f}$ that $F \times F \subset R_{U}$, which means that we find a number $n_{0} \in \mathbb{N}$ that $\left(a_{n}, a_{m}\right) \in R_{U}$ for all $n, m \geq n_{0}$.
As $U=G \backslash\left\{a_{n},-a_{n}\right\}_{n \in \mathbb{N}}$ for some standard neighborhood $G$ of $e$ and the sequence $\left\{a_{n}, n \in \mathbb{N}\right\}$

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converges to $e$, and thus it is Cauchy, in the standard topology, there is an $n_{0} \in \mathbb{N}$ that $a_{n}-a_{m} \in G$ for all $n, m \geq n_{0}$.

There is no $r \in \mathbb{N}$ that $a_{n}-a_{m}=a_{r}$.
In fact, it would mean that there is a $r \in \mathbb{N}$ that

$$
\frac{1}{p_{n}}-\frac{1}{p_{m}}=\frac{p_{m}-p_{n}}{p_{m} p_{n}}=\frac{1}{p_{r}},
$$

in other words

$$
\frac{p_{m} p_{n}}{p_{m}-p_{n}}=p_{r} .
$$

That means that $p_{m} p_{n}=p_{r}\left(p_{m}-p_{n}\right)$, which is a contradiction because $p_{m}, p_{n}, p_{r}$ are prime numbers, and $r \neq m, n$.

Thus $a_{n}-a_{m} \in U$ for all $n, m \geq n_{0}$, and $\mathfrak{f}$ is classic Cauchy, and, of course, it has no limit in $X$.

If there is a Hausdorff classic completion $Y$ of the space $X$ the filter $\mathfrak{f}$ is classic Cauchy on $Y$, see [3], Proposition 1, and it converges to a point $y \in Y, y \neq e$. We show that $e$ and $y$ cannot be separated.

Let there be disjoint open sets $G, H \subset Y$ such that $e \in G, y \in H$.
First remind that every neighborhood $G$ of $e$ intersects every neighborhood of every point $a_{n}, n \geq n_{0}$ for some $n_{0} \in \mathbb{N}$.

As $\mathfrak{f}$ converges to $y$, and thus the sequence $\left\{a_{n}, n \in \mathbb{N}\right\}$ also converges to $y$, we can take such $n_{0}$ that $a_{n} \in H$ for all $n \geq n_{0}$, and thus $H$ is a neighborhood of all $a_{n}, n \geq n_{0}$. That is a contradiction.

The quasi-topological group $X$ has no Hausdorff (classic, weak, semi-) completion containing $X$ as a subgroup, and, thus, it is not (classically, weakly, semi-) complete.

In [3], Theorem 5, a completion (weak reflection) is constructed for all non-complete semiuniform spaces. But this completion is not a quasi-topological group, because it contains the original space as an open dense subspace.

There still remains an important question:
Question 1 Do quasi-topological groups have (non-Hausdorff) group classic completions or completions?

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# NEW CLASSES OF DIGRAPHS WITH GAUSSIAN SPECTRA HÍC Pavel, (SK), POKORNÝ Milan, (SK) 


#### Abstract

In general, the spectrum of a digraph contains both real and complex eigenvalues. A digraph is called a Gaussian integral digraph if it has a Gaussian integral spectrum, which means that all eigenvalues are Gaussian integers. In the paper the digraphs $n \circ \overline{\vec{C}}_{4}(\{s, w\})$, $K_{1,2 n} * \vec{C}_{4}\{s, w\}, \vec{P}_{m}^{2}\left(K_{1, n}\right)$ are defined. The authors determine the characteristic polynomials of these digraphs and obtain necessary and sufficient conditions for these digraphs to be Gaussian integral. The discovery of these digraphs is a new contribution to the search of Gaussian integral digraphs.


Key words. Gaussian Integral Digraph, Graph Spectrum, Divisor of a Digraph
Mathematics Subject Classification: Primary 05C50

## 1 Introduction

A digraph $\vec{G}$ consists of a finite set $V$ of points $v_{1}, v_{2}, \ldots, v_{n}$ and a set of ordered pairs of distinct points, written $(u, v)$ or briefly $u v$, called arcs. The adjacency matrix $A=A(\vec{G})$ of a labeled digraph $\vec{G}$ is the $n \times n$ matrix $\left[a_{i j}\right]$ with $a_{i j}=1$ if $v_{i} v_{j}$ is an arc of $\vec{G}$, otherwise $a_{i j}=0$. The characteristic polynomial of $\vec{G}$ is

$$
P(\vec{G})=P(\vec{G} ; x)=\operatorname{det}(x I-A)=\sum_{i=0}^{n} a_{i} x^{n-i} .
$$

The sequence $\lambda_{0}, \lambda_{1}, \ldots, \lambda_{n-1}$ of the roots of $P(\vec{G})$ is called the spectrum of $\vec{G}$ and denoted by $\operatorname{Spec}(\vec{G})$. Contrary to non-directed graphs, whose spectra are real, the spectrum of the digraph $\vec{G}$ contains both real and complex eigenvalues. A complex number $a+b i$ is called a Gaussian integer if both $a$ and $b$ are integers. The set of all these numbers is denoted by $Z[i]$. A digraph is called a Gaussian integral digraph if it has a Gaussian integral spectrum, which means that all eigenvalues
are from $Z[i]$. Of course, if it comes about that all of them are real integers, such digraph will be called integral. Some results on Gaussian integral digraphs can be found in [4, 7]. All Gaussian integral digraphs in [4] are constructed from $\vec{C}_{4}$ using a generalized coalescence of $n^{4}$ copies of $\left(\vec{C}_{4}, \vec{P}(u)\right)$, where $\vec{P}(u)$ is a dipath of length 0,1 or 2. In [7] the authors studied which circulant digraphs are Gaussian integral.
Note that from Gaussian integral digraphs $\vec{G}_{1}$ and $\vec{G}_{2}$ we can produce arbitrarily large families of Gaussian integral digraphs by means of well known binary graph operations (see for example $[4,6])$. Another families of Gaussian integral digraphs can be constructed from Gaussian integral digraphs $\vec{G}_{1}, \vec{G}_{2}, \ldots, \vec{G}_{n}$ using NEPS (see [3]).
In this paper new classes of Gaussian integral digraphs $n \circ \overline{\vec{C}}_{4}(\{s, w\}), K_{1,2 n} * \vec{C}_{4}\{s, w\}$ and $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ are constructed. We determine the characteristic polynomials of these digraphs and obtain necessary and sufficient conditions for these digraphs to be Gaussian integral. The discovery of these digraphs is a new contribution to the search of digraphs with Gausian spectra.

## 2 Preliminaries

A dipath $u \rightarrow v$ in $\vec{G}$ is an alternating sequence of distinct points and arcs beginning at $u$ and ending at $v$. A dicycle is obtained from $u \rightarrow v$ dipath by adding the arc $(v, u)$. We say that $v$ is reachable from $u$ if there exists a dipath $u \rightarrow v$. A digraph $\vec{G}$ is strongly connected or more briefly strong if every two points are mutually reachable. If both $(u, v)$ and $(v, u)$ are in $\vec{G}$, they form a symmetric pair of arcs which is denoted by $\{u, v\}$. A digraph $\vec{G}$ is nonsymmetric if not every arc lies in a symmetric pair. Thus $\vec{G}$ is nonsymmetric if and only if the matrix $A(\vec{G})$ is nonsymmetric. Two nonisomorphic digraphs $\vec{G}_{1}$ and $\vec{G}_{2}$ are called cospectral if $\operatorname{Spec}\left(\vec{G}_{1}\right)=\operatorname{Spec}\left(\vec{G}_{2}\right)$. Let the characteristic polynomial of a digraph $\vec{G}$ be $P(\vec{G} ; x)=\sum_{i=0}^{n} a_{i} x^{n-i}$. Then by Sach's theorem [2;Theorem 1.2] the coefficients $a_{i}$ of $P(\vec{G} ; x)=\sum_{i=0}^{n} a_{i} x^{n-i}$ are given by $a_{i}=\sum_{\vec{G}(i) \subset \vec{G}}^{n}(-1)^{c(\vec{G}(i))}$, where the summation extends over all subdigraphs $\vec{G}(i)$ with $i$ points, whose components are dicycles and where the exponent $c(\vec{G}(i))$ is the number of dicycles in $\vec{G}(i)$. More details on the spectra of digraphs can be found in [1].
A partition of the vertex set $V(\vec{G})=\bigcup_{i=1}^{n} V_{i}$ of a digraph $\vec{G}$ is called an equitable partition if there exists a square matrix $M=\left(d_{i j}\right)$ of order $n$ such that for every $i, j \in\{1,2, \ldots, n\}$ and for every vertex $x \in V_{i}$ there are exactly $d_{i j}$ arcs joining $x$ to vertices in $V_{j}$. The digraph $\vec{D}$ with the
adjacency matrix $M=\left(d_{i j}\right)$ is called a front-divisor (divisor) of $\vec{G}$. Obviously, the vertices of $\vec{D}$ correspond to the classes $V_{i}$ of the equitable partition. The more important property of a frontdivisor (divisor) $\vec{D}$ of a digraph $\vec{G}$ is that the characteristic polynomial $P(\vec{D} ; x)$ divides the characteristic polynomial of $\vec{G}$ ([2; Theorem 4.7]), i.e. there exists a polynomial $P(\vec{C} ; x)$ such that $P(\vec{G} ; x)=P(\vec{D} ; x) \cdot P(\vec{C} ; x)$. It is proved in [2] that $P(\vec{C} ; x)$ is the characteristic polynomial of an integer matrix $M_{\vec{C}}$. The $\vec{C}$ is called a co-divisor of $\vec{G}$ and $P(\vec{C} ; x)$ is characteristic polynomial of $\vec{C}$. More details about the theory of divisors and co-divisors can be found in [2,5].

## 3 Strong Nonsymmetric Gaussian Digraphs $n \circ \overline{\vec{C}}_{4}(\{s, w\})$

The directed cycle $\vec{C}_{4}$ is obviously Gaussian with the spectrum $\operatorname{Spec}\left(\vec{C}_{4}\right)=( \pm 1, \pm i)$. The digraph $\overline{\vec{C}}_{4}$, which is complementary to the digraph $\vec{C}_{4}$, is also Gaussian with $\operatorname{Spec}\left(\overline{\vec{C}}_{4}\right)=(2,0,-1+i,-1-i)$.
Let $\quad \overline{\vec{C}}_{4}$ be the digraph with the point set $V\left(\overline{\vec{C}}_{4}\right)=\{s, w, u, v\}$ and the arc set $E\left(\overline{\vec{C}}_{4}\right)=\{\{s, w\},\{u, v\},(s, v),(v, w),(w, u),(u, s)\}$. Let us construct the digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$, created from $n$ copies of $\overline{\vec{C}}_{4}$ by identifying undirected edge $\{s, w\}$. The digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is depicted in Figure 3.1.


Figure 3.1: Digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$
The vertex set of $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is $V\left(n \circ \overline{\vec{C}}_{4}(\{s, w\})\right)=\left\{s, w, v_{i}, u_{i} ; i=1,2, \ldots, n\right\}$ and the arc set of $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is $E\left(n \circ \overline{\vec{C}}_{4}(\{s, w\})\right)=\left\{\{s, w\},\left\{v_{i}, u_{i}\right\},\left(s, v_{i}\right),\left(v_{i}, w\right),\left(w, u_{i}\right),\left(u_{i} s\right) ; i=1,2, \ldots, n\right\}$.
From theory of divisors and co-divisors of graphs follows that the divisor $\vec{D}$ of the digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ has the vertex set $V(\vec{D})=\left\{s, w,\left[u_{1}\right],\left[v_{1}\right]\right\}$ and the arc set
$E(\vec{D})=\left\{\{s, w\},\left\{\left[u_{1}\right],\left[v_{1}\right]\right\},\left(s,\left[v_{1}\right]\right)^{n},\left(w,\left[u_{1}\right]\right)^{n},\left(\left[v_{1}\right], w\right),\left(\left[u_{1}\right], s\right)\right\}$. The vertices $\left[u_{1}\right],\left[v_{1}\right]$ of the divisor correspond to the classes of the equitable partition. The co-divisor consists of $n-1$ disjunct nondirected edges. Both divisor and co-divisor of the digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ are depicted in Figure 3.2.


Figure 3.2: Divisor and co-divisor of $n \circ \overline{\vec{C}}_{4}(\{s, w\})$
The adjacency matrix of the divisor $\vec{D}$ of the digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is

$$
A(\vec{D})=\left(\begin{array}{llll}
0 & 1 & 0 & n \\
1 & 0 & n & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{array}\right)
$$

and its characteristic polynomial is $P(\vec{D} ; x)=x^{4}-2 x^{2}-4 n x+1-n^{2}$. The characteristic polynomial of the co-divisor of the digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is $P(\vec{C} ; x)=\left(x^{2}-1\right)^{n-1}$. Thus

$$
P\left(n \circ \overline{\vec{C}}_{4}(\{s, w\}) ; x\right)=P(\vec{D}) \cdot P(\vec{C})=\left(x^{4}-2 x^{2}-4 n x+1-n^{2}\right)\left(x^{2}-1\right)^{n-1}
$$

We get the following theorem.
Theorem 3.1 The digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is Gaussian if and only if the zeros of the equation $x^{4}-2 x^{2}-4 n x+1-n^{2}=0$ belong to $Z[i]$.

Corollary 3.2 The digraph $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is Gaussian if and only if $n=k^{2}$. The spectrum of $n \circ \overline{\vec{C}}_{4}(\{s, w\})$ is $\left\{1-k, 1+k,-1-k i,-1+k i, \pm 1^{\left(k^{2}-1\right)}\right\}$.
Proof.
Obviously $\quad x^{4}-2 x^{2}-4 n x+1-n^{2}=\left(x^{2}+1\right)^{2}-(2 x+n)^{2}=\left(x^{2}+2 x+1+n\right)\left(x^{2}-2 x+1-n\right)=$ $=\left((x+1)^{2}+n\right)\left((x-1)^{2}-n\right)$. The roots of $x^{4}-2 x^{2}-4 n x+1-n^{2}=0$ belong to $Z[i]$ if and only if

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the roots of $\left((x+1)^{2}+n\right)\left((x-1)^{2}-n\right)=0$ belong to $Z[i]$, which holds only if $n=k^{2}$. Thus $\operatorname{Spec}\left(n \circ \overline{\vec{C}}_{4}(\{s, w\})\right)=\left\{1-k, 1+k,-1-k i,-1+k i, \pm 1^{\left(k^{2}-1\right)}\right\}$.

## 4 Strong Nonsymmetric Gaussian Digraphs $K_{1,2 n} * \vec{C}_{4}\{s, w\}$

Let $\vec{C}_{4}\{s, w\}$ be an directed cycle on 4 vertices, where $s$ and $w$ is a pair of diagonal vertices that are not connected by an arc. $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ is created from $n$ copies of $\vec{C}_{4}\{s, w\}$, where the diagonal vertices $s, w$ are connected by nondirected edge to a new vertex $y$. The digraph $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ is depicted in Figure 4.1.


Figure 4.1: Digraph $K_{1,2 n} * \vec{C}_{4}\{s, w\}$
The following theorem holds.
Theorem 4.1 The digraph $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ is Gaussian if and only if the zeros of $x\left(x^{4}-2 n x^{2}-(2 n+1)\right)=0$ belong to $Z[i]$.
Proof.
The divisor of $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ has the vertex set $V(\vec{D})=\left\{y,\left[s_{1}\right],\left[u_{1}\right],\left[v_{1}\right],\left[w_{1}\right]\right\}$ and the arc set $E(\vec{D})=\left\{\left(y,\left[s_{1}\right]\right)^{n},\left(y,\left[w_{1}\right]\right)^{n},\left(\left[s_{1}\right], y\right),\left(\left[w_{1}\right], y\right),\left(\left[s_{1}\right],\left[u_{1}\right]\right),\left(\left[u_{1}\right],\left[w_{1}\right]\right),\left(\left[w_{1}\right],\left[v_{1}\right]\right),\left(\left[v_{1}\right],\left[s_{1}\right]\right)\right\}$. The characteristic polynomial of the divisor is $P(\vec{D} ; x)=x\left(x^{4}-2 n x^{2}-(2 n+1)\right)$. The co-divisor of $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ consists of $n-1$ copies of $\vec{C}_{4}$, so its characteristic polynomial $P(\vec{C} ; x)=\left(x^{4}-1\right)^{n-1}$ has only Gaussian zeros. The divisor and co-divisor of $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ are depicted in Figure 4.2.


Figure 4.2: Divisor and co-divisor of $K_{1,2 n} * \vec{C}_{4}\{s, w\}$
Since $P\left(K_{1,2 n} * \vec{C}_{4}\{s, w\} ; x\right)=P(\vec{D} ; x) \cdot P(\vec{C} ; x)$, the zeros of $P\left(K_{1,2 n} * \vec{C}_{4}\{s, w\} ; x\right)$ are Gaussian if and only if the zeros of $P(\vec{D} ; x)=x\left(x^{4}-2 n x^{2}-(2 n+1)\right)$ are Gaussian.

Corollary 4.2 For every $n=4 \cdot \sum_{i=0}^{k} i, k \in N_{0}$ the digraph $K_{1,2 n} * \vec{C}_{4}\{s, w\}$ is Gaussian and its spectrum is $\left\{0, \pm 1^{n-1}, \pm i^{n}, \pm\left(1+8 \cdot \sum_{i=0}^{k} i\right) ; k \in N_{0}\right\}$.
Proof.
It is sufficient to study whether the zeros of $x^{4}-2 n x^{2}-(2 n+1)=0$ are Gaussian. Using $x^{2}=y$ we have $y^{2}-2 n y-(2 n+1)=0$, whose zeros are $2 n+1$ and -1 . From $x^{2}=-1$ we get $x_{1,2}= \pm i$, from $x^{2}=2 n+1$ we get $n=4 \cdot \sum_{i=0}^{k} i, k \in N_{0}$ and $x_{3,4}= \pm\left(1+8 \cdot \sum_{i=0}^{k} i\right) ; k \in N_{0}$.

## 5 Strong Nonsymmetric Gaussian Digraphs $\vec{P}_{m}^{2}\left(K_{1, n}\right)$

Now we obtain a family of Gaussian digraphs from a star $K_{1, n}$. We will construct a digraph $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ by joining the endpoints of $K_{1, n}$ with $m$ new dipaths of length 2 . Furthermore, none of these dipaths of length 2 can create a dipath of length more than 2.

Example 5.1 The digraph $\vec{P}_{4}^{2}\left(K_{1,3}\right)$ is depicted in Figure 5.1. As the characteristic polynomial of $\vec{P}_{4}^{2}\left(K_{1,3}\right)$ is $P\left(\vec{P}_{4}^{2}\left(K_{1,3}\right) ; x\right)=x^{4}\left(x^{4}-3 x^{2}-4\right)=x^{4}(x-2)(x+2)(x-i)(x+i)$, the spectrum of $\vec{P}_{4}^{2}\left(K_{1,3}\right)$ is $\left\{ \pm 2, \pm i, 0^{4}\right\}$. So the digraph $\vec{P}_{4}^{2}\left(K_{1,3}\right)$ is Gaussian.


Figure 5.1: Non-isomorphic cospectral digraphs $\vec{P}_{4}^{2}\left(K_{1,3}\right)$
Remark 5.2 If we place four dipaths of length 2 from example 5.1 in different way, we get nonisomorphic digraph, which is cospectral to the original digraph. There exist five non-isomorphic cospectral digraphs $\vec{P}_{4}^{2}\left(K_{1,3}\right)$. They are depicted in figure 5.1.

Let us generalize the knowledge from example 5.1.
Theorem 5.3 A digraph $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ is Gaussian integral if and only if $m=a^{2} b^{2}, n=a^{2}-b^{2}$, $a>b>1, a, b \in N$. The spectrum of the digraph $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ is $\left\{ \pm a, \pm b i, 0^{n+m-3}\right\}$.
Proof.
The characteristic polynomial of $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ is $P\left(\vec{P}_{m}^{2}\left(K_{1, n}\right) ; x\right)=x^{n+m-3}\left(x^{4}-n x^{2}-m\right)$. It follows from Sach's Theorem that $a_{2}=-n, a_{4}=-m$ and $a_{i}=0$ for $i \neq 2, i \neq 4$. As the digraph $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ is Gaussian if and only if $\left(x^{4}-n x^{2}-m\right)=\left(x^{2}-a^{2}\right)\left(x^{2}+b^{2}\right) ; a, b \in N$, we get $a^{2} \cdot b^{2}=m, a^{2}-b^{2}=n, \quad a>b>1, \quad a, b \in N$. The spectrum of the digraph $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ is $\left\{ \pm a, \pm b i, 0^{n+m-3}\right\}$. So the digraph $\vec{P}_{m}^{2}\left(K_{1, n}\right)$ is Gaussian integral.

Question 5.4 Let $f(m, n)$ denote the number of cospectral Gaussian digraphs $\vec{P}_{m}^{2}\left(K_{1, n}\right)$. From remark 5.2 we have $f(4,3)=5$. Is there any explicit formula for $f(m, n)$ ?

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# THE QUEST FOR DE-GROOT-LIKE DUAL OF PRETOPOLOGICAL SYSTEMS WITH MATHEMATICA AS OF A TOOL OF VISUALIZATION 

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#### Abstract

In this paper we study the preframe structure, representing the opens of a possible pretopological system, and its behavior with respect to the certain de Groot-like dualization construction. We present some counterexamples, contributing to the discussion regarding the possibility of obtaining similar results as there are known for the topological spaces. We also present a Mathematica 7 compatible package, which demonstrates and visualize the problem for the finite posets of opens.


Key words and phrases. Preframe, pretopological system, de Groot-like dual for preframes and pretopological systems.
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## 1 Introduction and Status of the Problem

For a given topological space $(X, \tau)$, a topology $\tau^{d}$, generated by the family of all compact saturated sets used as its closed base, is called the de Groot dual of the original topology. Jimmie Lawson and Michael Mislove stated in [6] a problem, whether the sequence $\tau^{d}$, $\tau^{d d}$, $\tau^{d d d}, \ldots$, containing the iterated duals of $\tau$, is infinite or the process of taking duals terminates after finitely many steps with topologies that are dual to each other. The problem was solved by the second author, who in 2001 proved that for any topology it holds $\tau^{d d}=\tau^{d d d d}$ [4] (the result was first announced and communicated on Toposym, in Prague 2001) and in 2004 the result was improved (again by the second author) to its final form $\tau^{d}=\left(\tau \vee \tau^{d d}\right)^{d}[5]$. Note that from the last mentioned equality it follows that for any topology, $\tau^{d} \subseteq \tau^{d d d}$. It should be also noted that the paper [4] pointed out several natural questions regarding the dual topologies.

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Some of them were addressed by the recent paper [8] of Tomoo Yokoyama. However, the second part of the original question of J. Lawson and M. Mislove stated in [6], which topologies can arise as duals, still remains open.

The questions of J. Lawson and M. Mislove related to the de Groot dual arise from an alternate approach to the certain constructions of various semantic models in the theoretical computer science, where the dual and the patch topologies constitute an important tool of investigation. However, there is a much wider class of related algebraic and topological structures, like topological systems, frames, locales, and many other, having even a greater importance for the topic than the class of topological spaces itself. An interesting direction of research was introduced by Bernhard Banaschewski [1], who replaced the usual frame structure (for example, of open sets in a topological space) by a more general, partially ordered structure called preframe, where the suprema exist for all non-empty up-directed subcollections. B. Banaschewski founded this structure useful for his alternate proof [1] of a relatively familiar result of P. Johnstone [3] - the localic version of the well-known Tichonov Theorem.

It is natural and potentially useful for applications in the theoretical computer science to study the preframe structure in connections with a proper modification of the de Groot dual. Although we do not have an analogue of the results of the second author reached for the topological spaces yet, some first attempts are already contained in the our paper [2], where we defined a counterpart of the de Groot dual for a certain class of pretopological systems. Pretopologial systems form a slight generalization of the familiar notion of the topological systems (see, for example, [7] for the exact definition), where the frame structure of opens is replaced by the preframe structure. The pretopological systems, for which our modification of the de Groot dual is possible, are similar to the localic topological systems; also for them the abstract points of the system are fully determined by the structure of opens. We call these pretopological systems them compactly-localic. For the definitions and more detail, the reader is referred to [2].

In this paper we will concentrate on the preframe structure of the opens of the pretopological counterpart of the de Groot dual. It has been shown in [2], that under some circumstances, the opens of the dual may be represented as certain maps from $A$ to the Sierpiński frame 2, where $A$ is the poset representing the opens of the original pretopological system. If we denote the poset of such maps by $\langle A \rightarrow \mathbf{2}\rangle$, the sequence of the iterated duals then have the form of $\langle A \rightarrow \mathbf{2}\rangle,\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle,\langle\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle, \ldots$, etc. So far we do not know whether there exist (and hold) appropriate, full-featured counterparts of the results of the second author for the pretopological systems and how they should look. But some preliminary results and counterexamples presented in this paper illustrate the difficulties which should be overwhelmed in order to reach some final, positive result.

In parallel to the theoretical part of the paper, we present here also a Mathematica 7 compatible package IteratedDuals.m which calculates the first three elements of the previously mentioned sequence representing the iterated duals for a finite poset $A$. The package also displays their Hasse diagrams. Although the package itself has no theoretical importance for our investigation, it can serve as a useful visualization and demonstration tool, conducing to the reader's convenience and comfort. Another, alternate reason for presenting of the package (at the conference Aplimat) is to move the usage of the software Mathematica somewhat towards to the more theoretical disciplines, and demonstrate its utility - even for such theoretical
disciplines, as topology.
Now, let us recall some notions and make some denotations. We say that a partially ordered set (briefly a poset) $A$ is a preframe [1], if $A$ is closed under directed joins and finite meets (including the meet of the empty set), such that the binary meets distribute over the directed joins. It should be noted that by the usual definition, a directed set is non-empty, so the preframe need not have the least element - the supremum of the empty set (that is, $\bigvee \varnothing$ ). On the other hand, a preframe always has the greatest element $\wedge \varnothing$. By $\mathbf{2}=\{\perp, \top\}$ we denote the Sierpinski frame, consisting of the two elements, $T$ and $\perp$. Let $A$ be a set, then each mapping $f: A \rightarrow \mathbf{2}$ can be uniquely identified with its T-kernel, $\operatorname{Ker}_{\top} f=\{x \mid x \in A, f(x)=\top\}$. In this way, we can equip $\mathbf{2}^{A}$ with the partial order, given by the inclusion on the power set $2^{A}$. By False and True we denote the constant functions on $A$ identically equal to $\perp$ and T , respectively. Let $A, B$ be posets. We say that a mapping $f: A \rightarrow B$ is a morphism if it preserves the (non-empty) directed joins and finite meets (including the meet of the empty set), whenever they exist. The set of all morphisms $f: A \rightarrow \mathbf{2}$ we denote by $\langle A \rightarrow \mathbf{2}\rangle$. We consider it as a poset, naturally equipped with the order induced from the partially ordered set $\mathbf{2}^{A}$. Throughout this paper, if not otherwise stated, by the dual of a poset $A$ we mean the set $\langle A \rightarrow \mathbf{2}\rangle$ with this order.

## 2 Theoretical Results

We will start with the following proposition.
Proposition 2.1 Let $A$ be a poset. Then $\langle A \rightarrow \mathbf{2}\rangle$ forms a preframe of all morphisms of $A$ to 2.

Proof. We will show that $\langle A \rightarrow \mathbf{2}\rangle$ has the non-empty directed joins, all finite meets (including the meet of the empty set) and that the meets distribute over the directed joins.

Let $Y \subseteq\langle A \rightarrow \mathbf{2}\rangle$ be non-empty and directed. Let $f(a)=\bigvee_{y \in Y} y(a)$ for every $a \in A$. We will show that $f=\bigvee Y$ in $\langle A \rightarrow \mathbf{2}\rangle$. First, we must show that $f \in\langle A \rightarrow \mathbf{2}\rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in $A$. Then $f(\bigvee B)=\bigvee_{y \in Y} y(\bigvee B)=$ $\bigvee_{y \in Y} \bigvee_{b \in B} y(b)=\bigvee_{b \in B} \bigvee_{y \in Y} y(b)=\bigvee_{b \in B} f(b)$, so $f$ preserves non-empty directed joins. Let $C \subseteq A$ be non-empty and finite. Suppose that $\bigwedge C$ exists in $A$. Then $f(\bigwedge C)=\bigvee_{y \in Y} y(\bigwedge C)=$ $\bigvee_{y \in Y} \bigwedge_{c \in C} y(c)=\top$ implies that there exist some $y_{1} \in Y$, such that for every $c \in C$ it follows $y_{1}(c)=\top$. Then $\top=\bigwedge_{c \in C} \bigvee_{y \in Y} y(c)=\bigwedge_{c \in C} f(c)$ which implies $f(\bigwedge C) \leq \bigwedge_{c \in C} f(c)$. Conversely, suppose that $\bigwedge_{c \in C} f(c)=\bigwedge_{c \in C} \bigvee_{y \in Y} y(c)=\top$. Then for every $c \in C$ there is some $y_{c} \in Y$ with $y_{c}(c)=T$. Since $Y$ is directed and $C$ is finite, there exist some $y_{1} \in Y$ such that $y_{1} \geq y_{c}$ for every $c \in C$. Hence, for every $c \in C$ it follows $y_{1}(c)=T$. Then $\top=\bigvee_{y \in Y} \bigwedge_{c \in C} y(c)=\bigvee_{y \in Y} y(\bigwedge C)=f(\bigwedge C)$ which implies that $f(\bigwedge C) \geq \bigwedge_{c \in C} f(c)$. Now we have $f(\bigwedge C)=\bigwedge_{c \in C} f(c)$, so $f$ preserves also non-empty finite meets. It remains to check the preservation of the empty meet. Suppose that $A$ has the greatest element $\Lambda \varnothing \in A$. Then $f(\bigwedge \varnothing)=\bigvee_{y \in Y} y(\bigwedge \varnothing)=\bigvee_{y \in Y} \top=\top$. Hence, $f$ is an element of $\langle A \rightarrow \mathbf{2}\rangle$, and, clearly, an upper bound of $Y$ in $\langle A \rightarrow \mathbf{2}\rangle$. Now, let $u \in\langle A \rightarrow \mathbf{2}\rangle$ be another upper bound of $Y$. Then, for
every $a \in A$ and every $y \in Y$ it follows that $u(a) \geq y(a)$, which gives $u(a) \geq \bigvee_{y \in Y} y(a)=f(a)$ and, consequently, $u \geq f$. So f is a correctly defined supremum of $Y$ in $\langle A \rightarrow \mathbf{2}\rangle$.

Suppose that $Z \subseteq\langle A \rightarrow \mathbf{2}\rangle$ is non-empty and finite. Let $g(a)=\bigwedge_{z \in Z} z(a)$ for every $a \in A$. We will show that $g=\bigwedge Z$ in $\langle A \rightarrow \mathbf{2}\rangle$. First, we must show that $g \in\langle A \rightarrow \mathbf{2}\rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in $A$. Then $g(\bigvee B)=\bigwedge_{z \in Z} z(\bigvee B)=$ $\bigwedge_{z \in Z} \bigvee_{b \in B} z(b)=\top$ implies that for every $z \in Z$ there is $b_{z} \in B$ with $z\left(b_{z}\right)=\top$. Since $Z$ is finite and $B$ is directed, there is some $b_{1} \in B$ such that $b_{1} \geq b_{z}$ for every $z \in Z$. Then $z\left(b_{1}\right)=\top$ for every $z \in Z$, which implies that $T=\bigvee_{b \in B} \bigwedge_{z \in Z} z(b)=\bigvee_{b \in B} g(b)$. Hence, $g(\bigvee B) \leq \bigvee_{b \in B} g(b)$. Conversely, suppose that $\bigvee_{b \in B} g(b)=\bigvee_{b \in B} \bigwedge_{z \in Z} z(b)=\top$. Then, there exists $b_{1} \in B$, such that $z\left(b_{1}\right)=\top$ for every $z \in Z$. Then $\top=\bigwedge_{z \in Z} \bigvee_{b \in B} z(b)=\bigwedge_{z \in Z} z(\bigvee B)=g(\bigvee B)$. It follows that $g(\bigvee B) \geq \bigvee_{b \in B} g(b)$ and hence, together with the previously proved (converse) inequality, we have $g(\bigvee B)=\bigvee_{b \in B} g(b)$. Now, let $C \subseteq A$ be non-empty and finite, having $\Lambda C \in A$. Then $g(\bigwedge C)=\bigwedge_{z \in Z} z(\bigwedge C)=\bigwedge_{z \in Z} \bigwedge_{c \in C} z(c)=\bigwedge_{c \in C} \bigwedge_{z \in Z} z(c)=\bigwedge_{c \in C} g(c)$. Finally, suppose that $A$ has the greatest element $\bigwedge \varnothing \in A$. Then $g(\bigwedge \varnothing)=\bigwedge_{z \in Z} z(\bigwedge \varnothing)=\bigwedge_{z \in Z} \top=\top$. It follows that $g$ is an element of $\langle A \rightarrow \mathbf{2}\rangle$, and, clearly, a lower bound of $Z$ in $\langle A \rightarrow \mathbf{2}\rangle$. Let $l \in\langle A \rightarrow \mathbf{2}\rangle$ be a lower bound of $Z$. Then, for every $a \in A$ and every $z \in Z$ we have $l(a) \leq z(a)$, which gives $l(a) \leq \bigwedge_{z \in Z} z(a)=g(a)$ and, consequently, $l \leq g$. Therefore, $g$ is a correctly defined infimum of $Z$ in $\langle A \rightarrow \mathbf{2}\rangle$. Moreover, the mapping True, constantly equal to $T$, obviously preserves all non-empty directed joins and all finite meets, so $\langle A \rightarrow \mathbf{2}\rangle$ also has the greatest element. Note that True does not preserve the empty join, but it is not required.

Finally, we will show that binary meets distribute over directed joins in $\langle A \rightarrow \mathbf{2}\rangle$. Let $x \in\langle A \rightarrow \mathbf{2}\rangle$ and $Y \subseteq\langle A \rightarrow \mathbf{2}\rangle$ be directed. Then $(x \wedge(\bigvee Y))(a)=x(a) \wedge(\bigvee Y)(a)=$ $x(a) \wedge\left(\bigvee_{y \in Y} y(a)\right)=\bigvee_{y \in Y}(x(a) \wedge y(a))=\bigvee_{y \in Y}((x \wedge y)(a))=\left(\bigvee_{y \in Y}(x \wedge y)\right)(a)$ for every $a \in A$, which implies $x \wedge(\bigvee Y)=\bigvee_{y \in Y}(x \wedge y)$. By the definition, $\langle A \rightarrow \mathbf{2}\rangle$ is a preframe.

Note that it may happen that $\langle A \rightarrow \mathbf{2}\rangle$ has the bottom $\perp=\bigvee \varnothing$ although the constant mapping False : $A \rightarrow \mathbf{2}$ with the constant value $\perp$ need not be an element of $\langle A \rightarrow \mathbf{2}\rangle$. Let $A$ be a poset. Let us denote by $h_{A}: A \rightarrow\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle$ a mapping for which $h_{A}(a)(x)=x(a)$ for every $x \in\langle A \rightarrow \mathbf{2}\rangle$.

Proposition 2.2 Let $A$ be a poset. Then $h_{A}: A \rightarrow\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle$ is a morphism.

Proof. Suppose that there exists the greatest element $\Lambda \varnothing \in A$. It follows that $h_{A}(\Lambda \varnothing)(x)=$ $x(\bigwedge \varnothing)=\mathrm{T}$ for every morphism $x \in\langle A \rightarrow \mathbf{2}\rangle$, so $h_{A}(\bigwedge \varnothing)=\mathrm{T}$.

Let $B \subseteq A$ be non-empty and directed and suppose that there exists $\bigvee B \in A$. Let $x \in\langle A \rightarrow \mathbf{2}\rangle$. Then $h_{A}(\bigvee B)(x)=x(\bigvee B)=\bigvee_{b \in B} x(b)=\bigvee_{b \in B} h_{A}(b)(x)=\left(\bigvee_{b \in B} h_{A}(b)\right)(x)$, which implies that $h_{A}(\bigvee B)=\bigvee_{b \in B} h_{A}(b)$.

Let $C \subseteq A$ be non-empty, finite and assume that there exists $\Lambda C \in A$. Let $x \in\langle A \rightarrow \mathbf{2}\rangle$. It follows $h_{A}(\bigwedge C)(x)=x(\bigwedge C)=\bigwedge_{c \in C} x(c)=\bigwedge_{c \in C} h_{A}(c)(x)=\left(\bigwedge_{c \in C} h_{A}(c)\right)(x)$, which implies that $h_{A}(\bigwedge C)=\bigwedge_{c \in C} h_{A}(c)$.

Since $h_{A}$ preserves all non-empty directed joins and all finite meets, it follows that $h_{A}$ is a morphism.

Example 2.1 There exist a preframe $A$ such that $h_{A}$ is not an epimorphism.

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Construction. Let $A=\omega+1=\{1,2, \ldots, \omega\}$, where $\omega$ is the first infinite ordinal, with its natural linear order. Let $n^{\prime}: A \rightarrow \mathbf{2}$ be a mapping with the T-kernel $\{n, n+1, \ldots, \omega\}$ for every $n \in \mathbb{A}$ and $(\omega+1)^{\prime}$ be a mapping identically equal to $\perp$. The construction is illustrated by the figure:


Figure 1.
Since every morphism is an isotone mapping and the constant mapping with the empty T-kernel, $(\omega+1)^{\prime}=$ False, is not a morphism, it is not difficult to check that $\langle A \rightarrow \mathbf{2}\rangle=$ $\left\{\omega^{\prime}, \ldots, 2^{\prime}, 1^{\prime}\right\}$. Notice that $\langle A \rightarrow \mathbf{2}\rangle$ is linearly ordered by the set inclusion of the corresponding T-kernels of its elements. For every $x \in\langle A \rightarrow \mathbf{2}\rangle$ we put

$$
p(x)= \begin{cases}\top, & \text { for } x>\omega^{\prime} \\ \perp, & \text { for } x=\omega^{\prime}\end{cases}
$$

Obviously $p$ is a morphism, so $p \in\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle$. But for every $a \in A$ and every $x \in\langle A \rightarrow \mathbf{2}\rangle$ it follows

$$
h_{A}(a)(x)=x(a)=\left\{\begin{array}{l}
\top \text { for } x \geq a^{\prime} \\
\perp \text { for } x<a^{\prime}
\end{array}\right.
$$

Therefore, there is no $a \in A$ such that $p=h_{A}(a)$, which implies that $h_{A}$ is not a surjection.
Example 2.2 There exist a preframe $A$ such that $h_{A}$ is not a monomorphism.
Construction. Let $A=\{0,1, \ldots, \omega\}$ be the distributive lattice with the Hasse diagram given by the figure:


Figure 2.

That means, 0 is the bottom, $\omega$ is the top, $2 k$ has two successors $2 k+1,2 k+2$ and $2 k+1$ has a unique successor $2 k+3$ for every $k \in\{0,1, \ldots\}$. Since for every $Y \subseteq A$ infinite it follows $\bigvee Y=\omega$, the binary meets distribute over all joins. It follows that $A$ is preframe (moreover, a frame).

Let $x \in\langle A \rightarrow 2\rangle$. We will show that $x(0)=x(1)$. If $x(0)=\mathrm{T}$, then also $x(1)=x(0 \vee 1)=$ $x(0) \vee x(1)=\top \vee x(1)=\top$. Suppose that $x(2 k)=\perp$ for every $k \in\{0,1, \ldots\}$. The set $S=\{2,4, \ldots\}$ infinite and directed. It follows that $x(\omega)=x(\bigvee S)=\bigvee_{s \in S} x(s)=\bigvee_{s \in S} \perp=\perp$. Then $x(1)=x(1 \wedge \omega)=x(1) \wedge x(\omega)=x(1) \wedge \perp=\perp$. Finally, suppose that $k$ is the greatest number from $\{0,1, \ldots\}$ such that $x(2 k)=\perp$. But $\perp=x(2 k)=x((2 k+2) \wedge(2 k+1))=$ $x(2 k+2) \wedge x(2 k+1)=\top \wedge x(2 k+1)$, which implies $x(2 k+1)=\perp$. Then $x(1)=x(1 \wedge(2 k+1))=$ $x(1) \wedge x(2 k+1)=x(1) \wedge \perp=\perp$. Therefore, $h_{A}(0)=h_{A}(1)$ which implies that $h_{A}$ is not injective.

Proposition 2.3 Let $A$ be a finite preframe. Then $h_{A}: A \rightarrow\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle$ is an isomorphism.

Proof. Let $p \in\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle$. We put $x_{1}=\bigwedge \operatorname{Ker}_{\top}(p)$ and $a_{1}=\bigwedge \operatorname{Ker}_{\top}\left(x_{1}\right)$, where $\operatorname{Ker}_{\mathrm{T}}$ is a denotation for the T -kernel of a mapping with its co-domain equal to a subset of 2. Since $A$ is finite, also $\operatorname{Ker}_{\top}(p)$ is a finite set; say $\operatorname{Ker}_{\top}(p)=\left\{y_{1}, y_{2}, \ldots y_{k}\right\}$. Then $p\left(x_{1}\right)=p\left(y_{1} \wedge p_{2} \wedge \cdots \wedge p_{k}\right)=p\left(y_{1}\right) \wedge p\left(y_{2}\right) \wedge \cdots \wedge p\left(y_{k}\right)=\mathrm{T}$, which means that $x_{1}$ is the least element of $\operatorname{Ker}_{\top}(p)$. Similarly, $a_{1}$ is the least element of $\operatorname{Ker}_{\top}\left(x_{1}\right)$.

We claim that $h_{A}\left(a_{1}\right)=p$. Indeed, for every $y \in\langle A \rightarrow \mathbf{2}\rangle$ it follows $p(y)=T \Leftrightarrow y \in$ $\operatorname{Ker}_{\top}(p) \Leftrightarrow x_{1} \leq y \Leftrightarrow \operatorname{Ker}_{\top}\left(x_{1}\right) \subseteq \operatorname{Ker}_{\top}(y) \Leftrightarrow a_{1} \in \operatorname{Ker}_{\top}(y) \Leftrightarrow h_{A}\left(a_{1}\right)(y)=y\left(a_{1}\right)=\top$. Hence, $h_{A}\left(a_{1}\right)=p$ which means that $h_{A}$ is surjective.

Suppose that there exist $a_{2} \in A$ such that also $h_{A}\left(a_{2}\right)=p$. Then $x_{1}\left(a_{2}\right)=\top$, which implies that $a_{1} \leq a_{2}$. Suppose that $a_{1} \neq a_{2}$ and let

$$
z(a)= \begin{cases}\top, & \text { for } a \geq a_{2} \\ \perp, & \text { otherwise }\end{cases}
$$

Since $A$ is finite, $z$ obviously preserves all directed joins. Then $z(\bigwedge \varnothing)=\top$ since $a_{1}<a_{2} \leq \Lambda \varnothing$. Let $a, b \in A$. Then $z(a \wedge b)=\top \Leftrightarrow a \wedge b \geq a_{2} \Leftrightarrow a \geq a_{2}$ and $b \geq a_{2} \Leftrightarrow z(a)=\top$ and $z(b)=$ $\top \Leftrightarrow z(a) \wedge z(b)=\top$. Hence, $z(a \wedge b)=z(a) \wedge z(b)$. It follows that $z$ preserves also all finite meets. Then $z \in\langle A \rightarrow \mathbf{2}\rangle$, but $h_{A}\left(a_{1}\right)(z)=z\left(a_{1}\right)=\perp \neq \top=z\left(a_{2}\right)=h_{A}\left(a_{2}\right)(z)$, which is a contradiction. Therefore, $a_{1}=a_{2}$, which implies that $h_{A}$ is injective.

The following corollary is an immediate consequence of the previous proposition and Proposition 2.1.

Corollary 2.3 Let $A$ be a finite poset. Then its iterated duals, $\langle A \rightarrow \mathbf{2}\rangle$ and $\langle\langle\langle A \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle \rightarrow \mathbf{2}\rangle$, are isomorphic.

## 3 Interactions with Mathematica

For a visualization of the problem, we have developed a Mathematica package IteratedDuals.m, which can display the Hasse diagrams of the original, finite poset and its first three iterated duals. There are certainly several possibilities how to represent a poset in Mathematica. In our package, a poset is considered as embedded to a suitable power set, partially ordered by the inclusion. This representation of posets is well compatible with the approach used in the theoretical part of this paper, because the set of all maps from a poset $A$ to the Sierpiński frame $\mathbf{2}$ is naturally frame-isomorphic with the power set $\mathbf{2}^{A}$ equipped with the inclusion partial order.

In the package IteratedDuals.m, the elements of $\langle A \rightarrow \mathbf{2}\rangle$ are represented in two complementary formats. In the first one, the map $f: A \rightarrow \mathbf{2}$ is represented as a binary relation, that is, as an element of $A \times \mathbf{2}$. This representation is more suitable for verifying whether a map $f$ is a morphism; and since all our considered sets are finite, only the preservation of finite meets (including the meet of the empty set) is necessary to check. In the package, the "is-morphism-test" is provided by a combination of the following functions:

```
(* Check if an element is a lower bound of a set *)
IsLowerBound2TF[Element_, ElementSet_List]:=Module[{n,i, Passed},
n=Length[ElementSet];
Passed=True;
For[i=1, i<=n,i++,If [Element \cap ElementSet[[i]]!=Element,Passed=False]];
Return[Passed]];
(* Find all lower bounds of a set *)
LowerBounds[ElementSet_List, OrderedSet_List]:=Module[{n,i, LBounds},
n=Length[OrderedSet];
LBounds={};
For[i=1,i<=n,i++,If[IsLowerBound2TF[OrderedSet[[i]], ElementSet],
LBounds=Append[LBounds, OrderedSet[[i]] ]]];
Return[Union[LBounds]] ];
(* Check whether the infimum of a set in an ordered set exists *)
MeetExists[ElementSet_List,OrderedSet_List]:=Module[{LBounds,MeetCandidate,ESet, MExists},
MExists=False;
ESet=Union[ElementSet];
LBounds=LowerBounds[ESet, OrderedSet];
MeetCandidate=Union[Flatten[LBounds,1]];
If[ESet \cap OrderedSet==ESet,If [MemberQ[LBounds,MeetCandidate],MExists=True]];
Return[MExists] ];
(* Calculate the infimum - meet of a set in an ordered set *)
Meet[ElementSet_List,OrderedSet_List]:=Module[{LBounds,MeetCandidate,ESet},
ESet=Union[ElementSet];
LBounds=LowerBounds[ESet, OrderedSet];
MeetCandidate=Union[Flatten[LBounds,1]];
If[ESet \cap OrderedSet==ESet,If[MemberQ[LBounds, MeetCandidate],
Return[MeetCandidate]]]];
(* Check if a map is a morphism *)
IsMorphism2TF[Mapping_List]:=Module[{X, i, j, n, MeetTemp, Passed},
```


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```
X=Domain [Mapping] ;
n=Length [Mapping];
Passed=True;
For[i=1, i<=n,i++,
For[j=1, j<=n, j++,
If [MeetExists[{Mapping[[i]][[1]],Mapping[[j]][[1]]}, X],
If [!MemberQ[Mapping,{Meet[{Mapping[[i]][[1]],Mapping[[j]][[1]]}, X],
Mapping[[i]][[2]] ^ Mapping[[j]][[2]] }], Passed=False]] ]];
If[MeetExists[{},X],
If[!MemberQ[Mapping, {Meet[{},X],True}], Passed=False]];
Return[Passed] ];
```

In the second format, the map $f: A \rightarrow \mathbf{2}$ is represented by its $T$-kernel (which coincides, in this case, with another notion, in some literature referred as the support of $f$ ) - the set of all elements in $A$ for which the value of $f$ is $T$. This representation is more suitable and efficient for working with the partial order on $\langle A \rightarrow \mathbf{2}\rangle$, and displaying the corresponding Hasse diagrams. For the conversion from the first format to the second one, we use the following functions:

```
(* Find the support of a map *)
Support[Mapping_List]:=Module[{Function,n,i, Supp},
Function=Union [Mapping];
n=Length[Function];
Supp={};
For[i=1, i<=n,i++,If[Function[[i]][[2]]==True, Supp=Append[Supp,
Function[[i]][[1]]]]];
Return[Union[Supp]] ];
(* Convert maps to their supports *)
Maps2Supports[x_List]:=Module[{Maps, n, SetofSupports},
Maps=Union[x];
n=Length [Maps];
SetofSupports={};
For[i=1, i<=n,i++, SetofSupports=Append[SetofSupports,Support[Maps[[i]]] ]];
Return[Union[SetofSupports]] ];
```

For displaying the Hasse diagrams, our package internally uses the functions HasseDiagram and ShowLabeledGraph, which are included in the standard Mathematica package Combinatorica. This package is automatically loaded with IteratedDuals. Below there is an example of the usage:

```
<< IteratedDuals
```

Poset $=\{\{0,1\},\{1,2\},\{2,0\},\{0\},\{ \}\} ;$
ShowDuals[Poset]
Poset and its three iterated duals:


The Legend:
The elements of the first dual:
$a=\{ \}$
$\mathrm{b}=\{\{0,1\}\}$
$c=\{\{0,2\}\}$
$\mathrm{d}=\{\{1,2\}\}$
$e=\{\{0\},\{0,1\},\{0,2\}\}$
$f=\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}$
The elements of the second dual:
$\mathrm{aa}=\{\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}\}$
$\mathrm{bb}=\{\{\{1,2\}\},\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}\}$
$c \mathrm{c}=\{\{\{0\},\{0,1\},\{0,2\}\},\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}\}$
$d d=\{\{\{0,1\}\},\{\{0\},\{0,1\},\{0,2\}\},\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}\}$
$\mathrm{ee}=\{\{\{0,2\}\},\{\{0\},\{0,1\},\{0,2\}\},\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}\}$
$f f=\{\{ \},\{\{0,1\}\},\{\{0,2\}\},\{\{1,2\}\},\{\{0\},\{0,1\},\{0,2\}\},\{\{ \},\{0\},\{0,1\},\{0,2\},\{1,2\}\}\}$
(the output is shortened)
Note that the package IteratedDuals can be installed by the usual way to the appropriate directory and it is compatible with Mathematica 7.0. The package will be freely available and included with the Aplimat conference materials.

## 4 Conclusion

Although for a finite poset $A$ we have an adequate counterpart of the result $\tau^{d} \subseteq \tau^{d d d}$ proved by the second author for the topological spaces (as it is shown in Proposition 2.3), the counterexamples in Example 2.1 and Example 2.2 demonstrate that a requested general result cannot be reached in this form. The reason for the negative conclusion possibly could lie in the fact, that unlike in the topological case, the points of the corresponding pretopological system are modified in each step of taking the dual. The fact of modifying the underlying set of points we should take into our considerations and appropriately adjust the way how the posets of dualized opens are compared. We will concentrate on this topic in more detail in our forthcoming paper.

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# COUNTABLE ACTIONS OF HYPERSTRUCTURES <br> AND PSEUDO-METRIZABLE UNIFORM SPACES 

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#### Abstract

There is generalized concept of general hyperstructures (A. D. Nezhad, R. S. Hashemi) onto the case of countable many basic sets (spaces) called general $\omega$ hyperstructures. This construction is then applied to pseudo-metrizable uniform spaces; in particular there is proved that isomorphism of certain general $\omega$-hyperstructures of tolerance spaces with the same carrier set implies uniform equivalence of pseudo-uniform spaces induced by countable families of tolerances forming bases of uniformities.

Key words and phrases. General hyperoperation, countable sequence of tolerances, general $\omega$-hyperstructure, ordered set, pseudo-metrizable uniform space. Mathematics Subject Classification. Primary 20N20; Secondary 54E15.


Throughout this paper, the symbol $X, Y$ will denote two non-empty sets, where $P^{*}(X \cup Y)$ denotes the set of all non-empty subsets of $X \cup Y$.

A general hyperstructure is formed by two non-empty sets $X, Y$ together with a hyperoperation,

$$
\begin{gathered}
*: X \times Y \longrightarrow P^{*}(X \cup Y) \\
(x, y) \mapsto x * y \subseteq(X \cup Y) \backslash \emptyset .
\end{gathered}
$$

Remark. A general hyperoperation $*: X \times Y \longrightarrow P^{*}(X \cup Y)$ yields a map of powersets determined by this hyperoperation.

Thus the map $\otimes: P^{*}(X) \times P^{*}(Y) \longrightarrow P^{*}(X \cup Y)$ is defined by $A \otimes B=\underset{a \in A, b \in B}{\bigcup} a * b$.
Conversely an general hyperoperation on $P^{*}(X) \times P^{*}(Y)$ yields a general hyperoperation on $X \times Y$, defined by $x * y=\{x\} \otimes\{y\}$.

In the above definition if $A \subseteq X, B \subseteq Y, x \in X, y \in Y$, then we define,

$$
\begin{gathered}
A * y=A *\{y\}=\bigcup_{a \in A} a * y, \\
x * B=\{x\} * B=\bigcup_{b \in B} x * b, \\
A \otimes B=\bigcup_{a \in A, b \in B} a * b .
\end{gathered}
$$

Remark. If $X=Y=H$, then we obtain the classical hyperstructure theory.
The concept of general hyperstructure with a hyperoperation which is a mapping

$$
*: X \times Y \longrightarrow P^{*}(X \cup Y)
$$

mentioned above (used by A. D. Nezhad and R. S. Hashemi) allows straightforward generalization onto case of "hyperoperation of an arbitrary finite arity" in the following way:

Definition 1 [4] Let $n \in \mathbb{N}$ be an arbitrary positive integer, $n \geq 1$. Let $\left\{X_{k} ; k=1, \ldots, n\right\}$ be a system of non-empty sets. By a general n-hyperstructure we mean the pair

$$
\left(\left\{X_{k} ; k=1, \ldots, n\right\}, *_{n}\right),
$$

where $*_{n}: \prod_{k=1}^{n} X_{k} \rightarrow \mathcal{P}^{*}\left(\bigcup_{k=1}^{n} X_{k}\right)$ is a mapping assigning to any $n$-tuple $\left(x_{1}, \ldots, x_{n}\right) \in \prod_{k=1}^{n} X_{k}$ a non-empty subset $*_{n}\left(x_{1}, \ldots, x_{n}\right) \subset \bigcup_{k=1}^{n} X_{k}$.

Similarly as above, with this hyperoperation there is associated a mapping of power sets

$$
\otimes_{n}: \prod_{k=1}^{n} \mathcal{P}^{*}\left(X_{k}\right) \rightarrow \mathcal{P}^{*}\left(\bigcup_{k=1}^{n} X_{k}\right)
$$

defined by

$$
\otimes_{n}\left(A_{1}, \ldots, A_{n}\right)=\bigcup\left\{*_{n}\left(x_{1}, \ldots, x_{n}\right) ;\left(x_{1}, \ldots, x_{n}\right) \in \prod_{k=1}^{n} A_{k}\right\} .
$$

This construction is based on an idea of Nezhad and Hashemi for $n=2$. Hyperstructures with $n$-ary hyperoperations are investigated among others in [1, 23]. These hyperstructures allow the below straightforward generalization which, however, is in close connection with various mathematical structures. Some of below presented results are in a close connections with $[1,14,16]$ and of course [23].

Let us define the $\omega$-general -hyperstructure:
Definition 2 Let $\omega$ be the smallest infinite countable ordinal. Let $\left\{X_{k} ; k \in \omega\right\}$ be a system of non-empty sets. By an $\omega$-general -hyperstructure we mean the pair $\left(\left\{X_{k} ; k \in \omega\right\}, *_{\omega}\right)$, where $*_{\omega}: \prod_{k \in \omega} X_{k} \rightarrow \mathcal{P}^{*}\left(\bigcup_{k=1}^{\omega} X_{k}\right)$ is a mapping assigning to any sequence $\left\{x_{k}\right\}_{k \in \omega} \in \prod_{k \in \omega} X_{k}$ a nonempty subset $*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right) \subset \bigcup_{k \in \omega} X_{k}$.

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Similarly as above, with this hyperoperation there is associated a mapping of power sets

$$
\otimes_{\omega}: \prod_{k \in \omega} \mathcal{P}^{*}\left(X_{k}\right) \rightarrow \mathcal{P}^{*}\left(\bigcup_{k \in \omega} X_{k}\right)
$$

defined by

$$
\otimes_{\omega}\left(\left\{A_{k}\right\}_{k \in \omega}\right)=\bigcup\left\{*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right) ;\left(\left\{x_{k}\right\}_{k \in \omega}\right) \in \prod_{k \in \omega} A_{k}\right\} .
$$

Definition 3 Let $\mathbb{G}_{1}(\omega)=\left(\left\{X_{k} ; k \in \omega\right\}, *_{\omega}\right), \mathbb{G}_{2}(\omega)=\left(\left\{Y_{k} ; k \in \omega\right\}, \bullet_{\omega}\right)$, be a pair of general $\omega$-hyperstructures. By a good homomorphism $H: \mathbb{G}_{1}(\omega) \rightarrow \mathbb{G}_{2}(\omega)$ we mean any system of mappings $H=\left\{h_{k}: X_{k} \rightarrow Y_{k}\right\}$ such that the following diagram is commutative:


Here $\prod_{k \in \omega} h_{k}\left(\left\{x_{k}\right\}_{k \in \omega}\right)=\left\{h_{k}\left(x_{k}\right)\right\}_{k \in \omega}$ for any sequence $\left\{x_{k}\right\}_{k \in \omega}$ and $\varphi^{\sharp}: \mathcal{P}^{*}\left(\bigcup_{k \in \omega} X_{k}\right) \rightarrow \mathcal{P}^{*}\left(\bigcup_{k \in \omega} Y_{k}\right)$ is the lifting of a mapping $\varphi: \bigcup_{k \in \omega} X_{k} \rightarrow \bigcup_{k \in \omega} Y_{k}$ defined by the mathematical induction. For $x \in X_{0}$ we put $\varphi(x)=h_{0}(x)$. Suppose $\varphi: \bigcup_{j=0}^{k} X_{j} \rightarrow \bigcup_{j=0}^{k} Y_{j}$ is well-defined. Then for any $x \in X_{k+1} \backslash \bigcup_{j=0}^{k} X_{j}$ we put $\varphi(x)=h_{k+1}(x)$. Then using mathematical induction the mapping $\varphi: \bigcup_{k \in \omega} X_{k} \rightarrow \bigcup_{k \in \omega} Y_{k}$ is well-defined.

If all mappings $h_{k} \in H$ are bijections (or isomorphism if all $H_{k}, Y_{k}$ are endowed with some structures) we call the $H$ the isomorphism of $\omega$-hyperstructures $G_{1}(\omega), G_{2}(\omega)$.

As a certain generalization of the general $n$-hyperstructure from [4], Example 3.2, we will construct the following structure:

Example 4 Consider a countable system of pairwise disjoint ordered sets $\left(X_{k}, \leq_{k}\right), k \in \omega$ and for $x \in X_{k}$ let us denote $[x)_{k}=\left\{y \in X_{k} ; x \leq_{k} y\right\}$, i.e. $[x)_{k}$ is the principal end generated by the element $x$ within the ordered set $\left(X_{k}, \leq_{k}\right)$. Further, put

$$
*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right)=\bigcup_{k \in \omega}\left[x_{k}\right)_{k}
$$

for any sequence $*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right) \in \prod_{k \omega} X_{k}$. Then $*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right) \subseteq \bigcup_{k \in \omega} X_{k}$, thus

$$
\mathbb{G}(\omega)=\left(\left\{X_{k} ; k \in \omega\right\}, *_{\omega}\right)
$$

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is a general $\omega$-hyperstructure in the sense of the above definition.
If $\mathbb{H}(\omega)=\left(\left\{Y_{k} ; k=1, \ldots, n\right\}, \bullet_{n}\right)$ is a general $\omega$-hyperstructure such that $\left(Y_{k}, \preceq_{k}\right), k \in \omega$ are pairwise disjoint ordered sets and

$$
\boldsymbol{\bullet}_{\omega}\left(\left\{y_{k}\right\}_{k \in \omega}\right)=\bigcup_{k \in \omega}\left[y_{k}\right)_{k} \subseteq \bigcup_{k \in \omega} Y_{k}
$$

for any sequence $\left\{y_{k}\right\}_{k \in \omega} \in \prod_{k \omega} Y_{k}$ we consider a system $h_{k}:\left(X_{k}, \leq_{k}\right) \rightarrow\left(Y_{k}, \preceq_{k}\right), k \in \omega$, of strongly isotone mappings, i.e. for any $x \in X_{k}$ there holds $h_{k}\left(\left[x_{k}\right)_{k}\right)=\left[h_{k}\left(x_{k}\right)\right)_{k}$. Then denoting $H=\left\{h_{k}: X_{k} \rightarrow Y_{k} ; k \in \omega\right\}$ we obtain that $H$ is a good homomorphism of the general $\omega$-hyperstructure; $\mathbb{G}(n)$ into the general $\omega$-hyperstructure $\mathbb{H}(\omega)$. Indeed, consider an arbitrary sequence $\left\{x_{k}\right\}_{k \in \omega} \in \prod_{k \in \omega} X_{k}$. As above denote by $\varphi: \mathcal{P}^{*}\left(\bigcup_{k \in \omega} X_{k}\right) \rightarrow \mathcal{P}^{*}\left(\bigcup_{k \in \omega} Y_{k}\right)$ the lifting of the mapping $\varphi: \bigcup_{k=1}^{n} X_{k} \rightarrow \bigcup_{k \in \omega} Y_{k}$ induced by the system $\left\{h_{k}: X_{k} \rightarrow Y_{k} ; k \in \omega\right\}$-here in such a way that $\varphi \mid X_{k}=h_{k}$. Then for any sequence $\left\{x_{k}\right\}_{k \in \omega} \in \prod_{k \in \omega} X_{k}$ we have

$$
\begin{aligned}
\varphi\left(*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right)\right) & =\varphi\left(\bigcup_{k \in \omega}\left[x_{k}\right)_{k}\right)=\bigcup_{k \in \omega} \varphi\left(\left[x_{k}\right)_{k}\right)=\bigcup_{k \in \omega} h_{k}\left(\left[x_{k}\right)_{k}\right) \\
& =\bigcup_{k \in \omega}\left[h_{k}\left(x_{k}\right)_{k}\right)=\bullet_{\omega}\left(\left\{h_{k}\left(x_{k}\right)\right\}_{k \in \omega}\right) \\
& =\boldsymbol{\bullet}_{\omega}\left(\prod_{k \in \omega} h_{k}\left(\left\{x_{k}\right\}_{k \in \omega}\right)\right),
\end{aligned}
$$

i.e. $\varphi \circ *_{\omega}=\bullet_{\omega} \circ \prod_{k \in \omega} h_{k}$, thus the diagram

$$
\begin{align*}
& \prod X_{k} \stackrel{*_{\omega}}{\longrightarrow} \mathcal{P}^{*}\left(\bigcup_{k \in \omega} X_{k}\right) \\
& \prod_{k \in \omega} h_{k} \downarrow  \tag{D2}\\
& \prod Y_{k} \xrightarrow{\bullet} \\
& \downarrow^{\sharp} \mathcal{P}^{*}\left(\bigcup_{k \in \omega} Y_{k}\right)
\end{align*}
$$

is commutative.
From the above example there follows immediately the following assertion.
Proposition 5 Let $\left(X_{k}, \leq_{k}\right),\left(Y_{k}, \preceq_{k}\right), k \in \omega$, be two countable collections of pairwise disjoint ordered sets and $\mathbb{G}(\omega), \mathbb{H}(\omega)$ be the corresponding $\omega$-general hyperstructures. Suppose $\left(X_{k}, \leq_{k}\right) \cong\left(Y_{k}, \preceq_{k}\right)$ for each $k \in \omega$ and $h_{k}:\left(X_{k}, \leq_{k}\right) \rightarrow\left(Y_{k}, \preceq_{k}\right)$ are corresponding orderisomorphisms. Then we have $\mathbb{G}(\omega) \cong \mathbb{H}(\omega)$.

It is well-known fact that there are several properties of metric spaces which are not topological but are closely connected with topological properties. For example, the property of
being a Cauchy sequence, which is not a topological invariant and with this term connected concept of a complete metric space, further concept of a uniformly continuous functions and other notions. Thus the other direction of generalization of metric spaces different from the concept of a topological space is the uniform spaces theory based on the mathematical construct employed in studying uniformity properties. Roughly speaking, a uniform space is defined to be a set $X$ together with a family of subsets of the cartesian square $X \times X$ which satisfies certain natural conditions. The approach used in our contribution is due to André Weil [22]. Compare also the excellent monograph [19]. Recall first the basic notions. A filter $\mathcal{F}$ in a set $S$ is a family of non-void subsets of $S$ such that
i) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ and
ii) if $A \in \mathcal{F}$ and $A \subset B \subset S$, then $B \in \mathcal{F}$.

A uniformity for a set $X$ is a filter $\mathcal{U}$ of binary reflexive relations on $X$ such that for any $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{F}$ and if $U \in \mathcal{U}$, then there exists a relation $V \in \mathcal{F}$ such that $V \circ V \subset U$. Relation $U \in \mathcal{U}$ are called entourages or neighbourhoods of the diagonal $\Delta \subset X \times X$. The pair $(X, \mathcal{U})$ is a uniform space. It is not generally true that the union or the intersection of two uniformities for $X$ is a uniformity. However, the union of a collection of uniformities generates a uniformity in a rather natural sense. A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ if and only for each $U \in \mathcal{U}$ there is a relation $V \in \mathcal{B}$ such that $V \subset U$. If $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ them $\mathcal{B}$ determines $\mathcal{U}$ entirely; clearly for a subset $U \subset X \times X$ we have $U \in \mathcal{U}$ if and only if there exists $V \in \mathcal{B}$ such that $V \subset U$. In the monograph [19] there is a contained the following characterization theorem (a slightly reformulated by us):

Theorem ([19, Ch. 6, Th.2, p.177]). A family $\mathcal{B}$ of subsets $X \times X$ is a base for some uniformity for $X$ if and only if

1. $\Delta_{X} \subset U$ for each $U \in \mathcal{B}$ (i.e. any relation $U \in \mathcal{B}$ is reflexive),
2. if $U \in \mathcal{B}$ then there exists a relation $V \in \mathcal{B}$ such that $V \in \subset U^{-1}$,
3. if $U \in \mathcal{B}$ then there exists a relation $V \in \mathcal{B}$ such that $V \circ V \subset \mathcal{U}$,
4. if $U, V \in \mathcal{B}$ then there exists $W \in \mathcal{B}$ with the property $W \subset U \cap V$.

Since the concept of a uniformity generalizes such properties of metric (or pseudo-metric) spaces which cannot be expressed in terms of topological spaces, one of crucial question in the theory of uniformity spaces is the metrization problem. So, in [19, Ch. 6, p.184-188] it is shown that every uniformity is derived from the family of its uniformly continuous pseudo-metrics, moreover a uniformity can be derived from a single pseudo-metric if and only if the uniformity has a countable base.

Recall that a function $d: X \times X \rightarrow \mathbb{R}$ is said to be a pseudo-metric on a set $X$ and the pair ( $X, d$ ) is said to be a pseudo-metric space if

1. $d(x, y) \geq 0, d(x, x)=0$ for any pair $x, y \in X$,
2. $d(x, y)=d(y, x), x, y \in X$,

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3. the triangle inequality is satisfied, (i.e. $d(x, y)+d(y, z) \geq d(x, z), x, y, z \in X)$.

Each pseudo-metric $d$ for a set $X$ generates a uniformity in the following way: For each positive number $r \in \mathbb{R}$ let

$$
V_{d, r}=\{[x, y] ; d(x, y)<r\} .
$$

Clearly $\left(V_{d, r}\right)^{-1}=V_{d, r}, V_{d, r} \cap V_{d, s}=V_{d, t}$, where $t=\min \{r, s\}$ and $V_{d, r} \circ V_{d, r} \subseteq V_{d, 2 r}$. It follows that the family of all sets of the form $V_{d, r}$ is a base for a uniformity for the set $X$.

This uniformity is called the pseudo-metric uniformity or the uniformity generated by $d$. A uniform space ( $X, \mathcal{U}$ ) is said to be a pseudo-metrizable (or metrizable) if there is a pseudo-metric (metric) $d$, respectively, such that $\mathcal{U}$ is the uniformity generated by $d$.

A mapping $f$ of a uniform space $(X, \mathcal{U})$ into a uniform space $(Y, \mathcal{U})$ is said to be uniformly continuous if for each relation $(V \in \mathcal{V})$ the relation $\{[x, y] ;[f(x), f(y)] \in V\}$ is a member of $\mathcal{U}$. This condition may be rephrased in several ways.

Defining a mapping $f \times f: X \times X \rightarrow Y \times Y$ by $(f \times f)(x, y)=[f(x), f(y)] \in Y \times Y$ (where $f: X \rightarrow Y$ is a map) it is easy to see that the following conditions are equivalent:

1. $f:(X, \mathcal{U}) \rightarrow(Y, \mathcal{V})$ is uniformly continuous,
2. for each $Y \in \mathcal{V}$ there is $U \in \mathcal{U}$, such that $(f \times f)(U) \in V$,
3. $(f \times f)^{-1}(\mathcal{V}) \subset \mathcal{U}$, i.e. $(f \times f)^{-1}(V) \in \mathcal{U}$, for any entourage $V \in \mathcal{V}$,
4. if $\mathcal{B} \subset \mathcal{V}$ is a base for the uniformity $\mathcal{V}$ then for each $V \in \mathcal{B}$ we have $(f \times f)^{-1}(V) \in \mathcal{U}$.

Recall that in the case of usual uniformity $\mathcal{U}$ on the set $\mathbb{R}$ of all real numbers the above conditions are equivalent to the concept of usually uniformly continuous function: For any $\epsilon>0$ there exists a number $\delta>0$ such that, $x, y \in \mathbb{R},|x-y|<\delta$ we have $|f(x)-f(y)|<\epsilon$.

Evidently the composition of two uniformly continuous mapping is again uniformly continuous. If $f$ is one-to-one map of $X$ onto $Y$ and both $f$ and $f^{-1}$ are uniformly continuous, then $f$ is called a uniform isomorphism and spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are said to be uniformly equivalent, in [2] and $f:(X, \mathcal{U}) \rightarrow(Y, \mathcal{V})$ is called a uniform homomorphism and spaces $(X, \mathcal{U})$, $(Y, \mathcal{V})$ are called uniform homeomorphic.

The composition of two uniform homeomorphisms, the inverse of a uniform homeomorphism, and the identity mapping of a uniform space onto itself are all uniform homeomorphisms, and consequently the class of all uniform spaces is divided into equivalence subclasses, consisting of uniform homeomorphic spaces. One of classical result of the uniform spaces theory in the well-known metrization lemma 12 ([19, p.185]) which says that if $\left\{U_{n} ; n \in \omega\right\}$ is a sequence of reflexive binary relations on $X$ such that $U_{0}=X \times X$ and $U_{n+1}^{3} \subset U_{n}$ for each $n$, then there exists a function

$$
d: X \times X \rightarrow \mathbb{R}_{0}^{+} \quad(=\{r \in \mathbb{R} ; r \geq 0\})
$$

satisfying the triangle inequality and the condition

$$
U_{n} \subset\left\{[x, y] ; d(x, y) 2^{-n}\right\} \subset U_{n-1}
$$

for each $n \in \omega, n>0$. Moreover if each relation $U_{n}$ is symmetric, then there is a pseudometric satisfying the above condition. Now, if a uniform space $(X, \mathcal{U})$ has a countable base $\left\{V_{n} ; n \in \omega\right\}$ then it is possible to construct by induction a family $\left\{U_{n} ; n \in \omega\right\}$ of symmetric reflective relations $U_{n} \subset X \times X$ such that $U_{n+1}^{3} \subset U_{n-1}, U_{n} \subset V_{n}$ for each positive integer $n$. (Here, as above $U_{n}^{3}=U_{n} \circ U_{n} \circ U_{n}$.) The family $\left\{U_{n} ; n \in \omega\right\}$ is then a base for the uniformity $\mathcal{U}$, and upon applying the above metrization lemma it follows that the uniform space $(X, \mathcal{U})$ is pseudo-metrizable. Thus, there holds:

Theorem 6 ([19, Metrization theorem 13, p.186]). A uniform space is pseudo-metrizable if and only if its uniformity has a countable base.

As a certain application of constructions contained in the first part of our contribution it the following theorem. Notice, that a tolerance space $(X, T)$ is a set $X$ endowed with a reflexive and symmetric binary relation $T \subset X \times X$, i.e. a tolerance relation. Algebraic theory of tolerances is contained e.g. in [3] and in many other papers. Further, by a product $\prod_{n \in \omega}\left(X, T_{n}\right)$ of a family of tolerance spaces $\left\{\left(X, T_{n}\right) ; n \in \omega\right\}$ with the same carrier $X$ we mean the direct product defined by

$$
\prod_{n \in \omega}\left(X, T_{n}\right)=\left(X^{n}, T\right)
$$

where for a pair of sequences $\left\{x_{n}\right\}_{n \in \omega},\left\{y_{n}\right\}_{n \in \omega} \in X^{\omega}$ we have $\left\{x_{n}\right\}_{n \in \omega} T\left\{y_{n}\right\}_{n \in \omega}$ whenever $x_{n} T_{n} y_{n}$ for any $n \in \omega$. The union $\bigcup_{n \in \omega}\left(X, T_{n}\right)=\left(X, \bigcup_{n \in \omega} T_{n}\right)$. Notice, that on a set $X$ evidently arbitrary union of tolerances is a tolerance relation on this set $X$.

Definition 7 Uniform spaces $(X, \mathcal{U}),(Y, \mathcal{V})$ are said to be base-equivalent if for some bases $\mathcal{B}_{\mathcal{U}}, \mathcal{B}_{\mathcal{V}}$ of uniformities $\mathcal{U}, \mathcal{V}$, respectively there exists a bijection $\xi: X \times X \rightarrow Y \times Y$ such that $\mathcal{B}_{\mathcal{V}}=\xi^{\sharp}\left(\mathcal{B}_{\mathcal{U}}\right)$, where $\xi^{\sharp}: \mathcal{P}(X \times X) \rightarrow \mathcal{P}(Y \times Y)$ is the lifting of the bijecton $\xi$.

Theorem 8 Let $\left\{\left(X, T_{n}\right) ; n \in \omega\right\},\left\{\left(Y, S_{n}\right) ; n \in \omega\right\}$ be sequences of tolerance spaces (with non-void carrier sets $X, Y$, respectively) satisfying the conditions $T_{0}=X \times X, S_{0}=Y \times Y$ and $U_{n+1}^{3} \subset U_{n-1}$ for $U_{n} \in\left\{T_{n}, S_{n}\right\}, n \in \omega, n>0$. Then $\left\{\left(X, T_{n}\right) ; n \in \omega\right\},\left\{\left(Y, S_{n}\right) ; n \in \omega\right\}$ are bases for the uniformity. If $(X, \mathcal{U}),(Y, \mathcal{V})$ are corresponding generated uniform spaces, then they are pseudo-metrizable. Moreover, if general $\omega$-hyperstructures

$$
\begin{aligned}
& \mathbb{G}(\omega)=\left\{\prod_{n \in \omega}\left(X, T_{n}\right) ; *_{\omega}, \mathcal{P}^{*}\left(\bigcup_{n \in \omega}\left(X, T_{n}\right)\right)\right\}, \\
& \mathbb{H}(\omega)=\left\{\prod_{n \in \omega}\left(Y, S_{n}\right) ; \bullet_{\omega}, \mathcal{P}^{*}\left(\bigcup_{n \in \omega}\left(Y, S_{n}\right)\right)\right\}
\end{aligned}
$$

with $*_{\omega}\left(\left\{x_{k}\right\}_{k \in \omega}\right)=\bigcup_{k \in \omega} T_{k}\left(x_{k}\right)$ for any sequence $\left\{x_{k}\right\}_{k \in \omega} \in \prod_{n \in \omega}\left(X, T_{n}\right)$ and similarly in the case $\bullet_{\omega}: \prod_{n \in \omega}\left(Y, S_{n}\right) \rightarrow \mathcal{P}^{*}\left(\bigcup_{n \in \omega}\left(Y, S_{n}\right)\right)$, are isomorphic, i.e. $\mathbb{G}(\omega) \cong \mathbb{H}(\omega)$ (all mappings $h_{n}:\left(X, T_{n}\right) \rightarrow\left(Y, S_{n}\right), n \in \omega$ are tolerance isomorhpism), the uniform spaces $(X, \mathcal{U}),(Y, \mathcal{V})$ are base-equivalent.

Proof. By the above considerations countable system of tolerance relations $\left\{T_{n} ; n \in \omega\right\}$, $\left\{S_{n} ; n \in \omega\right\}$ on sets $X, Y$, respectively satisfying in this theorem required conditions are based of uniformities, say $\mathcal{U} \subset X \times X, \mathcal{V} \subset Y \times Y$. By the metrization Theorem 6 both uniform spaces $(X, \mathcal{U}),(Y, \mathcal{V})$ are metrizable.

Now suppose general $\omega$-hyperstructures $\mathbb{G}(\omega), \mathbb{H}(\omega)$ are isomorphic. Suppose

$$
H=\left\{h_{n}: X_{n} \rightarrow Y_{n} ; n \in \omega\right\}
$$

is an isomorphism of $\mathbb{G}(\omega)$ onto $\mathbb{H}(\omega)$, where $h_{n}:\left(X, T_{n}\right) \rightarrow\left(Y, S_{n}\right)$ is an isomorphism of tolerance spaces for any $n \in \omega$. According to the definition of isomorphisms of general $\omega$ hyperstructures $\mathbb{G}(\omega), \mathbb{H}(\omega)$ there exists a bijection $\varphi: X \rightarrow Y$ satisfying the equality $\varphi \circ *_{\omega}=$ $\bullet \omega \circ \prod_{n \in \omega} h_{n}$. Then the lifting $(\varphi \times \varphi)^{\sharp}: \mathcal{P}^{*}(X \times X) \rightarrow \mathcal{P}^{*}(Y \times Y)$ of the mapping $\varphi \times \varphi: X \times X \rightarrow$ $Y \times Y$ maps one-to-one the base $\mathcal{B}_{\mathcal{U}}=\left\{T_{n} ; n \in \omega\right\}$ of the uniformity $\mathcal{U}$ onto the base $\mathcal{B}_{\mathcal{V}}$ of the uniformity $\mathcal{V}$, consequently uniform spaces $(X, \mathcal{U}),(Y, \mathcal{V})$ are base-equivalent.

The problem of existence of a uniform homeomorphism $\psi:(X, \mathcal{U}) \rightarrow(Y, \mathcal{V})$, in particular, its effective construction will be investigated in a forhtcomming paper.

As it has been mentioned above we use the approach to uniform spaces based on a concept of a filter of vicinities of a diagonal which was based by André Weil [22]. This aproach is presented in many monographs - among others let us mention [19, 21]. Horst Herrlich in his second book of the series [17] presented the covering approach but Weil theory is mentioned, e.g. in Definition 4.4.12.

Except papers $[15,23]$ and many other papers devoted to the thema there is a pioneering paper [18] investigated very general multialgebras already in 1966.

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# SOLVABILITY OF CERTAIN GROUPS OF SECOND-ORDER LINEAR DIFFERENTIAL OPERATORS 

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#### Abstract

There are investigated groups of ordinary second-order linear differential operators with respect to the problem of their solvability. In particular, using the stabilization of the commutant chain of the basic group of operators multiplication of which is motivated by composition of affine transformations of linear spaces we have obtained that the mentioned group is solvable. Key words and phrases. Linear ordinary second-order differential operator, groups of linear differential operators, solution spaces of linear ordinary second-order differential equations, subnormal series of a group, chain of commutants of a group, solvable group. Mathematics Subject Classification. Primary 47D03, 20F12, 20F16.


There is well-known fact from the literature [4], [15]-[21] that Otakar Borůvka started in the fifthies of the last century the systematic study of global properties of the ordinary second-order linear differential equations [4]. Further important results generalizing Borůvka's approach were obtained by František Neuman [15]-[21]. The just mentioned school is characteristical by applications not only analytic methods in those investigations, but also algebraic, topological and geometric tools. Local methods and results are not sufficient when studying problems of a global nature, such as boundedness, periodicity, asymptotic and oscillatory behavior, nonvanishing solutions, and consequently the factorization of linear differential operators [4], as well as many other things [21].

In this contribution we are concern ourselves onto certain algebraic aspects and properties of systems of ordinary linear differential second-order operators which are left hand sides of corresponding linear homogeneous differential equations. In particular, in connection with the theory of series of groups motivated by the Galois theory of solvability of algebraic equations and the modern theory of extensions of fields we show that the group of linear second-order

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differential operators considered and investigated in our early papers is solvable (in the Galois sense). It is to be noted that there exists a large and deep Galois theory of linear differential equations published in many papers as well as in the book [22] (and in other papers of authors of the mentioned monography), but results obtained here seems to be independent on the mentioned theory and not contained between these results.

As usually, $\mathbb{C}^{k}(J)$ stands for the commutative ring of all real functions of one variable defined on an open interval $J$ of real numbers, and having there continuous derivatives up to order $k \geq 0$. Instead of $\mathbb{C}^{0}(J)$ we write only $\mathbb{C}(J)$; this symbol denotes the ring of all continuous functions on $J, \mathbb{C}_{+}(J)$ is its subsemiring of all positive continuous functions. In accordance with [15]-[19] denote by $\mathbb{A}_{2}(J)$ the set of ordinary linear homogeneous differential equations of the second order

$$
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0
$$

such that $p \in \mathbb{C}_{+}(J), q \in \mathbb{C}(J)$. Let us denote by Id the identity operator and $\mathbb{D}=\frac{d}{d x}$. By $L(p, q)$ will be denoted the differential operator

$$
L(p, q)=D^{2}+p(x) D+q(x) \mathrm{Id} ;
$$

thus the above equation has the form $L(p, q) y=0$. Further, we denote by

$$
\mathbb{L} \mathbb{A}_{2}(J)=\left\{L(p, q): \mathbb{C}^{2}(J) \rightarrow \mathbb{C}(J) ; p \in \mathbb{C}_{+}(J), q \in \mathbb{C}(J)\right\}
$$

the set of all such differential operators.
For $r \in \mathbb{R}$ we denote by $\chi_{r}: J \rightarrow \mathbb{R}$ the constant function with the value $r$. In [5] it is contained the proof of the following assertion.

Proposition 1 Let $J \subseteq \mathbb{R}$ be an open interval,

$$
\mathbb{L} \mathbb{A}_{1}(J)=\left\{L(p, q) ; p \in \mathbb{C}_{+}(J), q \in \mathbb{C}(J)\right\} .
$$

For any pair of differential operators $L\left(p_{1}, q_{1}\right), L\left(p_{2}, q_{2}\right) \in \mathbb{L} \mathbb{A}_{2}(J)$ define

$$
L\left(p_{1}, q_{1}\right) \cdot L\left(p_{2}, q_{2}\right)=L\left(p_{1} p_{1}, p_{1} q_{2}+q_{1}\right) .
$$

Then $\left(\mathbb{L} \mathbb{A}_{1}(J), \cdot\right)$ is a noncommutative group with the unit element $L\left(\chi_{1}, \chi_{0}\right)$.
In what follows we write simply $L(1,0)$ instead of $L\left(\chi_{1}, \chi_{0}\right)$.
Now recall one of basic and important concepts of the group theory, i.e. the concept of a normal subgroup of a group (called also invariant). So, a subgroup $H$ of a group $G$ is called normal (or invariant) if for any element $a \in G$ there holds $a^{-1} \cdot H \cdot a \subseteq H$ or equivalently $a^{-1} \cdot H \cdot a=H$. This means that the subgroup $H$ is invariant with respect to inner automorphisms of the group $G$. As usually we write $H \triangleleft G$. Denoting

$$
\mathbb{L}_{11} \mathbb{A}_{2}(J)=\left\{L(p, q) ; L(p, q) \in \mathbb{L}_{\mathbb{A}_{2}}(J), p(x) \equiv 1\right\}
$$

thus

$$
\mathbb{L}_{11} \mathbb{A}_{2}(J)=\{L(1, q) ; q \in \mathbb{C}(J)\}
$$

we can recall the theorem proved in [3].

Theorem 1 Let $J \subseteq \mathbb{R}$ be an open interval. Then $\left(\mathbb{L}_{11} \mathbb{A}_{2}(J), \cdot\right)$ is a normal commutation subgroup of the group $\left(\mathbb{L}_{\mathbb{A}_{2}}(J), \cdot\right)$, i.e. $\left(\mathbb{L}_{11} \mathbb{A}_{2}(J), \cdot\right) \triangleleft\left(\mathbb{L}_{\mathbb{A}_{2}}(J), \cdot\right)$ and $\left(\mathbb{L}_{11} \mathbb{A}_{2}(J), \cdot\right) \cong(\mathbb{C}(J),+)$.

The above formulated theorem can be extended in this way:
Denote $\mathbb{L}_{C 1} \mathbb{A}_{2}(J)=\left\{L(r, q) ; L(r, q) \in \mathbb{L}_{\mathbb{A}_{2}}(J), r \in \mathbb{R}, r \neq 0\right\}$, i.e. $L(r, q) y=0$ are secondorder linear differential equations with constant coefficients at the first derivative of functions $y(x)$, thus equations of the form

$$
y^{\prime \prime}+r \cdot y^{\prime}+q(x) \cdot y=0, \quad r \in \mathbb{R}, r \neq 0, q \in \mathbb{C}(J)
$$

Theorem 2 Let $J \subseteq \mathbb{R}$ be an open interval. There holds

$$
\left(\mathbb{L}_{11} \mathbb{A}_{2}(J), \cdot\right) \triangleleft\left(\mathbb{L}_{C 1} \mathbb{A}_{2}(J), \cdot\right) \triangleleft\left(\mathbb{L}_{\mathbb{A}_{2}}(J), \cdot\right)
$$

Proof. First of all we show that $\left(\mathbb{L}_{C 1} \mathbb{A}_{2}(J), \cdot\right)$ is a subgroup of the group $\left(\mathbb{L} \mathbb{A}_{2}(J), \cdot\right)$. Suppose $L(r, q), L(s, u) \in \mathbb{L}_{C 1} \mathbb{A}_{2}(J)$. Then

$$
\begin{aligned}
L(r, q) \cdot L^{-1}(s, u)= & L(r, q) \cdot L\left(\frac{1}{s},-\frac{u}{s}\right)=L\left(\frac{r}{s},-\frac{r}{s} u+q\right) \\
& \in \mathbb{L}_{C 1} \mathbb{A}_{2}(J), \quad L(1,0) \in \mathbb{L}_{C 1} \mathbb{A}_{2}(J),
\end{aligned}
$$

thus $\mathbb{L}_{C 1} \mathbb{A}_{2}(J)$ is a subgroup of the group $\mathbb{L}_{\mathbb{A}_{2}} J$. Now suppose $L(1, q) \in \mathbb{L}_{11} \mathbb{A}_{2}(J), L\left(r_{1}, q_{1}\right)$, $L\left(r_{2}, q_{2}\right) \in \mathbb{L}_{C 1} \mathbb{A}_{2}(J), L(u, v) \in \mathbb{L}_{\mathbb{A}_{2}}(J)$ are arbitrary differential operators. We have

$$
\begin{aligned}
L\left(r_{1}, q_{1}\right) \cdot L(1, p) \cdot L^{-1}\left(r_{1}, q_{1}\right) & =L\left(r_{1}, r_{1} p+q_{1}\right) \cdot L\left(\frac{1}{r_{1}},-\frac{g_{1}}{r_{1}}\right) \\
& =L\left(1, r_{1} p\right) \in \mathbb{L}_{11} \mathbb{A}_{2}(J),
\end{aligned}
$$

thus

$$
L\left(r_{1}, q_{1}\right) \cdot \mathbb{L}_{11} \mathbb{A}_{2}(J) \cdot L^{-1}\left(r_{1}, q_{1}\right) \subseteq \mathbb{L}_{11} \mathbb{A}_{2}(J)
$$

Further

$$
\begin{aligned}
L(u, v) \cdot L\left(r_{1}, q_{1}\right) \cdot L^{-1}(u, v) & =L\left(r_{1} u, v+u q_{1}\right) \cdot L\left(\frac{1}{u},-\frac{v}{u}\right) \\
& =L\left(r_{1},\left(1-r_{1}\right) v+u q_{1}\right) \in \mathbb{L}_{C 1} \mathbb{A}_{2}(J)
\end{aligned}
$$

hence

$$
L(u, v) \cdot \mathbb{L}_{C 1} \mathbb{A}_{2}(J) \cdot L^{-1}(u, v) \subseteq \mathbb{L}_{C 1} \mathbb{A}_{2}(J)
$$

for any operator $L(u, v) \in \mathbb{L} \mathbb{A}_{2}(J)$. From the above considerations there follows

$$
\left(\mathbb{L}_{11} \mathbb{A}_{2}(J), \cdot\right) \triangleleft\left(\mathbb{L}_{C 1} \mathbb{A}_{2}(J), \cdot\right) \triangleleft\left(\mathbb{L}_{\mathbb{A}_{2}}(J), \cdot\right)
$$

Now recall some basic concepts from the theory of group series which will be needed in the following considerations. For more details see e.g. [11, 12, 13, 14].

A sequence of subgroups $H_{i}(i=0,1, \ldots, n)$ of a group $G$ such that

$$
\begin{equation*}
\mathbf{1}=H_{0} \triangleleft H_{1} \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_{n}=G \tag{H}
\end{equation*}
$$

is said to be a subnormal series of the group $G$, the corresponding factor-groups $H_{i} / H_{i-1}$ $(i=1,2, \ldots, n)$ are termed factors of the series $(\mathcal{H})$.

Definition 3 ([14, 10.22], [13, §57,p.361]) If $G$ is a group then its subnormal series

$$
\mathbf{1}=H_{0} \triangleleft H_{1} \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_{n}=G
$$

is said to be solvable, if all factors $H_{i} / H_{i-1}(i=1,2, \ldots, n)$ are commutative thus abelian groups. If a group $G$ possesses at least one solvabel subnormal series then $G$ is called a solvable group.

Remark 4 Evidently, an arbitrary abelian group is solvable, however there exist solvable groups which are not abelian.

In the literature one can find well-known results concerning solvable groups. Let us mention some of them: First of all it is famous Feit-Thompson theorem or odd order theorem which states that every finite group of odd order is solvable. It was proved by Walter Feit and John Griggs Thompson $(1962,1963)-[11,12]$. Further, every finite group of order $|G| \leq 100$ except 60 is solvable. (For these informations and for other details both authors are very obliged to Professor Václav Havel.) Any $p$-group $G$, i.e. $|G|=p^{n}$ for some prime number $p$ is solvable. If $K \triangleleft G$, then the group $G$ is solvable if and only if both groups $K$ and $G / K$ are solvable. A group is solvable if and only if it possesses a subnormal series all factors of which are solvable. In particular a direct product of a finite number of solvable groups is a solvable group. Of course, any finite group $G$ of the degree 5 and more is not solvable. Other important informations can be found in the literature [11, 12, 13, 14].

According to the above definition we can prove the solvability of the group $G=\mathbb{L} \mathbb{A}_{2}(J)$ in such a way that we construct a subnormal series of the group $\mathbb{L}_{2}(J)$ with abelian factors. In fact, we are able to construct the so-called underlying or backing chain of groupoids of fractions of continuous functions $f \in \mathbb{C}(J)$ without zero points within $J$ - in detail the chain of non-commutative groupoids

$$
\{1\}=G_{0}(J) \subset G_{1}(J) \subset \cdots \subset G_{j}(J) \subset G_{n-1}(J) \subset \mathbb{C}(J)
$$

in which binary operations are defined by the rule $(f \cdot g)(x)=\frac{f(x)}{g(x)}, x \in J$. Moreover, denoting by $(f)^{m}$ the usual $m$-th power of the dimensional $f$, i.e. $(f)^{m}(x)=[f(x)]^{m}, x \in J$, we can prove using the mathematical induction that

$$
G_{j}(J)=\left\{\prod_{k=1}^{j}\left(f_{k}\right)^{m_{k}} ; m_{k} \in \mathbb{Z}, f_{k} \in \mathbb{C}(J), f_{k}(x) \neq 0, x \in J, k=1,2, \ldots, j\right\}
$$

Further, to any groupoid $G_{j}(J)$ we assign a subgroup $\mathbb{H}_{j} \mathbb{A}_{2}(J)$ in such a way that

$$
\{L(1,0)\}=\mathbb{H}_{0} \mathbb{A}_{2}(J) \triangleleft \mathbb{H}_{1} \mathbb{A}_{2}(J) \triangleleft \cdots \triangleleft \mathbb{H}_{n} \mathbb{A}_{2}(J)=\mathbb{L} \mathbb{A}_{2}(J)
$$

is a subnormal series of the group $\mathbb{L}_{\mathbb{A}_{2}}(J)$ which is also subinvariant - this means that all groups $\mathbb{H}_{j} \mathbb{A}_{2}(J)$ are normal (i.e. invariant) subgroups of the group $\mathbb{L} \mathbb{A}_{2}(J)$. However the just mentioned construction and verifying of the fact that all factors $\mathbb{H}_{j} \mathbb{A}_{2}(J) / \mathbb{H}_{j-1} \mathbb{A}_{2}(J)$, $j=1,2, \ldots, n$ are abelian needs some effort. Much more simpler way is based on chains of commutants of the group $\mathbb{L}_{\mathbb{A}_{2}}(J)$.

Recall that a subgroup $G^{\prime}$ of a group $G$ generated by the set of commutators

$$
[a, b]=a^{-1} b^{-1} a b
$$

of all pairs $[a, b] \in G \times G$ is called the commutant of the group $G$. The commutant $G^{\prime \prime}$ of the group $G^{\prime}$ is called the second commutant of the group $G$. The usual notation is $G^{\prime}=G^{(1)}$, $G^{\prime \prime}=G^{(2)}$, etc. In general $G^{(n+1)}=\left(G^{(n)}\right)^{\prime}$ The commutant $G^{(n)}$ is also termed as the $n$-th derivative of the group $G$ and the chain of commutants of the group $G$

$$
G=G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \cdots
$$

is also called a derived chain of the group $G$.
Theorem 5 ([14, Theorem 10.31, p. 172]) Let $G$ be a group. The following conditions are equivalent:
(i) The group $G$ is solvable.
(ii) There exists a positive integer $n$ such that $G^{(n)}=\mathbf{1}$.
(iii) There exists a subnormal series of the group $G$, which is solvable.

Now, we are coming to the main result of our contribution.
Theorem 6 Let $J \subseteq \mathbb{R}$ be an open interval. Then $\{L(1,0)\}=\mathbf{1}=\mathbb{L}_{\mathbb{A}_{2}}(J)^{\prime \prime} \triangleleft \mathbb{L} \mathbb{A}_{2}(J)^{\prime}=$ $\mathbb{L}_{11} \mathbb{A}_{2}(J) \triangleleft \mathbb{L} \mathbb{A}_{2}(J)$, thus the group $\left(\mathbb{L}_{\mathbb{A}_{2}}(J), \cdot\right)$ is solvable.

Proof. According to Theorem 1 we have $\mathbb{L}_{11} \mathbb{A}_{2}(J) \triangleleft \mathbb{L}_{\mathbb{A}_{2}}(J)$. Moreover, the commutator $[L(p, q), L(u, v)]$ of an arbitrary pair $L(p, q), L(u, v) \in \mathbb{L}_{\mathbb{A}_{2}}(J)$ satisfies

$$
\begin{gathered}
{[L(p, q), L(u, v)]=L^{-1}(p, q) \cdot L^{-1}(u, v) \cdot L(p, q) \cdot L(u, v)} \\
=L\left(\frac{1}{p},-\frac{q}{p}\right) \cdot L\left(\frac{1}{u},-\frac{v}{u}\right) \cdot L(p u, p v+q) \\
=L\left(\frac{1}{p u},-\frac{v+q u}{p u}\right) \cdot L(p u, p v+q)=L\left(1, \frac{v(p-1)+q(1-u)}{p u}\right) \in \mathbb{L}_{11} \mathbb{A}_{2}(J),
\end{gathered}
$$

thus the subgroup of $\mathbb{L}_{\mathbb{A}_{2}}(J)$ generated by the set of commutators $[L(p, q), L(u, v)], L(p, q)$, $L(u, v) \in \mathbb{L}_{\mathbb{A}_{2}}(J)$, i.e. the commutant $\mathbb{L}_{\mathbb{A}_{2}}(J)^{\prime}$ is exactly the group $\mathbb{L}_{11} \mathbb{A}_{2}(J)$. Further, for any pair of operators

$$
L(1, q), L(1, v) \in \mathbb{L}_{11} \mathbb{A}_{2}(J)=\mathbb{L} \mathbb{A}_{2}(J)^{\prime}
$$

the commutator

$$
\begin{gathered}
{[L(1, q), L(1, v)]=L^{-1}(1, q) \cdot L^{-1}(1, v) \cdot L(1, q) \cdot L(1, v)} \\
=L(1,-q) \cdot L(1,-v) \cdot L(1, v+q) \\
=L(1,-v-q) \cdot L(1, v+q)=L(1,0)
\end{gathered}
$$

i.e.

$$
\mathbb{L}_{11} \mathbb{A}_{2}(J)^{\prime}=\mathbb{L} \mathbb{A}_{2}(J)^{\prime \prime}=\{L(1,0)\}=\mathbf{1}
$$

therefore the theorem holds.
In accordance with paper [2] let us denote by $\mathcal{F}$ the system of all pairs of linearly independent functions $\varphi_{1}, \varphi_{2} \in \mathbb{C}^{2}(J)$ such that

$$
\frac{\left|\begin{array}{ll}
\varphi_{1}^{\prime \prime} & \varphi_{2}^{\prime \prime} \\
\varphi_{1} & \varphi_{1}
\end{array}\right|}{W\left[\varphi_{1}, \varphi_{2}\right]} \neq 0
$$

where $W\left[\varphi_{1}, \varphi_{2}\right]$ is the Wronski determinant (different from zero). Further, let $\mathbb{G}(\mathcal{F})$ be the system of all two-dimensional vector spaces $V\left(\varphi_{1}, \varphi_{2}\right)$ with the base $\left[\varphi_{1}, \varphi_{2}\right] \in \mathcal{F}$. For any pair of spaces $V\left(\varphi_{1}, \varphi_{2}\right) \in \mathbb{G}(\mathcal{F}), V\left(\psi_{1}, \psi_{2}\right) \in \mathbb{G}(\mathcal{F})$ we define

$$
V\left(\varphi_{1}, \varphi_{2}\right) \cdot V\left(\psi_{1}, \psi_{2}\right)=V\left(\omega_{1}, \omega_{2}\right)
$$

where $\left\{\omega_{1}, \omega_{2}\right\}$ is the base of the space $V\left(\omega_{1}, \omega_{2}\right)$ such that $y=\omega_{1}(x), y=\omega_{2}(x), x \in J$ is a fundamental solution system of the second-order differential equation

$$
y^{\prime \prime}+\frac{D\left[\varphi_{1}, \varphi_{2}\right] \cdot D\left[\psi_{1}, \psi_{2}\right]}{W\left[\varphi_{1}, \varphi_{2}\right] \cdot W\left[\psi_{1}, \psi_{2}\right]} \cdot y^{\prime}+\frac{D\left[\varphi_{1}, \varphi_{2}\right] \cdot W\left[\varphi_{1}^{\prime}, \varphi_{2}^{\prime}\right]+W\left[\varphi_{1}^{\prime}, \varphi_{2}^{\prime}\right] \cdot W\left[\psi_{1}, \psi_{2}\right]}{W\left(\varphi_{1}, \varphi_{2}\right) \cdot W\left[\psi_{1}, \psi_{2}\right]} \cdot y=0
$$

with $D\left[\varphi_{1}, \varphi_{2}\right]=\left|\begin{array}{cc}\varphi_{1}^{\prime \prime} & \varphi_{2}^{\prime \prime} \\ \varphi_{1} & \varphi_{2}\end{array}\right|, D\left[\psi_{1}, \psi_{2}\right]=\left|\begin{array}{ll}\psi_{1}^{\prime \prime} & \psi_{2}^{\prime \prime} \\ \psi_{1} & \psi_{2}\end{array}\right|$. There is proved in paper [2] that $(\mathbb{G}(\mathcal{F}), \cdot)$ is a non-commutative group of vector spaces with the neutral element $V\left(1, e^{-x}\right)$. Using the one-to-one correspondence between groups $(\mathbb{G}(\mathcal{F}), \cdot),\left(\mathbb{L}_{\mathbb{A}_{2}}(J), \cdot\right)([16])$ we obtain with respect to the above results (Theorem 6) that the group $(\mathbb{G}(\mathcal{F}), \cdot)$ is solvable. Notice, that this problem with other details will be investigated in a forthcomming paper.

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# ON FINITE CONGRUENCE-SIMPLE SEMIRINGS 

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#### Abstract

Congruence-simple algebras form the basic building block of algebra for construction of more complex structures. However, their role in the use of algebraic structures in cryptology where they are used primarily for modification of DiffieHellman algorithm is fundamental. It is undoubtedly very interesting in this context that in contrast to simple groups and areas that have piqued special interest of mathematicians for decades, we only very rarely come across results of studies in congruence-simple semirings. The aim of this article is to summarize the results achieved in the area of finite congruence-simple semirings over the past decade and to pinpoint the open questions that are yet to be answered.


Key words. Congruence-simple semirings, finite semirings.
Mathematics Subject Classification: Primary 16Y60, 08A30; Secondary 11 T71.

## 1 Introduction

The first reference to semiring (i.e. a universal algebra with two associative binary operations, where one of them distributes over the other) as an algebraic structure can be found in the article of Vandiver from 1934 ([1]). However, one can come across semirings, even though not scientifically defined at that time, also in works of mathematicians published before this date (the reader is referred to ( [2], [3], [4] and [5]) for background, basic, and more advanced properties of, and comments, historical remarks, and further references on semirings). Before 1934, mathematicians worked with semirings of positive real, rational and whole numbers and more generally also with semirings of positive elements in ordered rings. We meet semirings in various applications in theoretical computer science and algorithm theory.

Congruence-simple algebras form the basic building block of algebra for construction of more complex structures. However, their role in the use of algebraic structures in cryptology where they are used primarily for modification of Diffie-Hellman algorithm is fundamental. It is undoubtedly very interesting in this context that in contrast to simple groups and areas that have piqued special

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interest of mathematicians for decades, we only very rarely come across results of studies in congruence-simple semirings. The aim of this article is to summarize the results achieved in the area of finite congruence-simple semirings over the past decade and to pinpoint the open questions that are yet to be answered.

## 2 Commutative Finite Congruence-Simple Semirings

## Definition

A set $R$ with two binary operations + and $\cdot$ is called a semiring if $(R,+)$ is a commutative monoid, $\left(R r^{*}\right)$ is a semigroup, and the distributive laws $x \cdot(y+z)=x \cdot y+x \cdot z$ and $(x \| y) \cdot z=x \cdot z \| y \cdot z$ hold for all $x, y, z \subset R$.

## Definition

An equivalence relation $\sim$ on a semiring $R$ is called congruence if $x \sim y$ implies
$a+x \sim a+y, a x \sim a y, x a \sim y a$, for all $x, y, a \in R$.
The semiring R is called congruence-simple if its only congruences are $\sim=i d_{R}$ and $\sim=R \times R$.

The properties of commutative congruence-simple semirings were published in 2001 ([6]). The summary article presents not only the properties of finite congruence-simple semirings but also of finitely generated semirings and non-finite semirings. For more details, see the article. What follows here is only a brief description of the basic results relevant to the finite case.

| + | 0 | 1 |
| :---: | :--- | :--- |
| 0 | 0 | 0 |
| 1 | 0 | 0 |


| $Z_{1}$ |  |  |
| ---: | ---: | :--- |
|  |  |  |
|  |  | 0 |
| 0 | 0 | 0 |
| 1 | 0 | 0 |


| $Z_{2}$ |  |  |  |  |  |
| :---: | :--- | :--- | :--- | :--- | :--- |
| + | 0 | 1 |  | 0 | 1 |
| 0 | 0 | 0 |  |  |  |
| 1 | 0 | 0 |  | 0 | 0 |


| + | 0 | 1 |
| :---: | :--- | :--- |
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$Z_{3}$

| $\cdot$ | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 0 |
| 1 | 0 | 0 |


| $Z_{4}$ |  |  |  |  |  |
| :---: | :--- | :--- | :--- | :--- | :--- |
| + | 0 | 1 |  | 0 | 1 |
| 0 | 0 | 0 |  | 0 | 1 |
| 1 | 1 | 1 | 1 |  |  |


| + | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$Z_{5}$

| $\cdot$ | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 1 |
| 1 | 1 | 1 |

## Theorem 1

The following conditions are equivalent for a semiring $S$ :

- $S$ is finite and congruence-simple.
- $S$ is finite and ideal-simple.
- $S$ is isomorphic to one of the following semirings:
- The two-element semirings $Z_{1}, Z_{2}, Z_{3}, Z_{4}$ and $Z_{5}$;
- finite fields;


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- zero-multiplication rings of finite prime order;
- the semirings (semifields) $V(G)$ (defined below), $G$ being a finite abelian group.


## Definition

For a multiplicative abelian group $G$, set $V(G)=G \cup\{\infty\}$. Extend the multiplication of $G$ to $V(G)$ by the rule $x . \infty=\infty \cdot x=\infty$ for all $x \in V(G)$. Define an addition on $V(G)$ by the rules $x\|x=x, x\| y=\infty$ for all $x, y \subset V(G)$ with $x \neq y$.

For proof see [6].

## 2 Additively commutative Finite Congruence-Simple Semirings

The results relevant to commutative finite congruence-simple semirings were soon taken up by Ch . Monico who characterized finite congruence-simple semirings for the case when the additive group is commutative ([7]). First he constructed a new class of finite congruence-simple semirings.

## Theorem 2

Let $I=\{1,2, \ldots, m\}, A=\{1,2, \ldots, n\}$, and $P=\left(p_{i j}\right)$ be an $n \times m$ matrix of 1 's and 0 's such that no row or column is identically zero, no two rows are identical, and no two columns are identical. Let $S=(I \times \Lambda) \cup\{\infty\}$ and define a binary relation on $S$ by $(i, \lambda) \cdot(j, \mu)=(i, \mu)$ if $p \lambda j=1$, and $(i, \lambda) \cdot(j, \mu)=\infty$ otherwise, $(i, \lambda) \cdot \infty=\infty \cdot(i, \lambda)=\infty \cdot \infty=\infty$.

Then $S$ is a congruence-free semigroup of order $m n+1$. Conversely, every finite congruence-free semigroup with an absorbing element is isomorphic to one of this kind.

Subsequently he formulated a new classification theorem of a finite, additively commutative, congruence-simple semiring:

## Theorem 3.

Let $S$ be a finite, additively commutative, congruence-simple semiring. Then one of the following holds:
(1) $|\mathrm{S}|=2$;
(2) $\mathrm{S} \cong \operatorname{Mat}_{n}\left(\mathrm{~F}_{\mathrm{q}}\right)$ for some finite field $\mathrm{F}_{\mathrm{q}}$ and some $\mathrm{n} \geq 1$;
(3) S is a zero multiplication ring of prime order;
(4) $S$ is additively idempotent;
(5) ( $S,{ }^{\prime}$ ) is a semigroup as in Theorem 2 with absorbing element $\infty \in S$ and $S+S=\{\infty\}$.

Simultaneously he showed that there are additively commutative and idempotent finite congruence-simple semirings which are not multiplicatively commutative:

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| + | a | 1 | b |
| :---: | :---: | :---: | :---: |
| a | a | 1 | b |
| 1 | 1 | 1 | b |
| b | b | b | b |


| $\cdot$ | a | 1 | b |
| :---: | :---: | :---: | :---: |
| a | a | a | b |
| 1 | a | 1 | b |
| b | a | b | b |

However, it was an isolated case of this type of semiring and Ch. Monico himself failed to construct any other. Moreover, in his dissertation thesis he formulates the following hypothesis on the basis of his studies of thousands of semirings: This semiring is additively idempotent with a one. However, observe that the element $a$ is `almost' a zero and \(b\) is `almost' an infinity. We conjecture that, except for some such semirings of order 3, the additively idempotent c-simple semirings are those of the form $V(G)$ for finite groups $G$. This conjecture is based on an admittedly small amount of empirical evidence; specically, the random generation of several thousand additively commutative semirings with small order (<15). ([8])

This hypothesis however turned out to be erroneous and the class of additively idempotent finite, additively commutative, congruence-simple semiring is much richer.

## 3. Additively commutative Finite Congruence-Simple Semirings with Zero

In 2008 J. Zumbrägel fully characterized additively commutative finite congruence-simple semirings with zero ([9]).

Zumbrägel first shows the relation between congruence simple semirings and dense subsemirings of endomorphism rings of finite idempotent commutative monoid, where dense subsemiring is defined as follows:

## Definition

Let $M$ be an idempotent commutative monoid. A subsemiring $S \subseteq \operatorname{End}(M)$ is called dense if it contains for all $a, b \in M$ the endomorphism $e_{a, b} \in \operatorname{End}(M)$, defined by $e_{a, b}(x):=0$ if $x+a=a(x \in$ M ) and $\mathrm{e}_{\mathrm{a}, \mathrm{b}}(\mathrm{x}):=\mathrm{b}$ otherwise.

## Theorem 4.

Let R be a finite semiring with zero which is not a ring. Then the following are equivalent:
(1) $R$ is congruence-simple.
(2) $|R| \leq 2$ or $R$ is isomorphic to a dense subsemiring $S \subseteq \operatorname{End}(M)$, where $(M,+)$ is a finite idempotent commutative monoid.

For proof see ([9]).


| $M$ | $\mathcal{S R}(M)$ |
| :---: | :---: |
|  | $\left\{R_{70}\right\}$ |
|  | $\left\{R_{50, a}\right\}$ |
|  | $\left\{R_{50, b}\right\}$ |
|  | $\left\{R_{43}, R_{42}\right\}$ |
|  | $\begin{aligned} & \left\{R_{50, c}, R_{47}, \quad R_{46, a}, \quad R_{46, b},\right. \\ & \left.R_{46,,}, R_{45}, R_{44}\right\} \\ & \text { (where } R_{46, a}, R_{46, b} \text { and } R_{46, c} \\ & \text { are isomorphic) } \end{aligned}$ |

The smallest lattices together with the corresponding endomorphism semirings
This confirms that the smallest non-trivial congruence simple semiring with zero has 6 elements. The following table gives its form:

| + | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 1 | 1 | 1 | 5 |
| 2 | 2 | 1 | 2 | 3 | 4 | 5 |
| 3 | 3 | 1 | 3 | 3 | 1 | 5 |
| 4 | 4 | 1 | 4 | 1 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 |


| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 0 | 0 | 2 | 3 |
| 3 | 0 | 3 | 2 | 3 | 2 | 3 |
| 4 | 0 | 4 | 0 | 0 | 4 | 5 |
| 5 | 0 | 5 | 4 | 5 | 4 | 5 |

## 4. New Additively commutative Finite Congruence-Simple Semiring

However, this characteristics of additively idempotent finite, additively commutative, congruence-simple semiring has turned out to be incomplete. There are other additively

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commutative finite congruence-simple semirings which do not have 0 and have more than 3 elements. An example of such a semiring was constructed by J. Jančařík:

| + | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 2 | 2 | 2 | 4 | 4 | 5 |
| 3 | 3 | 3 | 4 | 3 | 4 | 5 |
| 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 |


| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 3 | 3 | 5 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 2 | 2 | 2 | 4 | 4 | 5 |
| 3 | 0 | 3 | 5 | 3 | 5 | 5 |
| 4 | 1 | 4 | 5 | 4 | 5 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 |

This semiring is additively commutative and idempotent finite congruence-simple semirings with infinity.

## 5. Conclusion

Finite Congruence-Simple Semirings have been subject to research for more than 10 years. The results presented in this article indicate the way to a detailed classification. The task of finding a complete classification of all types of these semirings seems to be extremely difficult. One of the subtasks is a full characterization of the additively idempotent finite, additively commutative, congruence-simple semiring. The result presented in this paper clearly shows that this subclass has not been yet fully characterized and its complete characterization is still an open question.

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# RELATIVELY UNIFORM CONVERGENCE IN DUALLY RESIDUATED LATTICE ORDERED SEMIGROUPS 

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#### Abstract

In this paper the notion of a relatively uniform convergence in dually residuated lattice ordered semigroups is introduced and basic properties of this convergence are established.


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Swamy [13, 14, 15] introduced and studied dually residuated lattice ordered semigroups (notation DRl-semigroups) as a common abstraction of Boolean rings and lattice ordered groups. It was a solution of Birkhoff's problem No. 115 [1]. DRl-semigroups were also investigated by Kovař $[10,11]$ and by the author [8].

Relatively uniform convergence of sequences in vector lattices was studied by Birkhoff [1], Luxembourg and Zaanen [12]. Such type of convergence under the name convergence with a regulator has been also investigated by Vulikh [17] and Veksler [16]. Černák [2, 3] and Černák and Lihova [4] studied u-uniform convergence in lattice ordered groups. Relatively uniform convergence in lattice ordered groups was dealt with by Černák and Lihová [5] and Černák and Jakubík [7]. Weak relatively uniform convergence in lattice ordered groups was studied by Černák and Jakubík [6]. For the case of Riesz groups, weak relatively uniform convergence has been dealt with by the author in [9].

The purpose of this paper is to extend the concept of a relatively uniform convergence to dually residuated lattice ordered semigroups and to study basic properties of this convergence.

We review some notions and notations used in the paper.

A system $A=(A,+, \leq,-)$ is called a dually residuated lattice ordered semigroup if and only if
(1) $(A,+, \leq)$ is a commmutative lattice ordered semigroup with zero element 0 , i.e. $(\mathrm{A},+)$ is a commutative semigroup with zero 0 and $(A, \leq)$ is a lattice with lattice operations $\wedge$ and $\vee$ such that $a+(b \vee c)=(a+b) \vee(a+c)$ and $a+(b \wedge c)=(a+b) \wedge(a+c)$,
(2) given $a, b$ in $A$ there exists a least $x$ in $A$ such that $b+x \geq a$, and this $x$ is denoted by $a-b$,
(3) $(a-b) \vee 0+b \leq a \vee b$ for all $a, b \in A$,
(4) $(a-a) \leq 0$ for each $a \in A$.

Any DRl-semigroup can be equationally defined as an algebra with the binary operations $+, \vee, \wedge,-$, by replacing (2) by the equations:
$x+(y-x) \geq y, \quad x-y \leq(x \vee z)-y, \quad(x+y)-y \leq x \quad$ [13, Theorem 1].
Any abelian lattice ordered group is a DRl -semigroup.
For any $a$ and $b$ in a DRl-semigroup $A$, we shall write $|a-b|=(a-b) \vee(b-a)(|a-b|$ is called the symetric difference of $a$ and $b$.)
The symetric difference satisfies the following conditions:
(i) $|a-b| \geq 0,|a-b|=0$ if and only if $a=b$,
(ii) $|a-b|=|b-a|$,
(iii) $|a-c| \leq|a-b|+|b-c|$.

Any DRl-semigroup is an autometrized algebra with the symetric difference [13, Theorem 9].

If a DRl -semigroup $A$ is a lattice ordered group and $a, b, c \in A$, then notations $a-b$ and $|a-b|$ have the same meaning as in lattice ordered groups.

We denote $A^{+}=\{x \in A ; x \geq 0\}$ and use $N$ for the set of all positive integers.
A DRl-semigroup is said to be Archimedean if for each $x, y \in A^{+}, n x \leq y$ for each $n \in N$ implies $x=0$.

Luxemburg and Zaanen introduced notions of a u-uniform convergence and of a relatively uniform convergence of sequences in vector lattices in their monograph [12] as follows.

Definition 0.1 Let $V$ be a vector lattice, $u \in V^{+}$. It is said that a sequence $\left(x_{n}\right)$ in $V$ converges u-uniformly to an element $x \in V$, if the following condition is satisfied:
$\left(C_{1}\right)$ for each $\varepsilon \in R, \varepsilon \geq 0$ there exists $n_{0} \in N$, such that $\left|x_{n}-x\right| \leq \varepsilon u$, for each $n \in N, n \geq n_{0}$.
Definition 0.2 Let $V$ be a vector lattice. We say that a sequence $\left(x_{n}\right)$ in $V$ relatively uniformly converges to an element $x \in V$, whenever $\left(x_{n}\right)$ converges $u$-uniformly to $x$ for some $u \in V^{+}$.

Černák and Lihová [5] found out that $\varepsilon$ in the Definition 1 can be equivalently replaced by $\frac{1}{p}$, where $p \in N$, and adopted definitions of Luxemburg and Zaanen for lattice ordered groups.

Definition 0.3 Let $G$ be a lattice ordered group, $u \in G^{+}$. We say that a sequence $\left(x_{n}\right)$ in $G$ converges $u$-uniformly to an element $x \in G$, if the following condition is satisfied: $\left(C_{2}\right)$ for each $p \in N$ there exists $n_{0} \in N$ such that $p\left|x_{n}-x\right| \leq u$ for each $n \in N, n \geq n_{0}$.

Definition 0.4 Let $G$ be a lattice ordered group. We say that a sequence $\left(x_{n}\right)$ in $G$ relatively uniformly converges to an element $x \in G$, whenever $\left(x_{n}\right)$ converges $u$-uniformly to $x$ for some $u \in G^{+}$.

We can use definitions of Černák and Lihová of u-uniform convergence and of relatively uniform convergence above also for DRl-semigroups.

Definition 0.5 Let $A$ be a DRl-semigroup, $\left(x_{n}\right)$ a sequence in $A, u \in A^{+}$. It is said that a sequence $\left(x_{n}\right)$ in $A$ converges $u$-uniformly to an element $x \in A$, written $x_{n} \xrightarrow{u} x$, if the following condition is satisfied:
$\left(C_{3}\right)$ for each $p \in N$ there exists $n_{p} \in N$, such that $p\left|x_{n}-x\right| \leq u$ for each $n \in N, n \geq n_{p}$.
Definition 0.6 Let $A$ be a DRl-semigroup. We say that a sequence ( $x_{n}$ ) in $A$ relatively uniformly converges (briefly ru-coverges) to an element $x \in A$, in symbols $x_{n} \rightarrow x$, whenever there exists $u \in A^{+}$such that $x_{n} \xrightarrow{u} x$.

If $A$ is a lattice ordered group, our definitions are equivalent to the definitions of Černák and Lihová.

We shall often need the following assertions from [13].
Let $A$ be a DRl-semigroup, $a, b, c \in A$. Then
(A) $a \leq b$ implies $a-c \leq b-c$ and $c-b \leq c-a$ (Lemma 3),
(B) $(a \vee b)-c=(a-c) \vee(b-c) \quad$ (Lemma 4),
(C) $a-(b \wedge c)=(a-b) \vee(a-c) \quad$ (Lemma 5),
(D) $a-(b+c)=(a-b)-c=(a-c)-b$ (Lemma 6),
(E) $(a-b)+(b-c) \geq(a-c)$ (Lemma 12),
(F) $a-(b-c) \leq(a-b)+c$ and $(a+b)-c \leq(a-c)+b \quad$ (Lemma 13).

Theorem 0.7 Let $A$ be a DRl-semigroup, $a, b, c, d \in A$. Then
(i) $(a-b)+(c-d) \geq(a+c)-(b+d)$,
(ii) $|a-b|+|c-d| \geq|(a+c)-(b+d)|$,
(iii) $(a-b)+(c-d) \geq(c-b)-(d-a)$

Proof. (i) According to ( $D$ ) and ( $F$ ) we obtain $(a-b)+(c-d) \geq(a+(c-d))-b \geq((c-d)+a)-b \geq$ $((c+a)-d)-b=(c+a)-(b+d) \geq(a+c)-(b+d)$.
(ii) In view of (i) we have $|a-b|+|c-d|=(a-b) \vee(b-a)+(c-d) \vee(d-c)=((a-b)+(c-d)) \vee$ $((a-b)+(d-c)) \vee((b-a)+(c-d)) \vee((b-a)+(d-c)) \geq((a-b)+(c-d)) \vee((b-a)+(d-c)) \geq$ $((a+c)-(b+d)) \vee((b+d)-(a+c))=|(a+c)-(b+d)|$.
(iii) By (D) and (F) we get $(a-b)+(c-d) \geq(a+(c-d))-b \geq((c-d)+a)-b \geq$ $(c-(d-a))-b \geq(c-b)-(d-a)$.

Theorem 0.8 Let $A$ be an Archimedean DRl-semigroup, $\left(x_{n}\right)$ a sequence in $A, x, y \in A$. If $x_{n} \rightarrow x$ and $x_{n} \rightarrow y$, then $x=y$.
Proof. Since $x_{n} \rightarrow x$ and $x_{n} \rightarrow y$, there exist $u, v \in A^{+}$such that $x_{n} \xrightarrow{u} x, x_{n} \xrightarrow{v} y$.
Let $p \in N$. Then there exists $n_{p} \in N$ such that $u \geq p\left|x_{n}-x\right|, v \geq p\left|x_{n}-y\right|$ for each $n \in N, n \geq n_{p}$. From this it follows that $u+v \geq p\left|x_{n}-x\right|+p\left|x_{n}-y\right|=p\left(\left|x_{n}-x\right|+\left|x_{n}-y\right|\right)=$ $p\left(\left|x-x_{n}\right|+\left|x_{n}-y\right|\right) \geq p|x-y|$.

Then Archimedeanicity of A implies $|x-y|=0$. Hence $x=y$.

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Since each Archimedean lattice ordered group is abelian, each Archimedean lattice ordered group is an Archimedean DRl-semigroup. Hence Theorem 2 generalizes Lemma 2.5 from [5].

Theorem 0.9 Let $A$ be an Archimedean DRl-semigroup, $\left(x_{n}\right)$ and $\left(y_{n}\right)$ sequences in $A, x, y \in$ A. Let $x_{n} \rightarrow x$ and $x_{n} \rightarrow y$. Then
(i) $x_{n}+y_{n} \rightarrow x+y$,
(ii) $x_{n}-y_{n} \rightarrow x-y$,
(iii) $x_{n} \vee y_{n} \rightarrow x \vee y$,
(iv) $x_{n} \wedge y_{n} \rightarrow x \wedge y$.

Proof. Since $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$, there exist $u, v \in A^{+}$, such that $x_{n} \xrightarrow{u} x, y_{n} \xrightarrow{v} y$.
Let $p \in N$. Then there exists $n_{p} \in N$ such that $u \geq p\left|x_{n}-x\right|, v \geq p\left|y_{n}-y\right|$ for each $n \in N, n \geq n_{p}$.
(i) In view of Theorem 1 (ii) we get $u+v \geq p\left|x_{n}-x\right|+p\left|y_{n}-y\right|=p\left(\left|x_{n}-x\right|+\left|y_{n}-y\right|\right) \geq$ $p\left|\left(x_{n}+y_{n}\right)-(x+y)\right|$. Thus $x_{n}+y_{n} \xrightarrow{u+v} x+y$ and hence $x_{n}+y_{n} \rightarrow x+y$.
(ii) In view of Theorem 1 (iii) we have $u+v \geq p\left|x_{n}-x\right|+p\left|y_{n}-y\right|=p\left(\left|x_{n}-x\right|+\left|y_{n}-y\right|\right)=$ $p\left(\left(\left(x_{n}-x\right) \vee\left(x-x_{n}\right)\right)+\left(\left(y_{n}-y\right) \vee\left(y-y_{n}\right)\right)\right)=p\left(\left(\left(x_{n}-x\right)+\left(y_{n}-y\right)\right) \vee\left(\left(x_{n}-x\right)+\left(y-y_{n}\right)\right) \vee\right.$ $\left.\left(\left(x-x_{n}\right)+\left(y_{n}-y\right)\right) \vee\left(\left(x-x_{n}\right)+\left(y-y_{n}\right)\right)\right) \geq p\left(\left(\left(x_{n}-x\right)+\left(y-y_{n}\right)\right) \vee\left(\left(x-x_{n}\right)+\left(y_{n}-y\right)\right)\right)=$ $p\left(\left(\left(y-y_{n}\right)+\left(x_{n}-x\right)\right) \vee\left(\left(y_{n}-y\right)+\left(x-x_{n}\right)\right)\right) \geq p\left(\left(\left(x_{n}-y_{n}\right)-(x-y)\right) \vee\left((x-y)-\left(x_{n}-y_{n}\right)\right)\right)=$ $p\left|\left(x_{n}-y_{n}\right)-(x-y)\right|$. Therefore $x_{n}-y_{n} \xrightarrow{u+v} x-y$. Hence $x_{n}-y_{n} \rightarrow x-y$.
(iii) In view of (B) we have $u+v \geq p\left|x_{n}-x\right|+p\left|y_{n}-y\right|=p\left(\left|x_{n}-x\right|+\left|y_{n}-y\right|\right) \geq$ $p\left(\left|x_{n}-x\right| \vee\left|y_{n}-y\right|\right)=p\left(\left(x_{n}-x\right) \vee\left(x-x_{n}\right) \vee\left(y_{n}-y\right) \vee\left(y-y_{n}\right)\right)=p\left(\left(x_{n}-x\right) \vee\left(y_{n}-y\right) \vee\right.$ $\left.\left(x-x_{n}\right) \vee\left(y-y_{n}\right)\right) \geq p\left(\left(x_{n}-(x \vee y)\right) \vee\left(y_{n}-(x \vee y)\right) \vee\left(x-\left(\left(x_{n} \vee y_{n}\right)\right) \vee\left(y-\left(x_{n} \vee y_{n}\right)\right)\right)=\right.$ $p\left(\left(\left(x_{n} \vee y_{n}\right)-(x \vee y)\right) \vee\left((x \vee y)-\left(x_{n} \vee y_{n}\right)\right)\right)=p\left|\left(x_{n} \vee y_{n}\right)-(x \vee y)\right|$. Thus $x_{n} \vee y_{n} \xrightarrow{u+v} x \vee y$ and hence $x_{n} \vee y_{n} \rightarrow x \vee y$.
(iv) According to (C) we obtain $u+v \geq p\left|x_{n}-x\right|+p\left|y_{n}-y\right|=p\left(\left|x_{n}-x\right|+\left|y_{n}-y\right|\right) \geq$ $p\left(\left|x_{n}-x\right| \vee\left|y_{n}-y\right|\right)=p\left(\left(x_{n}-x\right) \vee\left(x-x_{n}\right) \vee\left(y_{n}-y\right) \vee\left(y-y_{n}\right)\right)=p\left(\left(x_{n}-x\right) \vee\left(y_{n}-y\right) \vee\right.$ $\left.\left.\left.\left(x-x_{n}\right) \vee\left(y-y_{n}\right)\right) \geq p\left(\left(x_{n} \wedge y_{n}\right)-x\right) \vee\left(\left(x_{n} \wedge y_{n}\right)-y\right)\right) \vee\left((x \wedge y)-x_{n}\right) \vee\left((x \wedge y)-y_{n}\right)\right)=$ $p\left(\left(\left(x_{n} \wedge y_{n}\right)-(x \wedge y)\right) \vee\left((x \wedge y)-\left(x_{n} \wedge y_{n}\right)\right)\right)=p\left|\left(x_{n} \wedge y_{n}\right)-(x \wedge y)\right|$. Therefore $x_{n} \wedge y_{n} \xrightarrow{u+v} x \wedge y$. Hence $x_{n} \wedge y_{n} \rightarrow x \wedge y$.

Theorem 3 generalizes assertions (i), (ii) and (iii) of Lemma 2.8 of Černak and Lihová [5].
Theorem 0.10 Let $A$ be an Archimedean DRl-semigroup, $\left(x_{n}\right)$ and $\left(y_{n}\right)$ sequences in $A$. Let $x_{n} \rightarrow x$ and $x_{n} \rightarrow y$ and $x_{n} \leq y_{n}$ for each $n \in N$. Then $x \leq y$.
Proof. Since $x_{n} \vee y_{n}=y_{n} \rightarrow y$, in view of Theorem 3 (iii) we have $x \vee y=y$. This yields $x \leq y$.

Theorem 0.11 Let $A$ be an Archimedean DRl-semigroup, $\left(x_{n}\right),\left(y_{n}\right)$ and $\left(z_{n}\right)$ sequences in $A$, $a \in A$. Let $x_{n} \leq y_{n} \leq z_{n}$ for each $n \in N, x_{n} \rightarrow a, z_{n} \rightarrow a$. Then $y_{n} \rightarrow a$.
Proof. Since $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$, there exist $u, v \in A^{+}$, such that $x_{n} \xrightarrow{u} x, y_{n} \xrightarrow{v} y$.
Let $p \in N$. Then there exists $n_{p} \in N$ such that $u \geq p\left|x_{n}-a\right|, v \geq p\left|y_{n}-a\right|$ for each $n \in N, n \geq n_{p}$. This and Theorem 1 (ii) imply $u+v \geq p\left|x_{n}-a\right|+p\left|z_{n}-a\right|=p\left(\left|x_{n}-a\right|+\mid z_{n}-\right.$ $a \mid) \geq p\left(\left|x_{n}-a\right| \vee\left|z_{n}-a\right|\right)=p\left(\left(x_{n}-a\right) \vee\left(a-x_{n}\right) \vee\left(z_{n}-a\right) \vee\left(a-z_{n}\right)\right) \geq p\left(\left(z_{n}-a\right) \vee\left(a-x_{n}\right)\right)$.

In view of (A) from $x_{n} \leq y_{n} \leq z_{n}$ we get $y_{n}-a \leq z_{n}-a, a-y_{n} \leq a-x_{n}$. This yields $\left(y_{n}-a\right) \vee\left(a-y_{n}\right) \leq\left(z_{n}-a\right) \vee\left(a-x_{n}\right)$.

Therefore $u+v \geq p\left(\left(z_{n}-a\right) \vee\left(a-x_{n}\right)\right) \geq p\left(\left(y_{n}-a\right) \vee\left(a-y_{n}\right)\right)=p\left|y_{n}-a\right|$ for each $n \in N, n \geq n_{p}$. Thus $y_{n} \xrightarrow{u+v} a$. Hence $y_{n} \rightarrow a$.

Theorem 0.12 If a sequence ( $x_{n}$ ) in an Archimedean DRl-semigroup $A$ is ru-convergent, then the sequence $\left(x_{n}\right)$ is bounded in $A$.
Proof. Let $x_{n} \rightarrow x$. Then $x_{n} \xrightarrow{u} x$ for some $u \in A^{+}$. From this follows that for each $p \in N$ there exists $n_{p} \in N$ such that $u \geq p\left|x_{n}-x\right|$ for all $n \in N, n \geq n_{p}$. Hence for $p=1$ we get that there exists $n_{p 1} \in N$ such that $u \geq\left|x_{n}-x\right|$ for all $n \in N, n \geq n_{p 1}$. This yields $x-x_{n} \leq u$, $x_{n}-x \leq u$. From this we obtain $x \leq\left(x-x_{n}\right)+x_{n} \leq u+x_{n}, x_{n} \leq\left(x_{n}-x\right)+x \leq u+x$. In view of (F) from $x \leq\left(x-x_{n}\right)+x_{n} \leq u+x_{n}$ we get $x-u \leq\left(u+x_{n}\right)-u \leq x_{n}$. Therefore $x-u \leq x_{n} \leq u+x$ for all $n \in N, n \geq n_{p 1}$.

Let $a=(x-u) \wedge x_{1} \wedge x_{2} \wedge \ldots \wedge x_{n_{p 1}-1}$ and $b=(x+u) \wedge x_{1} \vee x_{2} \vee \ldots \vee x_{n_{p 1}-1}$. Thus $a \leq x_{k} \leq b$ for each $k \in N$. Therefore the sequence $\left(x_{n}\right)$ is bounded in $A$.

Definition 0.13 A sequence $\left(x_{n}\right)$ in a DRl-semigroup $A$ is called a Cauchy sequence with respect to the relatively uniform convergence (shortly ru-Cauchy sequence), if for some $u \in A^{+}$ and each $p \in N$ there exists $n_{p} \in N$ such that $u \geq p\left|x_{m}-x_{n}\right|$ for each $m, n \in N, m, n \geq n_{p}$.

Theorem 0.14 Any ru-convergent sequence in an Archimedean DRl-semigroup $A$ is a ruCauchy sequence.

Proof. If $x_{n} \xrightarrow{u} x$ for some $u \in A^{+}$, then for each $p \in N$ there exists $n_{p} \in N$ such that $u \geq p\left|x_{m}-x\right|, u \geq p\left|x_{n}-x\right|$ for all $m, n \in N, m, n \geq n_{p}$. From this we get $2 u \geq$ $p\left|x_{m}-x\right|+p\left|x_{n}-x\right|=k\left(\left|x_{m}-x\right|+\left|x-x_{n}\right|\right) \geq p\left|x_{m}-x_{n}\right|$ for each $m, n \in N, m, n \geq n_{p}$. Therefore $\left(x_{n}\right)$ is an ru-Cauchy sequence.

Let $C$ be the set of all ru-Cauchy sequences in an Archimedean DRl-semigroup $A$.
Theorem 0.15 Let $\left(x_{n}\right)$ and $\left(y_{n}\right) \in C$. Then
(i) $\left(x_{n}+y_{n}\right) \in C$,
(ii) $\left(x_{n}-y_{n}\right) \in C$,
(iii) $\left(x_{n} \vee y_{n}\right) \in C$,
(iv) $\left(x_{n} \wedge y_{n}\right) \in C$.

Proof. Let $\left(x_{n}\right)$ and $\left(y_{n}\right) \in C$. Let $u, v$ be elements from $A^{+}$such that for each $p \in N$ there exists $n_{p} \in N$ such that $u \geq p\left|x_{m}-x_{n}\right|, v \geq p\left|y_{m}-y_{n}\right|$ for each $m, n \in N, m, n \geq n_{p}$.
(i) In view of Theorem 1 (ii) we have $u+v \geq p\left|x_{m}-x_{n}\right|+p\left|y_{m}-y_{n}\right|=p\left(\left|x_{m}-x_{n}\right|+\left|y_{m}-y_{n}\right|\right) \geq$ $p\left|\left(x_{m}+y_{m}\right)-\left(x_{n}+y_{n}\right)\right|$ for each $m, n \in N, m, n \geq n_{p}$. Hence $\left(x_{n}+y_{n}\right) \in C$.
(ii) According to Theorem 1 (iii) we get $u+v \geq p\left|y_{m}-y_{n}\right|+p\left|x_{m}-x_{n}\right|=p\left(\left|y_{m}-y_{n}\right|+\right.$ $\left.\left|x_{n}-x_{m}\right|\right) \geq p\left(\left(\left(y_{m}-y_{n}\right) \vee\left(y_{n}-y_{m}\right)\right)+\left(\left(x_{n}-x_{m}\right) \vee\left(x_{m}-x_{n}\right)\right)\right)=p\left(\left(\left(y_{m}-y_{n}\right)+\left(x_{n}-\right.\right.\right.$ $\left.\left.x_{m}\right)\right) \vee\left(\left(y_{m}-y_{n}\right)+\left(x_{m}-x_{n}\right)\right) \vee\left(\left(y_{n}-y_{m}\right)+\left(\left(x_{n}-x_{m}\right)\right) \vee\left(\left(y_{n}-y_{m}\right)+\left(x_{m}-x_{n}\right)\right)\right) \geq$ $p\left(\left(\left(y_{m}-y_{n}\right)+\left(x_{n}-x_{m}\right)\right) \vee\left(\left(y_{n}-y_{m}\right)+\left(x_{m}-x_{n}\right)\right)\right) \geq p\left(\left(\left(x_{n}-y_{n}\right)-\left(x_{m}-y_{m}\right)\right) \vee\left(\left(x_{m}-\right.\right.\right.$ $\left.\left.\left.y_{m}\right)-\left(x_{n}-y_{n}\right)\right)\right)=p\left|\left(x_{m}-y_{m}\right)-\left(x_{n}-y_{n}\right)\right|$ for each $m, n \in N, m, n \geq n_{p}$. Therefore $\left(x_{n}-y_{n}\right) \in C$.
(iii) In view of (B) we have $u+v \geq p\left|x_{m}-x_{n}\right|+p\left|y_{m}-y_{n}\right|=p\left(\left|x_{m}-x_{n}\right|+\left|y_{m}-y_{n}\right|\right) \geq$
$p\left(\left|x_{m}-x_{n}\right| \vee\left|y_{m}-y_{n}\right|\right)=p\left(\left(x_{m}-x_{n}\right) \vee\left(x_{n}-x_{m}\right) \vee\left(y_{m}-y_{n}\right) \vee\left(y_{n}-y_{m}\right)\right)=p\left(\left(x_{m}-x_{n}\right) \vee\right.$ $\left.\left(y_{m}-y_{n}\right) \vee\left(x_{n}-x_{m}\right) \vee\left(y_{n}-y_{m}\right)\right) \geq p\left(\left(x_{m}-\left(x_{n} \vee y_{n}\right)\right) \vee\left(y_{m}-\left(x_{n} \vee y_{n}\right)\right) \vee\left(x_{n}-\left(\left(x_{m} \vee y_{m}\right)\right) \vee\right.\right.$ $\left.\left(y_{n}-\left(x_{m} \vee y_{m}\right)\right)\right)=p\left(\left(\left(x_{m} \vee y_{m}\right)-\left(x_{n} \vee y_{n}\right)\right) \vee\left(\left(x_{n} \vee y_{n}\right)-\left(x_{m} \vee y_{m}\right)\right)\right)=p\left|\left(x_{m} \vee y_{m}\right)-\left(x_{n} \vee y_{n}\right)\right|$ for each $m, n \in N, m, n \geq n_{p}$. Thus $\left(x_{n} \vee y_{n}\right) \in C$.
(iv) According to (C) we obtain $u+v \geq p\left|x_{m}-x_{n}\right|+p\left|y_{m}-y_{n}\right|=p\left(\left|x_{m}-x_{n}\right|+\left|y_{m}-y_{n}\right|\right) \geq$ $p\left(\left|x_{m}-x_{n}\right| \vee\left|y_{m}-y_{n}\right|\right)=p\left(\left(x_{m}-x_{n}\right) \vee\left(x_{n}-x_{m}\right) \vee\left(y_{m}-y_{n}\right) \vee\left(y_{n}-y_{m}\right)\right)=p\left(\left(x_{m}-x_{n}\right) \vee\right.$ $\left.\left.\left(y_{m}-y_{n}\right) \vee\left(x_{n}-x_{m}\right) \vee\left(y_{n}-y_{m}\right)\right) \geq p\left(\left(x_{m} \wedge y_{m}\right)-x_{n}\right) \vee\left(\left(x_{m} \wedge y_{m}\right)-y_{n}\right)\right) \vee\left(\left(x_{n} \wedge y_{n}\right)-x_{m}\right) \vee$ $\left.\left(\left(x_{n} \wedge y_{n}\right)-y_{m}\right)\right)=p\left(\left(\left(x_{m} \wedge y_{m}\right)-\left(x_{n} \wedge y_{n}\right)\right) \vee\left(\left(x_{n} \wedge y_{n}\right)-\left(x_{m} \wedge y_{m}\right)\right)\right)=p \mid\left(\left(x_{m} \wedge y_{m}\right)-\left(x_{n} \wedge y_{n}\right) \mid\right.$ for each $m, n \in N, m, n \geq n_{p}$. Hence $\left(x_{n} \wedge y_{n}\right) \in C$.

Theorem 3 generalizes Lemma 2.16 of Černak and Lihová [5].
Let $\left(x_{n}\right),\left(y_{n}\right) \in C$. We put $\left(x_{n}\right)+\left(y_{n}\right)=\left(x_{n}+y_{n}\right)$. Further, we set $\left(x_{n}\right) \leq\left(y_{n}\right)$ if and only if if $x_{n} \leq y_{n}$ for each $n \in N$. If $x_{n}=0$ for each $n \in N$, then we denote the sequence $\left(x_{n}\right)$ by (0). Then Theorem 8 implies the following assertion.

Theorem $0.16(C,+, \leq)$ is a DRl-semigroup with zero (0) and lattice operations $\vee$ and $\wedge$ such that $\left(x_{n}\right) \vee\left(y_{n}\right)=\left(x_{n} \vee y_{n}\right),\left(x_{n}\right) \wedge\left(y_{n}\right)=\left(x_{n} \wedge y_{n}\right)$ for all $\left(x_{n}\right),\left(y_{n}\right) \in C$. Further, $\left(x_{n}\right)-\left(y_{n}\right)=\left(x_{n}-y_{n}\right)$ for all $\left(x_{n}\right),\left(y_{n}\right) \in C$.

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## SOMOS SEQUENCES

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#### Abstract

This article deals with sequences which were considered by Michael Somos when he investigated specific elliptic curves. There also are considered sequences published in the journal Crux and in the collection of problems from Moscow Mathematical Olympiad.


This article mainly deals with sequences which were considered by Michael Somos when he investigated specific elliptic curves. But, we were first inspired by a task published in the journal Crux [4] and in the collection of problems from Moscow Mathematical Olympiad [7].

First, we got acquainted with the problem from Russian Mathematical Olympiad which was used in the competition in the year 1963. This task run as follows:

The sequence $\left(a_{n}\right)$ is given recursively:

$$
a_{1}=a_{2}=1, \quad a_{n+2}=\frac{a_{n+1}^{2}+2}{a_{n}} \quad(n \geq 1) .
$$

Prove that each sequence element is an integer number.
Then we read two similar problems in the journal Crux in the year 1989. There were published following tasks:

The sequences $\left(a_{n}\right)$ are given recursively:

$$
\begin{gathered}
a_{1}=a_{2}=1, \quad a_{n+2}=\frac{a_{n+1}^{2}+1}{a_{n}} \quad(n \geq 1), \\
a_{1}=a_{2}=a_{3}=1, \quad a_{n+3}=\frac{a_{n+2} a_{n+1}+1}{a_{n}} \quad(n \geq 1) .
\end{gathered}
$$

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Prove that each sequence element in both cases is an integer number.

These two last sequences belong among so called Somos sequences. Their definition is as follows [4]:

The sequence $\left(a_{n}\right)$ given recursively:

$$
\begin{gathered}
a_{1}=a_{2}=\cdots=a_{k}=1, \\
a_{n+k}=\frac{a_{n+\frac{k}{2}}^{2}+1}{a_{n}}(n \geq 1), k=2, \\
a_{n+k}=\frac{a_{n+\frac{k-1}{2}} a_{n+\frac{k+1}{2}}+1}{a_{n}}(n \geq 1), \quad k=3, \\
a_{n+k}=\frac{a_{n+k-1} a_{n+1}+a_{n+k-2} a_{n+2}+\cdots+a_{n+\frac{k}{2}}^{2}}{a_{n}}(n \geq 1), k \geq 4 \text { is even, } \\
a_{n+k}=\frac{a_{n+k-1} a_{n+1}+a_{n+k-2} a_{n+2}+\cdots+a_{n+\frac{k-1}{2}} a_{n+\frac{k+1}{2}}}{a_{n}}(n \geq 1), k \geq 5 \text { is odd, }
\end{gathered}
$$

is called the Somos sequence of order $k$.

Now we will introduce our own solving process of above-mentioned tasks.

Problem 1. The sequence $\left(a_{n}\right)$ is given recursively:

$$
a_{1}=a_{2}=1, \quad a_{n+2}=\frac{a_{n+1}^{2}+1}{a_{n}} \quad(n \geq 1) .
$$

Prove that each sequence element is an integer number.
Solution. Several first sequence elements are:

$$
a_{3}=2, \quad a_{4}=5, \quad a_{5}=13, \quad a_{6}=34, \quad a_{7}=89, \quad a_{8}=233, \quad a_{9}=610, \quad a_{10}=1597, \ldots
$$

Let us suppose that the sequence elements $a_{1}, a_{2}, \ldots, a_{n}$ are integers for each integer number $n \geq 4$. We prove that the number $a_{n+1}$ is integer as well.

From the equality

$$
a_{n-1} a_{n-3}=a_{n-2}^{2}+1,
$$

it follows

$$
\begin{equation*}
a_{n-2}^{2}+1 \equiv 0 \quad\left(\bmod a_{n-1}\right) \tag{1}
\end{equation*}
$$

From the equality

$$
a_{n} a_{n-2}=a_{n-1}^{2}+1,
$$

it follows

$$
\begin{equation*}
a_{n} a_{n-2} \equiv 1 \quad\left(\bmod a_{n-1}\right) . \tag{2}
\end{equation*}
$$

Using equalities (1)-(2), we recieve

$$
\begin{gathered}
a_{n}^{2} a_{n-2}^{2} \equiv 1 \quad\left(\bmod a_{n-1}\right) \\
a_{n}^{2} \cdot(-1) \equiv 1 \quad\left(\bmod a_{n-1}\right), \\
a_{n}^{2}+1 \equiv 0 \quad\left(\bmod a_{n-1}\right) .
\end{gathered}
$$

It means that the number

$$
a_{n+1}=\frac{a_{n}^{2}+1}{a_{n-1}}
$$

is integer.

Note 1. Similarly, each element of the sequence $\left(a_{n}\right)$ given recursively:

$$
a_{1}=a_{2}=1, \quad a_{n+2}=\frac{a_{n+1}^{2}+m}{a_{n}} \quad(n \geq 1)
$$

is an integer number for all positive integers $m$.
The proof is the same as in the problem 1.
The calculator for finding many first elements such sequences for the first 100 numbers $m$ is on the webpage [7].

Problem 2. The sequence $\left(a_{n}\right)$ is given recursively:

$$
a_{1}=a_{2}=a_{3}=1, \quad a_{n+3}=\frac{a_{n+2} a_{n+1}+1}{a_{n}} \quad(n \geq 1) .
$$

Prove that each sequence element is an integer number.
Solution. Several first sequence elements are:

$$
a_{4}=2, \quad a_{5}=3, \quad a_{6}=7, \quad a_{7}=11, \quad a_{8}=26, \quad a_{9}=41, \quad a_{10}=97, \ldots
$$

First, we prove that each number $\gamma_{n}=\frac{a_{n}+a_{n+2}}{a_{n+1}}$ is integer for each integer number $n$. Since $a_{4}=2$, it holds

$$
\gamma_{1}=\frac{a_{1}+a_{3}}{a_{2}}=2, \quad \gamma_{2}=\frac{a_{2}+a_{4}}{a_{3}}=3 .
$$

For each $n \geq 3$, it holds

$$
\begin{gathered}
\gamma_{n}=\frac{a_{n}+a_{n+2}}{a_{n+1}}=\frac{a_{n}+\frac{a_{n+1} a_{n}+1}{a_{n-1}}}{a_{n+1}}=\frac{a_{n} a_{n-1}+a_{n} a_{n+1}+1}{a_{n-1} a_{n+1}}= \\
=\frac{a_{n+1} a_{n-2}+a_{n} a_{n+1}}{a_{n-1} a_{n+1}}=\frac{a_{n-2}+a_{n}}{a_{n-1}}=\gamma_{n-2} .
\end{gathered}
$$

For each integer number $k$, it holds

$$
\gamma_{2 k}=\gamma_{2}=3, \quad \gamma_{2 k+1}=\gamma_{1}=2
$$

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$$
\begin{gathered}
\frac{a_{2 k+2}+a_{2 k}}{a_{2 k+1}}=3, \quad \frac{a_{2 k+3}+a_{2 k+1}}{a_{2 k+2}}=2, \\
a_{2 k+2}=3 a_{2 k+1}-a_{2 k}, \quad a_{2 k+3}=2 a_{2 k+2}-a_{2 k+1} .
\end{gathered}
$$

It follows that each sequence element $a_{n}$ is an integer number.

Note 2. Similarly, each element of the sequence $\left(a_{n}\right)$ given recursively:

$$
a_{1}=a_{2}=a_{3}=1, \quad a_{n+3}=\frac{a_{n+2} a_{n+1}+m}{a_{n}} \quad(n \geq 1)
$$

is an integer number for all positive integers $m$.
The proof is the same as in the problem 2.

Problem 3. The sequence $\left(a_{n}\right)$ is given recursively:

$$
a_{1}=a_{2}=a_{3}=a_{4}=1, \quad a_{n+4}=\frac{a_{n+3} a_{n+1}+a_{n+2}^{2}}{a_{n}} \quad(n \geq 1)
$$

Prove that each sequence element is an integer number.
Solution. First, we calculate the fifth, sixth, seventh, and eighth sequence element:

$$
a_{5}=2, \quad a_{6}=3, \quad a_{7}=7, \quad a_{8}=23
$$

For the greatest common divisor of the first eight sequence elements, it holds

$$
D\left(a_{m}, a_{m-1}\right)=D\left(a_{m}, a_{m-2}\right)=1, \quad 3 \leq m \leq 8
$$

Let us now suppose that the sequence elements $a_{1}, a_{2}, \ldots, a_{n}$ are integers for each integer number $n \geq 8$, and for each integer number $m, 3 \leq m \leq n$, it holds that $D\left(a_{m}, a_{m-1}\right)=$ $D\left(a_{m}, a_{m-2}\right)=1$. We prove that the number $a_{n+1}$ is integer as well, and satisfies equalities $D\left(a_{n+1}, a_{n}\right)=D\left(a_{n+1}, a_{n-1}\right)=1$.

From the equality

$$
a_{n-3} a_{n-7}=a_{n-4} a_{n-6}+a_{n-5}^{2}
$$

it follows

$$
\begin{equation*}
a_{n-4} a_{n-6}+a_{n-5}^{2} \equiv 0 \quad\left(\bmod a_{n-3}\right) \tag{3}
\end{equation*}
$$

From the equality

$$
a_{n-2} a_{n-6}=a_{n-3} a_{n-5}+a_{n-4}^{2}
$$

it follows

$$
\begin{equation*}
a_{n-2} a_{n-6} \equiv a_{n-4}^{2} \quad\left(\bmod a_{n-3}\right) \tag{4}
\end{equation*}
$$

From the equality

$$
a_{n-1} a_{n-5}=a_{n-2} a_{n-4}+a_{n-3}^{2}
$$

it follows

$$
\begin{equation*}
a_{n-1} a_{n-5} \equiv a_{n-2} a_{n-4} \quad\left(\bmod a_{n-3}\right) \tag{5}
\end{equation*}
$$

From the equality

$$
a_{n} a_{n-4}=a_{n-1} a_{n-3}+a_{n-2}^{2},
$$

it follows

$$
\begin{equation*}
a_{n} a_{n-4} \equiv a_{n-2}^{2} \quad\left(\bmod a_{n-3}\right) . \tag{6}
\end{equation*}
$$

Using equalities (3)-(6), we recieve

$$
\begin{gathered}
a_{n-5}^{2} a_{n-4}\left(a_{n} a_{n-2}+a_{n-1}^{2}\right)= \\
=a_{n-5}^{2} a_{n-2}\left(a_{n} a_{n-4}\right)+\left(a_{n-1} a_{n-5}\right)^{2} a_{n-4} \equiv \\
\equiv a_{n-5}^{2} a_{n-2} a_{n-2}^{2}+\left(a_{n-2} a_{n-4}\right)^{2} a_{n-4}= \\
=a_{n-5}^{2} a_{n-2}^{3}+a_{n-2}^{2} a_{n-4} a_{n-4}^{2} \equiv \\
\equiv a_{n-5}^{2} a_{n-2}^{3}+a_{n-2}^{2} a_{n-4}\left(a_{n-2} a_{n-6}\right)= \\
=a_{n-2}^{3}\left(a_{n-5}^{2}+a_{n-4} a_{n-6}\right) \equiv \\
\equiv a_{n-2}^{3} \cdot 0 \equiv 0 \quad\left(\bmod a_{n-3}\right) .
\end{gathered}
$$

It is seen that the number $a_{n-5}^{2} a_{n-4}\left(a_{n} a_{n-2}+a_{n-1}^{2}\right)$ is divided by the number $a_{n-3}$. Since

$$
D\left(a_{n-3}, a_{n-4}\right)=D\left(a_{n-3}, a_{n-5}\right)=1
$$

the number $a_{n} a_{n-2}+a_{n-1}^{2}$ is divided by the number $a_{n-3}$. It follows from thereof that the number

$$
a_{n+1}=\frac{a_{n} a_{n-2}+a_{n-1}^{2}}{a_{n-3}}
$$

is integer.
It remains to prove that $D\left(a_{n+1}, a_{n}\right)=D\left(a_{n+1}, a_{n-1}\right)=1$. We suppose that $D\left(a_{n+1}, a_{n}\right)>1$ and let $p$ be such a prime number for which $p \mid a_{n+1}$ and $p \mid a_{n}$. From the equality

$$
a_{n+1} a_{n-3}=a_{n} a_{n-2}+a_{n-1}^{2}
$$

it holds $p \mid a_{n-1}$. But it is in contradiction with the condition of $D\left(a_{n}, a_{n-1}\right)=1$.
Let us assume further that $D\left(a_{n+1}, a_{n-1}\right)>1$, and let $p$ be such a prime number for which $p \mid a_{n+1}$ and $p \mid a_{n-1}$. From the equality

$$
a_{n+1} a_{n-3}=a_{n} a_{n-2}+a_{n-1}^{2},
$$

it follows $p \mid a_{n}$ or $p \mid a_{n-2}$. But it is in contradiction with the condition of $D\left(a_{n}, a_{n-1}\right)=$ $D\left(a_{n-1}, a_{n-2}\right)=1$.

Problem 4. The sequence $\left(a_{n}\right)$ is given recursively:

$$
a_{1}=a_{2}=a_{3}=a_{4}=a_{5}=1, \quad a_{n+5}=\frac{a_{n+4} a_{n+1}+a_{n+2} a_{n+3}}{a_{n}} \quad(n \geq 1)
$$

Prove that each sequence element is an integer number.
Solution. First, we calculate the first ten sequence elements:

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$$
a_{6}=2, \quad a_{7}=3, \quad a_{8}=5, \quad a_{9}=11, \quad a_{10}=37 .
$$

For the greatest common divisor of the first ten sequence elements, it holds

$$
D\left(a_{m}, a_{m-1}\right)=D\left(a_{m}, a_{m-2}\right)=D\left(a_{m}, a_{m-3}\right)=1, \quad 4 \leq m \leq 10 .
$$

Let us now suppose that the sequence elements $a_{1}, a_{2}, \ldots, a_{n}$ are integer for each integer number $n \geq 10$, and for each integer number $m, 4 \leq m \leq n$, it holds that $D\left(a_{m}, a_{m-1}\right)=$ $D\left(a_{m}, a_{m-2}\right)=D\left(a_{m}, a_{m-3}\right)=1$. We prove that the number $a_{n+1}$ is integer as well, and satisfies equalities $D\left(a_{n+1}, a_{n}\right)=D\left(a_{n+1}, a_{n-1}\right)=D\left(a_{n+1}, a_{n-2}\right)=1$.

From the equality

$$
a_{n} a_{n-5}=a_{n-1} a_{n-4}+a_{n-2} a_{n-3},
$$

it follows

$$
\begin{equation*}
a_{n} a_{n-5} \equiv a_{n-2} a_{n-3} \quad\left(\bmod a_{n-4}\right) . \tag{7}
\end{equation*}
$$

From the equality

$$
a_{n-1} a_{n-6}=a_{n-2} a_{n-5}+a_{n-3} a_{n-4},
$$

it follows

$$
\begin{equation*}
a_{n-1} a_{n-6} \equiv a_{n-2} a_{n-5} \quad\left(\bmod a_{n-4}\right) . \tag{8}
\end{equation*}
$$

From the equality

$$
a_{n-2} a_{n-7}=a_{n-3} a_{n-6}+a_{n-4} a_{n-5},
$$

it follows

$$
\begin{equation*}
a_{n-2} a_{n-7} \equiv a_{n-3} a_{n-6} \quad\left(\bmod a_{n-4}\right) . \tag{9}
\end{equation*}
$$

From the equality

$$
a_{n-3} a_{n-8}=a_{n-4} a_{n-7}+a_{n-5} a_{n-6},
$$

it follows

$$
\begin{equation*}
a_{n-3} a_{n-8} \equiv a_{n-5} a_{n-6} \quad\left(\bmod a_{n-4}\right) . \tag{10}
\end{equation*}
$$

From the equality

$$
a_{n-4} a_{n-9}=a_{n-5} a_{n-8}+a_{n-6} a_{n-7},
$$

it follows

$$
\begin{equation*}
a_{n-5} a_{n-8}+a_{n-6} a_{n-7} \equiv 0 \quad\left(\bmod a_{n-4}\right) . \tag{11}
\end{equation*}
$$

Using equalities (7)-(11), we recieve

$$
\begin{gathered}
a_{n-5} a_{n-6} a_{n-7}\left(a_{n} a_{n-3}+a_{n-1} a_{n-2}\right)= \\
=\left(a_{n} a_{n-5}\right) a_{n-6} a_{n-7} a_{n-3}+a_{n-5} a_{n-7} a_{n-2}\left(a_{n-1} a_{n-6}\right) \equiv \\
\equiv a_{n-2} a_{n-3} a_{n-6} a_{n-7} a_{n-3}+a_{n-5} a_{n-7} a_{n-2} a_{n-2} a_{n-5}= \\
=a_{n-2}\left(a_{n-6} a_{n-7} a_{n-3}^{2}+a_{n-5}^{2}\left(a_{n-2} a_{n-7}\right)\right) \equiv
\end{gathered}
$$

$$
\begin{gathered}
\equiv a_{n-2}\left(a_{n-6} a_{n-7} a_{n-3}^{2}+a_{n-5}^{2} a_{n-3} a_{n-6}\right)= \\
=a_{n-2}\left(a_{n-6} a_{n-7} a_{n-3}^{2}+\left(a_{n-5} a_{n-6}\right) a_{n-5} a_{n-3}\right) \equiv \\
\equiv a_{n-2}\left(a_{n-6} a_{n-7} a_{n-3}^{2}+a_{n-3} a_{n-8} a_{n-5} a_{n-3}\right)= \\
\quad=a_{n-2} a_{n-3}^{2}\left(a_{n-6} a_{n-7}+\left(a_{n-5} a_{n-8}\right) \equiv\right. \\
\quad \equiv a_{n-2} a_{n-3}^{2} \cdot 0 \equiv 0 \quad\left(\bmod a_{n-4}\right)
\end{gathered}
$$

It is seen that the number $a_{n-5} a_{n-6} a_{n-7}\left(a_{n} a_{n-3}+a_{n-1} a_{n-2}\right)$ is divided by the number $a_{n-4}$. Since

$$
D\left(a_{n-4}, a_{n-5}\right)=D\left(a_{n-4}, a_{n-6}\right)=D\left(a_{n-4}, a_{n-7}\right)=1
$$

the number $a_{n} a_{n-3}+a_{n-1} a_{n-2}$ is divided by the number $a_{n-4}$. It follows from thereof that the number

$$
a_{n+1}=\frac{a_{n} a_{n-3}+a_{n-1} a_{n-2}}{a_{n-4}}
$$

is integer.
It remains to prove that $D\left(a_{n+1}, a_{n}\right)=D\left(a_{n+1}, a_{n-1}\right)=D\left(a_{n+1}, a_{n-2}\right)=1$. It is possible to do so in the same way as in problem 2 using the equality

$$
a_{n+1} a_{n-4}=a_{n} a_{n-3}+a_{n-1} a_{n-2}
$$

Next two problems concern sequences given by the same definition as the first three problems, only the "dimension" of the recursion increases. Solutions to the next problems are analogous to those above.

Problem 5. The sequence $\left(a_{n}\right)$ is given recursively:

$$
a_{1}=a_{2}=a_{3}=a_{4}=a_{5}=a_{6}=1, \quad a_{n+6}=\frac{a_{n+5} a_{n+1}+a_{n+4} a_{n+2}+a_{n+3}^{2}}{a_{n}} \quad(n \geq 1)
$$

Several first sequence elements are

$$
3,5,9,23,75,421,1103,5047,41783,281527,2534423, \ldots
$$

Dean Hickerson proved that each element of this sequence is an integer number.

Problem 6. The sequence $\left(a_{n}\right)$ is given recursively:

$$
\begin{gathered}
a_{1}=a_{2}=a_{3}=a_{4}=a_{5}=a_{6}=a_{7}=1 \\
a_{n+7}=\frac{a_{n+6} a_{n+1}+a_{n+5} a_{n+2}+a_{n+4} a_{n+3}}{a_{n}} \quad(n \geq 1)
\end{gathered}
$$

Several first sequence elements are

$$
3,5,9,17,41,137,769,1925,7203,34081,227321, \ldots
$$

Raphael Robinson proved that each element of this sequence is an integer number.

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We continue further with the "dimension" of the recursion of considered sequences by using two other problems.

Problem 7. The sequence $\left(a_{n}\right)$ is given recursively:

$$
\begin{gathered}
a_{1}=a_{2}=a_{3}=a_{4}=a_{5}=a_{6}=a_{7}=a_{8}=1, \\
a_{n+8}=\frac{a_{n+7} a_{n+1}+a_{n+6} a_{n+2}+a_{n+5} a_{n+3}+a_{n+4}^{2}}{a_{n}} \quad(n \geq 1) .
\end{gathered}
$$

Several first sequence elements are

$$
4,7,13,25,61,187,775,5827,14815, \frac{420514}{7}, \ldots
$$

It can be seen that not all sequence elements are integer numbers.

Problem 8. The sequence $\left(a_{n}\right)$ is given recursively:

$$
\begin{gathered}
a_{1}=a_{2}=a_{3}=a_{4}=a_{5}=a_{6}=a_{7}=a_{8}=a_{9}=1, \\
a_{n+9}=\frac{a_{n+8} a_{n+1}+a_{n+7} a_{n+2}+a_{n+6} a_{n+3}+a_{n+5} a_{n+4}}{a_{n}} \quad(n \geq 1) .
\end{gathered}
$$

Several first sequence elements are

$$
4,7,13,25,49,115,355,1483,11137, \frac{755098}{7}, \ldots
$$

Again, it can be seen that not all sequence elements are integer numbers.

We have not investigated sequences of higher "dimensions", but, on the basis of previous results, it is possible to assume that not all sequence elements are integer numbers. Rather, integer numbers are only on several first positions in these sequences.

More informations can be found, for example, in articles [1], [2] and on webpages [3], [5], [6].

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# INTERVAL SOLUTIONS IN MAX-PLUS ALGEBRA <br> MYŠKOVÁ Helena, (SR) 


#### Abstract

In this paper, we shall deal with the solvability of interval systems of linear equations in the max-plus algebra. The max-plus algebra is the algebraic structures in which classical addition and multiplication are replaced by $\oplus$ and $\otimes$, where $a \oplus b=$ $\max \{a, b\}, a \otimes b=a+b$. Each system of linear equation we can write in the matrix form $A \otimes x=b$, where $A$ and $b$ are matrix and vector of suitable size. If we replace the matrix elements with matrix interval $\boldsymbol{A}=\langle\underline{A}, \bar{A}\rangle$ and vector elements by vector interval $\boldsymbol{b}=\langle\underline{b}, \bar{b}\rangle$, we get an interval system of linear equations. Several types of solutions of interval systems have been defined. If we require a solution to be from a given vector interval $\boldsymbol{x}=\langle\underline{x}, \bar{x}\rangle$, we can define several types of interval solutions. In this paper we give necessary and sufficient conditions for them.


Key words and phrases. Max-plus algebra, max-min algebra, interval system, interval solution.
Mathematics Subject Classification. 15A06; 65G30.

## 1 Preliminaries

Let $(B, \oplus, \otimes)$ be an algebraic structure with two binary operations. $(B, \oplus, \otimes)$ is called the max-plus algebra, if

$$
B=\mathbb{R} \cup\{\varepsilon\}, a \oplus b=\max \{a, b\}, a \otimes b=a+b
$$

where $\varepsilon=-\infty$.
Let $m, n$ be given positive integers. Denote by $M$ and $N$ the sets of indices $\{1,2, \ldots, m\}$, $\{1,2, \ldots, n\}$, respectively. The set of all $m \times n$ matrices over $B$ is denoted by $B(m, n)$ and the set of all column $n$-vectors over $B$ by $B(n)$.

For a given matrix $A \in B(m, n)$ and vector $x \in B(n)$ we have $[A \otimes x]_{i}=\max _{j \in N}\left\{a_{i j}+x_{j}\right\}$. We shall consider the ordering $\leq$ on the sets $B(m, n)$ and $B(n)$ defined as follows:

- for $A, B \in B(m, n): A \leq B$ if $a_{i j} \leq b_{i j}$ for all $i \in M, j \in N$,
- for $x, y \in B(n): x \leq y$ if $x_{j} \leq y_{j}$ for all $j \in N$.

It is easy to see that for each $A, B \in B(m, n)$ and for each $x, y \in B(n)$ holds:

$$
\text { if } A \leq B \text { and } x \leq y \text {, then } A \otimes x \leq B \otimes y \text {. }
$$

We call this property the monotonicity of the operation $\otimes$.
For the given matrix interval $\boldsymbol{A}=\langle\underline{A}, \overline{\bar{A}}\rangle$ with $\underline{A}, \bar{A} \in B(m, n), \underline{A} \leq \bar{A}$ and the given vector interval $\boldsymbol{b}=\langle\underline{b}, \bar{b}\rangle$ with $\underline{b}, \bar{b} \in B(m), \underline{b} \leq \bar{b}$ the notation

$$
\begin{equation*}
\boldsymbol{A} \otimes x=\boldsymbol{b} \tag{1}
\end{equation*}
$$

represents the set of all systems of linear max-plus equations of the form

$$
\begin{equation*}
A \otimes x=b \tag{2}
\end{equation*}
$$

such that $A \in \boldsymbol{A}, b \in \boldsymbol{b}$.
The set $\boldsymbol{A} \otimes x=\boldsymbol{b}$ will be called an interval system of max-plus linear equations. Each system of the form (2) is said to be a subsystem of system (1), if $A \in \boldsymbol{A}, b \in \boldsymbol{b}$.

We shall suppose that $b_{i}>\varepsilon$ for each $i \in M$ and $A$ does not contain a column with all entries equal to $\varepsilon$. An important role for solvability of (2) is played by a principal solution defined as follows:

$$
\begin{equation*}
x_{j}^{*}(A, b)=\min _{i \in M}\left\{b_{i}-a_{i j}\right\} \tag{3}
\end{equation*}
$$

for each $j \in N$.
Theorem 1.1 [3] Let $A \in B(m, n)$ and $b \in B(m)$ be given. Then the system $A \otimes x=b$ is solvable if and only if $x^{*}(A, b)$ is its solution.

Lemma $1.2[6]$ Let $A \in B(m, n)$ and $b \in B(m)$ be given. Then a vector $x \in B(n)$ satisfies the inequality $A \otimes x \leq b$ if and only if $x \leq x^{*}(A, b)$.

## 2 Types of solutions

In [7], there were defined the notions of a possible, tolerance and control solution of interval system of linear equations in a classical algebra. In [1] the existence of a possible and tolerance solution in the max-plus algebra has been studied. The necessary and sufficient condition for a given vector to be a control solution of interval system in the max-plus and max-min algebra was given in [5].

Definition 2.1 $A$ vector $x \in B(n)$ is a possible solution of interval system (1) if there exist $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that $A \otimes x=b$.

Theorem 2.2 [1] A vector $x \in B(n)$ is a possible solution of interval system (1) if and only if

$$
\begin{align*}
& \underline{A} \otimes x \leq \bar{b}  \tag{4}\\
& \bar{A} \otimes x \geq \underline{b} \tag{5}
\end{align*}
$$

Definition 2.3 $A$ vector $x \in B(n)$ is a tolerance solution of an interval system (1) if $A \otimes x \in \boldsymbol{b}$ holds for each $A \in \boldsymbol{A}$.

Theorem 2.4 [1] $A$ vector $x \in B(n)$ is a tolerance solution of an interval system (1) if and only if

$$
\begin{align*}
& \bar{A} \otimes x \leq \bar{b}  \tag{6}\\
& \underline{A} \otimes x \geq \underline{b} \tag{7}
\end{align*}
$$

Definition 2.5 $A$ vector $x \in B(n)$ is a control solution of interval system (1) if for each $b \in \boldsymbol{b}$ there exists $A \in \boldsymbol{A}$ such that $A \otimes x=b$.

Theorem 2.6 [5] $A$ vector $x \in B(n)$ is a control solution of interval system (1) if and only if it satisfies the system of inequalities

$$
\begin{align*}
& \underline{A} \otimes x \leq \underline{b},  \tag{8}\\
& \bar{A} \otimes x \geq \bar{b} . \tag{9}
\end{align*}
$$

Definition 2.7 $A$ vector $x \in B(n)$ is an algebraic solution of interval system(1) if

$$
\{A \otimes x ; A \in \boldsymbol{A}\}=\boldsymbol{b}
$$

Theorem 2.8 [5] $A$ vector $x \in B(n)$ is an algebraic solution of interval system (1) if and only if it satisfies the system of equalities

$$
\begin{align*}
& \underline{A} \otimes x=\underline{b},  \tag{10}\\
& \bar{A} \otimes x=\bar{b} . \tag{11}
\end{align*}
$$

Lemma 2.9 $A$ vector $x \in B(n)$ is an algebraic solution of (1) if and only if it is a tolerance solution of (1) and together it is a control solution of (1).

Proof. A vector $x \in B(n)$ is an algebraic solution of (1) if and only if $\{A \otimes x ; A \in \boldsymbol{A}\} \subset \boldsymbol{b}$ and $\boldsymbol{b} \subset\{A \otimes x ; A \in \boldsymbol{A}\}$, or equivalently, for each $A \in \boldsymbol{A}$ there exists $b \in \boldsymbol{b}$ such that $A \otimes x=b$ and for each $b \in \boldsymbol{b}$ there exists $A \in \boldsymbol{A}$ such that $A \otimes x=b$. By definitions of a tolerance solution and a control solution we get the assertion.

## 3 Interval solutions

Suppose that a solution of (2) can be not arbitrary, but it is required to be from the given interval vector $\boldsymbol{x}=\langle\underline{x}, \bar{x}\rangle, \underline{x}, \bar{x} \in B(n), \underline{x} \leq \bar{x}$.

We shall define several types of interval solutions and give necessary and sufficient conditions for them. At first, we introduce some necessary assertions.

For a given matrix $A$, vector $b$ and interval vector $\boldsymbol{x}$ we denote by $\hat{x}(A, b, \boldsymbol{x})$ the vector with entries

$$
\begin{equation*}
\hat{x}(A, b, \boldsymbol{x})_{j}=\min \left\{x_{j}^{*}(A, b), \bar{x}_{j}\right\} \tag{12}
\end{equation*}
$$

for each $j \in N$. The vector $\hat{x}(A, b, \boldsymbol{x})$ is called an interval-principal solution.
Theorem 3.1 Let $C, D \in B(m, n)$ and $c, d \in B(n)$. The system of inequalities

$$
\begin{align*}
& C \otimes x \leq c,  \tag{13}\\
& D \otimes x \geq d, \tag{14}
\end{align*}
$$

has a solution $x \in \boldsymbol{x}$ if and only if $\hat{x}(C, c, \boldsymbol{x}) \geq \underline{x}$ and

$$
\begin{equation*}
D \otimes \hat{x}(C, c, \boldsymbol{x}) \geq d \tag{15}
\end{equation*}
$$

Proof. The inequality $\hat{x}(C, c, \boldsymbol{x}) \geq \underline{x}$ implies $\hat{x}(C, c, \boldsymbol{x}) \in \boldsymbol{x}$. As $\hat{x}(C, c, \boldsymbol{x}) \leq x^{*}(C, c)$, from Lemma 1.2 it follows that the vector $\hat{x}(C, c, \boldsymbol{x})$ satisfies inequality (13). Inequality (15) means that $\hat{x}(C, c, \boldsymbol{x})$ is a solution of (14), so the system of inequalities (13), (14) has a solution from $\boldsymbol{x}$.

For the converse implication suppose that the system of inequalities (13), (14) has a solution $x \in \boldsymbol{x}$. The inequalities $x \leq x^{*}(C, c)$ and $x \leq \bar{x}$ imply $x \leq \hat{x}(C, c, \boldsymbol{x})$. The inequality $x \geq \underline{x}$ implies $\hat{x}(C, c, \boldsymbol{x}) \geq \underline{x}$. We get

$$
D \otimes \hat{x}(C, c, \boldsymbol{x}) \geq D \otimes x \geq d
$$

so inequality (15) is fulfilled.

Theorem 3.2 The system of inequalities (13), (14) is fulfilled for each $x \in \boldsymbol{x}$ if and only if

$$
\begin{align*}
& C \otimes \bar{x} \leq c,  \tag{16}\\
& D \otimes \underline{x} \geq d . \tag{17}
\end{align*}
$$

Proof.If $C \otimes \bar{x} \leq c$, then for each $x \in \boldsymbol{x}$ we have $C \otimes x \leq C \otimes \bar{x} \leq c$, so inequality (13) is fulfilled for each $x \in \boldsymbol{x}$. Similarly, inequality (17) implies that for each $x \in \boldsymbol{x}$ inequality (14) is held. So the system of inequalities (13), (14) is satisfied for each $x \in \boldsymbol{x}$.

The converse implication is trivial.

Definition 3.3 An interval vector $\boldsymbol{x}$ is
i) $a$ weak possible solution (WPS) of interval system (1) if there exist $x \in \boldsymbol{x}, A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that $A \otimes x=b$,
ii) $a$ strong possible solution (SPS) of interval system (1) if for each $x \in \boldsymbol{x}$ there exist $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that $A \otimes x=b$.

Theorem 3.4 An interval vector $\boldsymbol{x}$ is
i) a weak possible solution of interval system (1) if and only if $\hat{x}(\underline{A}, \bar{b}, \boldsymbol{x}) \geq \underline{x}$ and $\bar{A} \otimes \hat{x}(\underline{A}, \bar{b}, \boldsymbol{x}) \geq \underline{b}$,
ii) a strong possible solution of interval system (1) if and only if $\bar{A} \otimes \underline{x} \geq \underline{b}, \quad \underline{A} \otimes \bar{x} \leq \bar{b}$.

## Proof.

i) An interval vector $\boldsymbol{x}$ is a weak possible solution of (1) if and only if there exists $x \in \boldsymbol{x}$ such that $x$ is a possible solution of (1), i.e., there exists $x \in \boldsymbol{x}$ such that $x$ satisfies the system of inequalities (5), (4). By Theorem 3.1 the assertion follows.
ii) An interval vector $\boldsymbol{x}$ is a strong possible solution of (1) if and only if each vector $x \in \boldsymbol{x}$ is a possible solution of (1), i.e., for each $x \in \boldsymbol{x}$ the system of inequalities (5), (4) is held. By Theorem 3.2 we get $\bar{A} \otimes \underline{x} \geq \underline{b}, \quad \underline{A} \otimes \bar{x} \leq \bar{b}$.

Definition 3.5 An interval vector $\boldsymbol{x}$ is
i) $a$ weak tolerance solution (WTS) of interval system (1) if there exists $x \in \boldsymbol{x}$ such that for each $A \in \boldsymbol{A}$ there exists $b \in \boldsymbol{b}$ such that $A \otimes x=b$,
ii) $a$ strong tolerance solution (STS) of interval system (1) if for each $x \in \boldsymbol{x}$ and $A \in \boldsymbol{A}$ there exists $b \in \boldsymbol{b}$ such that $A \otimes x=b$.

Theorem 3.6 An interval vector $\boldsymbol{x}$ is
i) a weak tolerance solution of interval system (1) if and only if $\hat{x}(\bar{A}, \bar{b}, \boldsymbol{x}) \geq \underline{x}$ and $\underline{A} \otimes \hat{x}(\bar{A}, \bar{b}, \boldsymbol{x}) \geq \underline{b}$,
ii) a strong tolerance solution of interval system (1) if and only if $\bar{A} \otimes \bar{x} \leq \bar{b}, \underline{A} \otimes \underline{x} \geq \underline{b}$.

## Proof.

i) An interval vector $\boldsymbol{x}$ is a weak tolerance solution of (1) if and only if there exists $x \in \boldsymbol{x}$ such that $x$ is tolerance solution of (1), i.e., there exists $x \in \boldsymbol{x}$ such that $x$ satisfies the system of inequalities (7), (6). Using Theorem 3.1 we get $\hat{x}(\bar{A}, \bar{b}, \boldsymbol{x}) \geq \underline{x}$ and $\underline{A} \otimes \hat{x}(\bar{A}, \bar{b}, \boldsymbol{x}) \geq \underline{b}$.
ii) An interval vector $\boldsymbol{x}$ is a strong tolerance solution of (1) if and only if each vector $x \in \boldsymbol{x}$ is tolerance solution of (1), i.e., for each $x \in \boldsymbol{x}$ the system of inequalities (7), (6) is held. Using Theorem 3.2 we get $\bar{A} \otimes \bar{x} \leq \bar{b}, \quad \underline{A} \otimes \underline{x} \geq \underline{b}$.

Definition 3.7 An interval vector $\boldsymbol{x}$ is
i) $a$ weak control solution (WCS) of interval system (1) if there exists $x \in \boldsymbol{x}$ such that for each $b \in \boldsymbol{b}$ there exists $A \in \boldsymbol{A}$ such that $A \otimes x=b$,
ii) a strong control solution (SCS) of interval system (1) if for each $x \in \boldsymbol{x}$ and $b \in \boldsymbol{b}$ there exists $A \in \boldsymbol{A}$ such that $A \otimes x=b$.

Theorem 3.8 An interval vector $\boldsymbol{x}$ is
i) a weak control solution of interval system (1) if and only if $\hat{x}(\underline{A}, \underline{b}, \boldsymbol{x}) \geq \underline{x}$ and $\bar{A} \otimes \hat{x}(\underline{A}, \underline{b}, \boldsymbol{x}) \geq \bar{b}$,
ii) a strong control solution of interval system (1) if and only if $\underline{A} \otimes \bar{x} \leq \underline{b}, \quad \bar{A} \otimes \underline{x} \geq \bar{b}$.

## Proof.

i) An interval vector $\boldsymbol{x}$ is a weak control solution of (1) if and only if there exists $x \in \boldsymbol{x}$ such that $x$ is control solution of (1), i.e., there exists $x \in \boldsymbol{x}$ such that $x$ satisfies the system of inequalities (8), (9). Using Theorem 3.1 we get the necessary and sufficient condition $\hat{x}(\underline{A}, \underline{b}, \boldsymbol{x}) \geq \underline{x}$ and $\bar{A} \otimes \hat{x}(\underline{A}, \underline{b}, \boldsymbol{x}) \geq \bar{b}$.
ii) An interval vector $\boldsymbol{x}$ is a strong control solution of (1) if and only if each vector $x \in \boldsymbol{x}$ is control solution of (1), i.e., for each $x \in \boldsymbol{x}$ the system of inequalities (8), (9) is held. Using Theorem 3.2 we get $\underline{A} \otimes \bar{x} \leq \underline{b}, \quad \bar{A} \otimes \underline{x} \geq \bar{b}$.

Definition 3.9 An interval vector $\boldsymbol{x}$ is an strong algebraic solution (SAS) of interval system (1) if each vector $x \in \boldsymbol{x}$ is an algebraic solution of (1).

Theorem 3.10 An interval vector $\boldsymbol{x}$ is an strong algebraic solution of interval system (1) if and only if

$$
\begin{align*}
& \underline{A} \otimes \underline{x}=\underline{A} \otimes \bar{x}=\underline{b},  \tag{18}\\
& \bar{A} \otimes \underline{x}=\bar{A} \otimes \bar{x}=\bar{b} . \tag{19}
\end{align*}
$$

Proof. Equalities (18), (19) imply that equalities (10), (11) hold for each $x \in \boldsymbol{x}$. So for each $x \in \boldsymbol{x}$ the system of equalities (10), (11) is satisfied, which follows the assertion.

The converse implication is trivial.

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## 4 Relations between various types of interval solutions

Definitions of types of interval solutions imply relations among them.
Theorem 4.1 Let interval system (1) and an interval vector $\boldsymbol{x}$ be given. Then the following implications hold true
i) If $\boldsymbol{x}$ is a strong algebraic solution of (1), then $\boldsymbol{x}$ is a strong control interval solution and a strong tolerance interval solution of (1),
ii) If $\boldsymbol{x}$ is a strong control solution of (1), then $\boldsymbol{x}$ is a strong possible interval solution and weak control interval solution of (1),
iii) If $\boldsymbol{x}$ is a strong tolerance solution of (1), then $\boldsymbol{x}$ is a strong possible interval solution and weak tolerance interval solution of (1),
iv) If $\boldsymbol{x}$ is a strong possible solution of (1), then $\boldsymbol{x}$ is a weak possible interval solution of (1),
v) If $\boldsymbol{x}$ is a weak tolerance solution of (1), then $\boldsymbol{x}$ is a weak possible interval solution of (1),
vi) If $\boldsymbol{x}$ is a weak control solution of (1), then $\boldsymbol{x}$ is a weak possible interval solution of (1).

Proof. The implications in part $i$ ) follow from Lemma 2.9. Others implications follow directly from the definitions of types of interval solutions.

Remark 4.2 It is easy to show that the converse implications to those in Theorem 4.1 do not hold true.

Example 4.3 (SCS $\nRightarrow W T S)$ Let us have

$$
\boldsymbol{A}=\left(\begin{array}{cc}
\langle 1,11\rangle & \langle 3,6\rangle \\
\langle 5,9\rangle & \langle 2,4\rangle
\end{array}\right), \quad \boldsymbol{b}=\binom{\langle 10,13\rangle}{\langle 11,12\rangle}, \quad \boldsymbol{x}=\binom{\langle 3,6\rangle}{\langle 4,7\rangle} .
$$

As $\underline{A} \otimes \bar{x}=(10,11)^{T} \leq \underline{b}$ and $\bar{A} \otimes \underline{x}=(14,12)^{T} \geq \bar{b}$, the given interval vector $\boldsymbol{x}$ is a strong control solution of (1).

We check, if $\boldsymbol{x}$ is a weak tolerance solution. As $\hat{x}(\bar{A}, \bar{b}, \boldsymbol{x})=(2,7) \nsupseteq \underline{x}, \boldsymbol{x}$ is not a weak tolerance solution.

Example 4.4 (STS $\nRightarrow W C S)$ Let us have

$$
\boldsymbol{A}=\left(\begin{array}{cc}
\langle 1,11\rangle & \langle 3,6\rangle \\
\langle 5,9\rangle & \langle 2,4\rangle
\end{array}\right), \quad \boldsymbol{b}=\binom{\langle 7,18\rangle}{\langle 8,20\rangle}, \quad \boldsymbol{x}=\binom{\langle 3,6\rangle}{\langle 4,7\rangle} .
$$

As $\underline{A} \otimes \underline{x}=(7,8)^{T} \geq \underline{b}$ and $\bar{A} \otimes \bar{x}=(17,15)^{T} \leq \underline{b}$, the given interval vector $\boldsymbol{x}$ is a strong tolerance solution of (1).

We check, if $\boldsymbol{x}$ is a weak control solution. As $\hat{x}(\underline{A}, \underline{b}, \boldsymbol{x})=(3,4) \geq \underline{x}, \bar{A} \otimes \hat{x}(\underline{A}, \underline{b}, \boldsymbol{x})=$ $(14,12) \nsucceq \bar{b}, \boldsymbol{x}$ is not a weak control solution.

Example $4.5(W C S \nRightarrow S T S, W T S \nRightarrow S T S)$ Let us have

$$
\boldsymbol{A}=\left(\begin{array}{cc}
\langle 1,11\rangle & \langle 3,6\rangle \\
\langle 5,9\rangle & \langle 2,4\rangle
\end{array}\right), \quad \boldsymbol{b}=\binom{\langle 7,14\rangle}{\langle 8,12\rangle}, \quad \boldsymbol{x}=\binom{\langle 3,14\rangle}{\langle 4,10\rangle} .
$$

We have $\hat{x}(\underline{A}, \underline{b}, \boldsymbol{x})=(3,4)^{T} \geq \underline{x}$ and $\bar{A} \otimes \hat{x}(\underline{A}, \underline{b}, \boldsymbol{x})=(14,12) \geq \bar{b}$, so $\boldsymbol{x}$ is a weak control solution.

As $\hat{x}(\bar{A}, \bar{b}, \boldsymbol{x})=(3,8)^{T} \geq \underline{x}$ and $\underline{A} \otimes \hat{x}(\bar{A}, \bar{b}, \boldsymbol{x})=(11,10) \geq \underline{b}, \boldsymbol{x}$ is a weak tolerance solution.
We check, if $\boldsymbol{x}$ is a strong possible solution. As $A \otimes \bar{x}=(15,19)^{T} \not \leq \bar{b}, \boldsymbol{x}$ is not a strong possible solution.

Remark 4.6 The previous three examples show that implications $S C S \Rightarrow W T S, S T S \Rightarrow$ $W C S, W C S \Rightarrow S P S$ and $W T S \Rightarrow S P S$ are not fulfilled. It can be easily proved that if $S C S \nRightarrow W T S, \quad S T S \nRightarrow W C S, W C S \nRightarrow S P S$ and $W T S \nRightarrow S P S$, then also $S C S \nRightarrow S T S, S T S \nRightarrow S C S, W C S \nRightarrow W T S, W T S \nRightarrow W C S, S P S \nRightarrow W C S$, $S P S \nRightarrow W T S, W T S \nRightarrow S C S, W C S \nRightarrow S T S$.
E.g., if the implication $S C S \Rightarrow S T S$ holds true, then by $S T S \Rightarrow W T S$ (Theorem 4.1) we get $S C S \Rightarrow W T S$, a contradiction. The proofs of remaining non-implications are analogous.

Let us define on the set $\mathcal{T}$ of all above defined types of interval solutions a relation $R$ such that $T_{i} R T_{j}$ if and only if $T_{j}$ implies $T_{i}$ where $T_{i}$ and $T_{j}$ are types of interval solutions. The set $\mathcal{T}$ with the relation $R$ is a partially ordered set, since the relation $R$ is reflexive, antisymetric and transitive. So we can depict the set of all types of interval solutions by Hasse diagram. We use the previous assertions.


Figure 1: Hasse diagram of the relations between types of interval solutions

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# IMPORTANT ELEMENTS OF EL-HYPERSTRUCTURES 

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#### Abstract

The contribution deals with hyperstructure theory. There exists a way of creating semi-hypergroups and hypergroups (or rather transposition hypergroups) from (quasi-) ordered semigroups and groups. Even though it has been widely used by some authors, properties of hyperstructures created in this way have not yet been comprehensively studied. In this contribution their identities and inverses are studied. The last section of the article includes application of the new results on some previous results. It is to be noted that results of this article may be used whenever a hyperstructure is created from a (quasi-) ordered (semi-) groups, which may be the case of differential or transformation operators, preference relations used in economics, matrix theory, etc.


Key words and phrases. hyperstructure, identity, inverse, ordered group, quasi-ordered group
Mathematics Subject Classification. Primary 20N20

## 1 Motivation

A number of articles and contributions in the hyperstructure theory (especially by Czech authors such as Chvalina, Chvalinová, Hošková, Račková, Moučka or Novák) make use of the construction first used in [3] as Theorems 1.3 and 1.4 (chapter IV), pp. 146-147. Using these results known as "Ends lemma" (or "Ending lemma") we can form hyperstructures from ordered structures. Even though the lemma has been widely used, its possibilities and limits have not yet been comprehensively studied. ${ }^{1}$ Findings on some separate issues have been presented especially in [13] and [14].

This article aims at answering the following important questions: Are "Ends lemma"-based hyperstructures canonical hypergroups? What can be said about identities and inverses of "Ends

[^0]lemma"-based hyperstructures? Some results presented here have already been used in some other articles (e.g. [13]), yet they were not proved or explicitely included there. ${ }^{2}$

## 2 Preliminaries

Recall first some basic definitions and ideas from the hyperstructures theory. A hypergrupoid is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet: H \times H \rightarrow \mathcal{P}^{*}(H)$ is a binary hyperoperation on $H$. The symbol $\mathcal{P}^{*}(H)$ denotes the system of all nonempty subsets of $H$. If the associativity axiom $a \bullet(b \bullet c)=(a \bullet b) \bullet c$ holds for all $a, b, c \in H$, then the pair $(H, \bullet)$ is called a semihypergroup. If moreover the reproduction axiom: for any element $a \in H$ equalities $a \bullet H=H=H \bullet a$ hold, is satisfied, then the pair $(H, \bullet)$ is called a hypergroup. A hypergroup $(H, \bullet)$ is called a transposition hypergroup if it satisfies the following transposition axiom: For all $a, b, c, d \in H$ the relation $b \backslash a \approx c / d$ implies $a \bullet d \approx b \bullet c$, where $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$. The sets $b \backslash a=\{x \in H ; a \in b \bullet x\}$ and $c / d=\{x \in H ; c \in x \bullet d\}$ are called left and right extensions, or fractions, respectively. A commutative transposition hypergroup is called a join space.

An element $e \in H$, where $(H, \bullet)$ is a hyperstructure, is called an identity if for $\forall x \in H$ there holds $x \bullet e \ni x \in e \bullet x$. If for $\forall x \in H$ there holds $x \bullet e=\{x\}=e \bullet x$, then $e \in H$ is called a scalar identity. A commutative reversible ${ }^{3}$ hypergroup with a scalar identity such that a every element has a unique inverse ${ }^{4}$ is called a canonical hypergroup.

As far as the theory of ordered structures is concerned, we need to recall that by a quasiordered (semi)group we mean a triple $(G, \cdot, \leq$ ), where $(G, \cdot)$ is a (semi)group and $\leq$ is a reflexive and transitive binary relation on $G$ such that for any triple $x, y, z \in G$ with the property $x \leq y$ also $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ hold. Further, $[a)_{\leq}=\{x \in G ; a \leq x\}$ is a principal end generated by $a \in G$ while $(a]_{\leq}=\{x \in G ; x \leq a\}$ is a principal beginning generated by $a \in G$.

We are going to examine the "Ends lemma", which has the form of the following Theorems:
Theorem 2.1 ([3], Theorem 1.3, p. 146) Let $(S, \cdot, \leq)$ be an ordered semigroup. Binary hyperoperation $*: S \times S \rightarrow \mathcal{P}^{\prime}(S)$ defined by

$$
a * b=[a \cdot b)_{\leq}
$$

is associative. The semi-hypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

Further on the hyperstructure $(S, *)$ constructed in this way will be called the associated hyperstructure to the structure $(S, \cdot)$ or an EL-hyperstructure. ${ }^{5}$ Instead of $S$ the carrier set will be denoted by $H$.

[^1]Theorem 2.2 ([3], Theorem 1.4, p. 147) Let $(S, \cdot, \leq)$ be an ordered semigroup. The following conditions are equivalent:
$1^{0}$ For any pair $a, b \in S$ there exists a pair $c, c^{\prime} \in S$ such that $b \cdot c \leq a$ and $c^{\prime} \cdot b \leq a$
$2^{0}$ The associated semi-hypergroup $(S, *)$ is a hypergroup.
Remark 2.3 If $(S, \cdot, \leq)$ is an ordered group, then if we take $c=b^{-1} \cdot a$ and $c^{\prime}=a \cdot b^{-1}$, then condition $1^{0}$ is valid. Therefore, if $(S, \cdot, \leq)$ is an ordered group, then its associated hyperstructure is a hypergroup.

Remark 2.4 The wording of the above Theorems is the exact translation of theorems from [3]. The respective proofs, however, do not change in any way, if we regard quasi-ordered structures instead of ordered ones as the anti-symmetry of the relation $\leq$ is not needed. The often quoted version of the "Ends lemma" is therefore the version assuming quasi-ordered structures.

The following theorem extending the "Ends lemma" was proved by Račková in her Ph.D. thesis. The proof can be also found in [15].

Theorem 2.5 (Theorem 4, [15]) Let $(H, \cdot, \leq)$ be a quasi-ordered group and $(H, *)$ be the associated hypergroupoid. Then $(H, *)$ is the transposition hypergroup.

Remark 2.6 Naturally, if $(H, \cdot)$ is commutative, then $(H, *)$ is a join space.

## 3 New results

The result obtained by Račková means that the "Ends lemma" can be used in constructions of join spaces. Even though the "Ends lemma" was meant to construct semi-hypergroups and hypergroups from semigroups and groups, given Theorem 2.5 we may ask: Can we go any further? If "Ends lemma"-based join spaces had a scalar identity, they would be canonical hypergroups and could be used in constructions of (Krasner) hyperrings. Unfortunately, the answer is a negative one.

Theorem 3.1 Let $(H, \cdot, \leq)$ be a non-trivial quasi-ordered group, where the relation $\leq$ is not the identity relation, and let $(H, *)$ be its associated transposition hypergroup. Then $(H, *)$ does not have a scalar identity.

Proof. Suppose that in $(H, *)$ there exists a scalar identity $e$. This means that for all $a \in H$ there holds $[a \cdot e)_{\leq}=\{a\}=[e \cdot a)_{\leq}$. Since the relation $\leq$is a reflexive one, there is $x \in[x)_{\leq}$ for an arbitrary $x \in H$. This (among other implications) means that for all $a \in H$ there holds $a \cdot e=a=e \cdot a$. However, this implies that $e$ is the identity of $(H, \cdot)$. As a result $[a)_{\leq}=\{a\}$ for all $a \in H$, which means that $\leq$ is the identity relation, which is a contradiction.

Corollary 3.2 Let $(H, \cdot, \leq)$ be a non-trivial quasi-ordered group, where relation $\leq$ is not the identity relation, and let $(H, *)$ be its associated transposition hypergroup. Then - regardless of commutativity - $(H, *)$ cannot be a canonical hypergoup.

Remark 3.3 Notice that the theorem holds for semigroups as well. However, there is little use in stressing this fact as far as EL-hyperstructures are concerned because it is Corollary 3.2 that is the main practical use of Theorem 3.1.

Since the definition of (Krasner) hyperring is based on a canonical hypergroup, the "Ends lemma" may practically not be used when constructing them. However, thanks to Theorem 2.5 it can be used when constructing weaker structures such as hyperringoids. The respective sufficient and necessary conditions for the use of the "Ends lemma" to create ring-like hyperstructures were discussed in [13]. Let us now see whether $E L$-hyperstructures have at least any identities. If they do, what are they?

Theorem 3.4 Let $(H, *)$ be the associated semi-hypergroup of a quasi-ordered semigroup $(H, \cdot, \leq$ ) with the identity $u$. An element $e \in H$ is an identity of $(H, *)$ if and only if $e \leq u$.

Proof. " $\Rightarrow$ ": If $e \in H$ is an identity of an $E L$-semihypergroup $(H, *)$, then there holds $e \cdot a \leq a$ and $a \cdot e \leq a$ for $\forall a \in H$. Specifically, this holds for $a=u$. In this case we get $e \leq u$.
$" \Leftarrow "$ : Suppose that $e \leq u$. Since $(H, \cdot)$ is a quasi-ordered semigroup, this is equivalent to $e \cdot a \leq a$ for any $a \in H$, which means that for any $a \in H$ we have that $a \in[e \cdot a)_{\leq}=e * a$. In an analogous way we get that $a \in a * e$, i.e. $e$ is an identity of $(H, *)$.

Corollary 3.5 Let $(H, *)$ be the associated semi-hypergroup of a quasi-ordered semigroup $(H, \cdot, \leq$ ). The identity of $(H, \cdot)$ is an identity of $(H, *)$.

Lemma 3.6 Let $(H, *)$ be the associated join space of a quasi-ordered commutative group $(H, \cdot, \leq)$. If an element $e \in H$ is an identity of $(H, *)$, then $e \leq e^{-1}$.

Proof. If $e \in H$ is the identity of $(H, \cdot)$, the implication is obviously true. Therefore study the case that $e$ is not the identity of $(H, \cdot)$. If $e$ is an identity of $(H, *)$, then for $\forall a \in H$ there holds that $a \in a * e=[a \cdot e)_{\leq}$, i.e. $a \cdot e \leq a$. In a similar way there holds $e \cdot a \leq a$, i.e. $a \leq e^{-1} \cdot a$. This implies that for $\forall a \in H$ there holds $a \cdot e \leq a \leq e^{-1} \cdot a$, which in a commutative quasi-ordered group means that $e \leq e^{-1}$.

Description of identities is useful when describing inverse elements of hyperstructures. Let us now therefore focus on the issue of inverse elements of $E L$-hyperstructures.

Definition 3.7 Let $(H, \bullet)$ be a hypergroup endowed with at least one identity. An element $a^{\prime} \in H$ is called an inverse of $a \in H$ if there is an identity $e \in H$ such that $a \bullet a^{\prime} \ni e \in a^{\prime} \bullet a$. A hypergroup $(H, \bullet)$ is called regular if it has at least one identity and each element has at least one inverse. If $H$ is regular, we denote $E$ the set of its identities and $i(a)$ the set of inverses of an arbitrary element $a \in H$.

Theorem 3.8 Let $(H, *)$ be the associated transposition hypergroup of a quasi-ordered group $(H, \cdot, \leq)$. Then $(H, *)$ is regular. For an arbitrary $a \in H$ denote $a^{-1}$ its inverse in $(H, \cdot)$. Then $a^{-1}$ is an inverse of a in $(H, *)$.

Proof. Denote $u$ the identity of $(H, \cdot)$. Since $a$ and $a^{-1}$ are inverse elements in $(H, \cdot)$, there holds $a \cdot a^{-1}=a^{-1} \cdot a=u$, i.e. $a * a^{-1}=a^{-1} * a=[u)_{\leq}$. Since $u \in[u)_{\leq}$, we get that $a^{-1}$ is the inverse of $a$ in $(H, *)$. Since in a group every element has an inverse, there exists an inverse to every element in $(H, *)$, i.e. the hypergroup $(H, *)$ is regular.

Theorem 3.9 Let $(H, *)$ be the associated transposition hypergroup of a quasi-ordered group $(H, \cdot, \leq)$. Then for an arbitrary $a \in H$ there holds

$$
i(a)=\left\{a^{\prime} \in H ; a^{\prime} \leq a^{-1}\right\}=\left(a^{-1}\right]_{\leq},
$$

where $a^{-1}$ is an inverse of $a$ in $(H, \cdot)$.

Proof. Inverse elements to $a \in H$ in $(H, *)$ are defined as such elements $a^{\prime}$ for which there exists an identity $e$ in $(H, *)$ such that $e \in a * a^{\prime}$ and simultaneously $e \in a^{\prime} * a$, i.e. $a \cdot a^{\prime} \leq e$ and $a^{\prime} \cdot a \leq e$ for $E L$-hyperstructures. In order to prove the theorem, we have to prove the following implications:

1. If $a^{\prime} \leq a^{-1}$, then $a^{\prime}$ is the inverse of $a$ in $(H, *)$.

Suppose that $a^{\prime} \leq a^{-1}$. This means that $a^{\prime} \cdot a \leq a^{-1} \cdot a=u$, where $u$ is the identity of $(H, \cdot)$. It does not matter if we multiply from the left or from the right. Therefore if for an element $a^{\prime} \in H$ there holds that $a^{\prime} \leq a^{-1}$, then there exists an identity in $(H, *)$ with the desired property. This identity is the identity $u$ of the group $(H, \cdot)$, which is according to Corollary 3.5 - an identity of $(H, *)$.
2. If $a^{\prime} \in H$ is an inverse of $a$ in $(H, *)$, then $a^{\prime} \leq a^{-1}$.

Since $a, a^{\prime} \in H$ are inverses in $(H, *)$, there exists an identity $e \in H$ such that $e \in$ $a * a^{\prime} \cap a^{\prime} * a$. This means that there simultaneously holds $a \cdot a^{\prime} \leq e$ and $a^{\prime} \cdot a \leq e$. Denote $u$ the identity of $(H, \cdot)$. Since from Theorem 3.4 there holds that $e \leq u$ and due to transitivity of $\leq$, we altogether get that $a \cdot a^{\prime} \leq u$ and $a \cdot a^{\prime} \leq u$, which implies $a^{\prime} \leq a^{-1}$.

Remark 3.10 Theorem 3.4 and Theorem 3.9 suggest that in a general case both $E$ and $i(a)$ are infinite sets such as for e. g. the associated hypergroup of the group $(\mathbb{Q},+)$. This violates another defining axiom of a canonical hypergroup. In this respect cf. [9], Proposition 6, the application of which (in case that card $i(a)>1$ for any $a \in H$, where $H$ is an arbitrary ELtransposition hypergroup) suggests another line of proof of Theorem 3.1. Furthermore, as far as the concept of a fortified join space is concerned, the theorems suggest that EL-join spaces will be fortified join spaces only in some special cases. Notice that the concept of a fortified join space is used in the theory of multiautomata. For its definition cf. [2], p.231.

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## 4 Examples

The above results may be applied in a number of situations. One of them is the study of hyperstructures related to linear ordinary differential operators of the second order as e.g. in the article [4], Proposition 2 by Chvalina, Chvalinová. They can be also applied on e.g. results of contribution [12], which expands the ideas contained in [6], [7] and [8] (not included below), or on the study of symmetric matrices.

In paper [4] there are studied homogeneous second order linear differential equations

$$
\begin{equation*}
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0 \tag{1}
\end{equation*}
$$

such that $p \in C_{+}(I), q \in C(I)$, where $C^{k}(I)$ denotes the commutative ring of all continuous real functions of one variable defined on an open interval $I$ of reals with continuous derivatives up to order $k \geq 0$ (instead of $C^{0}(I)$ we write only $\left.C(I)\right)$, and $C_{+}(I)$ denotes its subsemiring of all positive continuous functions. The set of nonsingular ordinary differential equations (1) is denoted $\mathbb{A}_{2}$. The pair of functions $p, q$ is denoted $[p, q], D=\frac{\partial}{\partial x}$ and $I d$ is the identity operator. The notation $L(p, q)$ is reserved for the differential operator $L(p, q)=D^{2}+p(x) D+q(x) I d$, i.e. the notation $L(p, q)(y)=0$ is the equation (1). The set

$$
\mathbb{L}_{\mathbb{A}_{2}}(I)=\left\{L(p, q): C^{2}(I) \rightarrow C(I) ;[p, q] \in C_{+}(I) \times C(I)\right\}
$$

is the set of all such operators. Finally for an arbitrary $r \in \mathbb{R}$ the notation $\chi_{r}: I \rightarrow \mathbb{R}$ stands for the constant function with value $r$.

Proposition 1 of [4] states that if we define multiplication of operators by

$$
L\left(p_{1}, q_{1}\right) \cdot L\left(p_{2}, q_{2}\right)=L\left(p_{1} p_{2}, p_{1} q_{2}+q_{1}\right)
$$

and if we define that $L\left(p_{1}, q_{1}\right) \leq L\left(p_{2}, q_{2}\right)$ if

$$
p_{1}(x)=p_{2}(x), q_{1}(x) \leq q_{2}(x) \text { for any } x \in I,
$$

then $\left(\mathbb{L}_{\mathbb{A}_{2}}(I), \cdot, \leq\right)$ is a noncommutative ordered group with the unit element (identity) $L\left(\chi_{1}, \chi_{0}\right)$. Using the "Ends lemma" we get that if we put

$$
\begin{gathered}
L\left(p_{1}, q_{1}\right) * L\left(p_{2}, q_{2}\right)=\left\{L(p, q) \in \mathbb{L} \mathbb{A}_{2}(I) ; L\left(p_{1}, q_{1}\right) \cdot L\left(p_{2}, q_{2}\right) \leq L(p, q)\right\}= \\
=\left\{L\left(p_{1} p_{2}, q\right) ; q \in C(I), p_{1} q_{2}+q_{1} \leq q\right\},
\end{gathered}
$$

then $\left(\mathbb{L}_{\mathbb{A}_{2}}(I), *\right)$ is a transposition hypergroup ([4], Theorem 3).
Example 4.1 Using the results of section 3 we may formulate the following results:

1. In the transposition hypergroup $\left(\mathbb{L}_{\mathbb{A}_{2}}(I), *\right)$ there does not exist a scalar identity.
2. An operator $L(p, q)$ is an identity of $\left(\mathbb{L}_{2}(I), *\right)$ if and only if there holds $p(x)=1$, $q(x) \leq 0$ for $\forall x \in I$.
3. For an arbitrary operator $L(p, q) \in \mathbb{L}_{\mathbb{A}_{2}}(I)$ there is

$$
i(L(p, q))=\left\{L(r, s) \in \mathbb{L}_{\mathbb{A}_{2}}(I) ; r(x)=\frac{1}{p(x)}, s(x) \leq-\frac{q(x)}{p(x)}, \forall x \in I\right\} .
$$

Račková in [15] uses the "Ends lemma" to construct hyperstructures of symmetric matrices. She denotes $S$ the set of all $n \times n$ symmetric matrices, where $n \in \mathbb{N}$, and defines relation $\leq$ in the following way: for an arbitrary pair of matrices $A, B \in S$ define that $A \leq B$ if $B-A$ is a positive semidefinite matrix, i.e. if for an arbitrary $x \in \mathbb{R}^{n}, x \neq 0$, there holds $x^{T}(B-A) x \geq 0$, where $A, B$ are $n \times n$ matrices. Further she denotes + the usual addition of matrices and shows that the structure $(S,+, \leq)$ is a commutative ordered group. Thus the hyperstructure $(S, *)$, where $A * B=\{C \in S ; A+B \leq C\}$, is a join space. ${ }^{6}$

Example 4.2 Using the results of section 3 we may formulate the following results:

1. The join space $(S, *)$ is not a canonical hypergroup since there does not exist a scalar identity.
2. Every symmetric matrix $A=\left(a_{i j}\right)$ such that matrix $B=\left(b_{i j}\right)$, where $b_{i j}=-a_{i j}$ for $i \neq j$ and $b_{i j}=1-a_{i j}$ for $i=j$, is a positive semidefinite matrix, is an identity of the join space $(S, *)$.
3. The join space $(S, *)$ is regular hypergroup. For a symmetric matricx $A \in S$ we have

$$
i(A)=\left\{A^{\prime} \in S ; A^{-1}-A^{\prime} \text { is a positive semidefinite matrix }\right\} .
$$

Example 4.3 Suppose the commutative linear ordered group $(\mathbb{R},+)$. According to Theorem 3.4 a real number $r$ is an identity of the associated join space $(\mathbb{R}, *)$, where for an arbitrary $a, b \in \mathbb{R}$ we set that $a * b=\{x \in \mathbb{R}: a+b \leq x\}$, if and only if $r \leq 1$. According to Theorem 3.9 the set $i(r)$ of inverses of $r$ in $(\mathbb{R}, *)$ is $i(r)=\{x \in \mathbb{R}, x \leq-r\}$.

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# ON THE DETERMINANT OF A SPECIAL MATRIX WITH THE GIBONACCI NUMBERS 

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#### Abstract

Many authors investigated various matrices whose entries are the Fibonacci or the Lucas numbers. Some of them used methods leading to obtain certain real or complex factorizations of the mentioned numbers. Civciv (2008) computed the determinant of a five diagonal matrix with the Fibonacci numbers as its entries. Some of his results are put more precisely, completed and generalized in this contribution. Especially the determinant of a five diagonal matrix with the Gibonacci numbers is determined by using the eigenvalues of this matrix.


Key words. Fibonacci numbers, Gibonacci numbers, determinant, five - diagonal matrix, eigenvalues

Mathematics Subject Classification: Primary 11B39, 11C20; Secondary 15A15.

## 1. Introduction

Determinants of matrices with entries given as the Fibonacci or related numbers have a long history. Some application problems are often solved by using graphs or digraphs associated with this type of determinants (see more details in [6]). One of the main purpose of the investigation of matrices with the Fibonacci and the Lucas numbers is to derive of various factorizations of these numbers. The well - known Fibonacci numbers $F_{n}$ and Lucas numbers $L_{n}$ are defined as terms of the sequences given by the same recurrence with the different initial terms. Concretely, $F_{n+2}=F_{n+1}+F_{n}, F_{0}=0, F_{1}=1$ or $L_{n+2}=L_{n+1}+L_{n}, L_{0}=0, L_{1}=1$, respectively.
Cahill et al. [2] studied certain families of tridiagonal matrices and their correspondence to these sequences and derived the following complex factorizations

$$
F_{n}=\prod_{k=1}^{n-1}\left(1-2 i \cos \frac{k \pi}{n}\right), n \geq 2,
$$

and

$$
L_{n}=\prod_{k=1}^{n}\left(1-2 i \cos \frac{(2 k-1) \pi}{2 n}\right), n \geq 1
$$

They proved them by considering in what way these numbers can be connected to Chebyshev polynomials by determinants of sequences of suitable tridiagonal matrices.
Some results on factorization of the Fibonacci - like numbers $U_{n}$ and their squares are given in [7].
These factorizations were found using the circulant matrices, their determinants and eigenvalues. Then
$U_{n}=\frac{1}{\sqrt{p^{2}-4 q}} \prod_{k=1}^{n-1}\left(p-2 \sqrt{q} \cos \frac{k \pi}{n}\right) \quad$ and $\quad U_{n}^{2}=\frac{1}{p^{2}-4 q} \sum_{k=1}^{n}\left(p^{2}-2 q-2 q \cos \frac{2 k \pi}{n}\right)$,
where the numbers $U_{n}$ satisfy the recurrence, $U_{n+2}=p U_{n+1}-q U_{n}, U_{0}=0, U_{1}=1$ for arbitrary integer parameters $p, q$. But any similar factorizations for the Lucas - like numbers $V_{n}$, which are defined by the same recurrence with $V_{0}=2, V_{1}=p$, were not derived by the determinant of circulant matrices.
Civciv [4] investigated the following $k \times k$ five - diagonal matrix

$$
A_{k}=\left(\begin{array}{lllll}
1-F_{n} F_{n-1} & F_{n+1} & F_{n} F_{n-1} & \ldots & \\
-F_{n+1} & 1-2 F_{n} F_{n-1} & F_{n+1} & \ldots & \\
F_{n} F_{n-1} & -F_{n+1} & 1-2 F_{n} F_{n-1} & \cdots & \\
& \ldots \ldots . & & & \\
& \ldots . . & & 1-2 F_{n} F_{n-1} & F_{n+1} \\
& \ldots . . & -F_{n+1} & 1-F_{n} F_{n-1}
\end{array}\right) .
$$

He expressed the determinant of $A_{k}$ in the form

$$
\begin{aligned}
\operatorname{det} A_{k} & =\prod_{j=1}^{k}\left\{1-2 i F_{n+2} \cos \frac{\pi j}{k+1}-4 F_{n} F_{n-1} \cos ^{2} \frac{\pi j}{k+1}\right\}, \text { n odd } \\
& =\prod_{j=1}^{k}\left\{1-2 i F_{n-2} \cos \frac{\pi j}{k+1}+4 F_{n} F_{n-1} \cos ^{2} \frac{\pi j}{k+1}\right\}, n \text { even } .
\end{aligned}
$$

But this result is incorrect because of a small mistake in the end of derivation. The correct relation can be expressed in the form

$$
\begin{equation*}
\operatorname{det} A_{k}=\prod_{j=1}^{k}\left(1-2 i F_{n+1} \cos \frac{\pi j}{k+1}-4 F_{n} F_{n-1} \cos ^{2} \frac{\pi j}{k+1}\right) \tag{1}
\end{equation*}
$$

which we will prove in the next section.

## 2. The main results

There are many connections between determinants of tridiagonal matrices and the Fibonacci numbers and numbers which can be given as their generalization. But also some five-diagonal matrices and their determinants have this property.
First, consider a special case of the above - mentioned matrix $A_{k}$.

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## Theorem 1 ([4], Problem 1)

The relation

$$
\operatorname{det} B_{k}=\left|\begin{array}{ccccc}
0 & 2 & 1 & \ldots & \\
-2 & -1 & 2 & \ldots & \\
1 & -2 & -1 & \ldots & \\
& & \ldots . . & & \\
\ldots . . & & -1 & 2 \\
\ldots . . & & -2 & 0
\end{array}\right|=F_{k+1}^{2}
$$

holds for the $k \times k$ five - diagonal matrix $B_{k}$, where $k \geq 3$.

Proof. It is easy to see that the matrix $B_{k}$ is a special case of $A_{k}$ setting $n=2$. Therefore, we have immediately from (1)

$$
\operatorname{det} B_{k}=\prod_{j=1}^{k}\left(1+4 \cos ^{2} \frac{\pi j}{k+1}\right), \text { as } F_{0}=0, \quad F_{1}=F_{2}=1
$$

But the complex factorization of $F_{k+1}$ has the form

$$
F_{k+1}=\prod_{j=1}^{k}\left(1-2 i \cos \frac{\pi j}{k+1}\right) .
$$

We know that $\cos \alpha=-\cos (\pi-\alpha)$ for any real number $\alpha$, it also means that

$$
\cos \frac{\pi(k+1-j))}{k+1}=\cos \left(\pi-\frac{\pi j}{k+1}\right)=-\cos \frac{\pi j}{k+1} .
$$

Then

$$
\begin{aligned}
& F_{k+1}^{2}=\prod_{j=1}^{k}\left(1-2 i \cos \frac{\pi j}{k+1}\right)^{2}=\prod_{j=1}^{k}\left(1-2 i \cos \frac{\pi j}{k+1}\right) \prod_{j=1}^{k}\left(1+2 i \cos \frac{\pi(k+1-j)}{k+1}\right)= \\
& =\prod_{j=1}^{k}\left(1-2 i \cos \frac{\pi j}{k+1}\right) \prod_{j=1}^{k}\left(1+2 i \cos \frac{\pi j}{k+1}\right)=\prod_{j=1}^{k}\left(1-2 i \cos \frac{\pi j}{k+1}\right)\left(1+2 i \cos \frac{\pi j}{k+1}\right)= \\
& =\prod_{j=1}^{k}\left(1+4 \cos ^{2} \frac{\pi j}{k+1}\right)
\end{aligned}
$$

and the proof is over.
Now, consider so - called the generalized Fibonacci (also shortly Gibonacci) sequence $\left\{G_{n}\right\}$ which satisfy the basic recurrence $G_{n+2}=G_{n+1}+G_{n}$, but its initial terms can be arbitrary integers $G_{0}, G_{1}$. We will prove the following theorem by a similar way, as Civciv calculated the determinant of the matrix $A_{k}$.

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## Theorem 2.

Let $M_{k}$ is a five-diagonal square matrix of order $k \geq 3$ given as
where $G_{n}$ are the Gibonacci numbers, $n \geq 1$. Then
$\operatorname{det} M_{k}=\prod_{j=1}^{k}\left(1-2 i G_{n+1} \cos \frac{\pi j}{k+1}-4 G_{n} G_{n-1} \cos ^{2} \frac{\pi j}{k+1}\right)$.

## Proof.

It is easy to see that $M_{k}=P_{k} \cdot Q_{k}$, where $P_{k}, Q_{k}$ are the following square matrices of order $k$

It means that det $M_{k}=\operatorname{det} P_{k} \operatorname{det} Q_{k}$. As $T_{k}, Q_{k}$ are tridiagional matrices for their determinants the following recurrences hold $(k \geq 3)$

$$
\begin{gathered}
\operatorname{det} P_{k}=\operatorname{det} P_{k-1}+G_{n}^{2} \operatorname{det} P_{n-2}, \quad \operatorname{det} P_{1}=1, \operatorname{det} P_{2}=1+G_{n}^{2}, \\
\operatorname{det} Q_{k}=\operatorname{det} Q_{k-1}+G_{n-1}^{2} \operatorname{det} Q_{k-2}, \\
\operatorname{det} Q_{1}=1, \operatorname{det} Q_{2}=1+G_{n-1}^{2} .
\end{gathered}
$$

It is easy to see that $\operatorname{det} P_{k}$ is also equal to $\operatorname{det} \overline{P_{k}}$, where

$$
\operatorname{det} \overline{P_{k}}=\left|\begin{array}{ccccc}
1 & i G_{n} & 0 & \ldots & \\
i G_{n} & 1 & i G_{n} & \ldots & \\
0 & i G_{n} & 1 & \ldots & \\
& & \ldots \ldots & & \\
& \ldots \ldots & & 1 & i G_{n} \\
& \ldots \ldots & & i G_{n} & 1
\end{array}\right|
$$

and the numbers $G_{n}$ are only changed to $G_{n-1}$ for $\operatorname{det} Q_{k}$.

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We know that the determinant of a square matrix can be expressed as the product of its eigenvalues. Then, we can write

$$
\overline{P_{k}}=I+i G_{n}\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & \\
1 & 0 & 1 & \cdots & \\
0 & 1 & 0 & \ldots & \\
& \cdots & \cdots & & \\
\cdots & & 0 & 1 \\
\cdots & & & 1 & 0
\end{array}\right)=I+i G_{n} E
$$

where $I$ is the identity matrix of order $k$.
As the eigenvalues of the square matrix $E$ are the numbers $-2 \cos \frac{\pi j}{k+1}, j=1,2, \ldots, k$, (more details e.g. in [2]) the eigenvalues $\lambda_{j}$ of $\overline{P_{k}}$ have the form $\lambda_{j}=1-2 i G_{n} \cos \frac{\pi j}{k+1}$.

Thus, $\operatorname{det} \overline{P_{k}}=\operatorname{det} P_{k}=\prod_{j=1}^{k} \lambda_{j}=\prod_{j=1}^{k}\left(1-2 i G_{n} \cos \frac{\pi j}{k+1}\right)$ and similarly
$\operatorname{det} Q_{k}=\prod_{j=1}^{k}\left(1-2 i G_{n-1} \cos \frac{\pi j}{k+1}\right)$.
Then

$$
\begin{aligned}
& \operatorname{det} M_{k}=\operatorname{det} P_{k} \operatorname{det} Q_{k}=\prod_{j=1}^{k}\left(1-2 i G_{n} \cos \frac{\pi j}{k+1}\right)\left(1-2 i G_{n-1} \cos \frac{\pi j}{k+1}\right)= \\
& =\prod_{j=1}^{k}\left(1-2 i\left(G_{n}+G_{n-1}\right) \cos \frac{\pi j}{k+1}-4 G_{n} G_{n-1} \cos ^{2} \frac{\pi j}{k+1}\right)=\prod_{j=1}^{k}\left(1-2 i G_{n+1} \cos \frac{\pi j}{k+1}-4 G_{n} G_{n-1} \cos ^{2} \frac{\pi j}{k+1}\right)
\end{aligned}
$$

which completes the proof.

## 3. Concluding remarks

It is obvious that for $G_{n}=F_{n}$ we obtain the expression for $\operatorname{det} A_{k}$ given in the first section.
Every determinant can be represented by some weighted digraph or undirected graph. When a matrix has Fibonacci entries, combinatorial methods can bring deeper understanding to the calculation of its determinant. Therefore it is possible to find many situations in which are used connections between numbers satisfy the recurrence of the second order, determinants with such numbers and corresponding graphs.
Finally, we can also formulate some open problems. For example, is it possible to find the values $n$ for which $\operatorname{det} M_{k}$ is a real number? How to calculate the same determinant for the Fibonaci - like (or Lucas - like) numbers in place of $G_{n}$ ?

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# SECOND EXPLOSION OF REAL NUMBERS <br> SZALAY István, (HU) 


#### Abstract

Our universe is imaginable as a big open cube $R^{3}=\{(x, y, z): x ; y ; z \in R\}$, with the ordered field of real numbers $(R ;+; \bullet)$. Let us suppose that the beginning is the "big bang". We have no information about "big bang", but if we start from an existent "very small" given open cube, then by a suitable exploder function we have bigger and bigger cubes. Finally, the last one is $R^{3}$. An exciting question whether we are able to explode our universe, the cube $R^{3}$ ? For the answer, we have to solve the explosion problem of the ordered field $(R ;+; \bullet)$. The known solution is the ordered field of exploded real numbers $(\stackrel{\rightharpoonup}{R}, \nrightarrow-\infty$ ). The next question is the explosion of exploded real I


numbers, that is, to construe the ordered field $(\overline{\boxed{R}}) ;-\oplus--\infty-)$.
Key words and phrases. exploded numbers, second explosion, super-super operations.
Mathematics Subject Classification. Primary 60A05, 08A72; Secondary 28E10.

## 1 Introduction

First of all we mention that in the abstract sense the ordered fields $(\stackrel{\leftrightarrow}{R},-\underset{-\infty}{-\infty})$ and ।
$\qquad$ $\square$
$((\stackrel{\rightharpoonup}{R}) ;-\oplus--\varnothing-)$ were introduced in [1] moreover, the complex model of the ordered field $(\stackrel{\rightharpoonup}{R},-\nmid--\not-\infty)$ is given in [2]. In this model the set $\stackrel{\rightharpoonup}{R}$ of exploded real numbers is a proper subset of the set of complex numbers $C$. The most important facts are:

$$
\begin{equation*}
\stackrel{\rightharpoonup}{x}=(\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{|x|\}+i(\operatorname{sgn} x) \cdot[|x|], \quad x \in R, \tag{1}
\end{equation*}
$$

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where
$[x]$ is the greatest integer number which is less than or equal to $x$ and $\{x\}=x-[x]$. So,

$$
\begin{equation*}
\stackrel{\rightharpoonup}{R}=\{u \in C: u=\operatorname{Re} u+i \operatorname{Im} u ; \quad \operatorname{Im} u \in Z,(\operatorname{Re} u) \cdot(\operatorname{Im} u) \geq 0 \tag{2}
\end{equation*}
$$

where $Z$ is the set of integer numbers. We say that $u={ }^{\stackrel{\rightharpoonup}{R}} v, u, v \in \stackrel{\hookrightarrow}{R}$ if $u=v$ in the set $C$, that is $\operatorname{Re} u=\operatorname{Re} v \wedge \operatorname{Im} u=\operatorname{Im} v$. So, instead $"=\stackrel{\rightharpoonup}{R}$ " we use $"=$ ", only, Moreover

$$
\begin{equation*}
u=\operatorname{Im} u+\operatorname{th} \operatorname{Re} u, \quad u \in \stackrel{\rightharpoonup}{R} \tag{3}
\end{equation*}
$$

Definition (1) shows that if $x$ is an element of the open interval $(-1 ; 1)$, then $\stackrel{\rightharpoonup}{x} \in R$.
The inversion - identities are:

$$
\begin{equation*}
\stackrel{\rightharpoonup}{(u)}=u ; \quad u \in \stackrel{\longleftrightarrow}{R} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
(\stackrel{\stackrel{\rightharpoonup}{x})}{ }=x ; \quad x \in R \tag{5}
\end{equation*}
$$

The super-addition and super multiplication are:

$$
\begin{align*}
& u-\nrightarrow-v=\stackrel{\rightharpoonup}{u+v} ; u, v \in \stackrel{\rightharpoonup}{R}  \tag{6}\\
& u-\nsupseteq-v=\stackrel{\rightharpoonup}{u} \cdot \frac{v}{\square} ; \quad u, v \in \stackrel{\rightharpoonup}{R} \tag{7}
\end{align*}
$$

respectively.
The ordering is defined by the way: For any $u, v \in \stackrel{\rightharpoonup}{R}$

$$
u<^{\stackrel{\sqcup}{R}} v
$$

if and only if

$$
\operatorname{Im} u<\operatorname{Im} v
$$

or

$$
\begin{equation*}
\operatorname{Im} u=\operatorname{Im} v \quad \text { and } \quad \operatorname{Re} u<\operatorname{Re} v . \tag{8}
\end{equation*}
$$

Clearly, if $u, v \in R$, then $u<^{\stackrel{4}{R}} v \Longleftrightarrow u<v$. So, instead of " $<{ }^{R}$ " we use " $<$ ". Moreover,

$$
\begin{align*}
& u<v \Rightarrow u-\Varangle-w<v-\nmid-w ; u, v, w \in \stackrel{\rightharpoonup}{R}  \tag{9}\\
& u<v \Rightarrow u-\nless-w<v-\infty<u, v,(0<) w \in \stackrel{\rightharpoonup}{R} \tag{10}
\end{align*}
$$

are valid.
The ordered field $(\stackrel{\rightharpoonup}{R}-\Varangle-$, $-\ngtr-)$ is isomorphic with the ordered field $(R ;+; \bullet)$.
Now we detail the postulates and requirements for the explosion of exploded real numbers:
Postulate of extension:

$$
\begin{equation*}
\stackrel{\rightharpoonup}{R} \subset \overline{(\stackrel{\rightharpoonup}{R})} \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
\text { For any } u \in \stackrel{\rightharpoonup}{R} \text { there exists one and only one } \quad \stackrel{\rightharpoonup}{u}^{\prime} \overline{(\stackrel{\rightharpoonup}{R})} \text {. } \tag{12}
\end{equation*}
$$

Postulate of unambiguity:

$$
\begin{equation*}
\stackrel{\stackrel{\rightharpoonup}{4}}{u}=\stackrel{(R)}{\stackrel{\rightharpoonup}{v}} \Longleftrightarrow u=v ; \quad u, v \in \stackrel{\stackrel{\rightharpoonup}{R}}{R} . \tag{13}
\end{equation*}
$$

Postulate of ordering:

$$
\begin{equation*}
\stackrel{\stackrel{\rightharpoonup}{4}}{u} \stackrel{\stackrel{\rightharpoonup}{4}}{\stackrel{\rightharpoonup}{v}} \stackrel{\stackrel{1}{v}}{v} \Longleftrightarrow u<v ; \quad u, v \in \stackrel{\stackrel{\rightharpoonup}{R}}{ } \tag{14}
\end{equation*}
$$

Postulate of super-super addition:

Postulate of super-super multiplication:

Requirement of equality for explodeds of exploded real numbers:

$$
u={ }^{\stackrel{( }{\square})} v \Longleftrightarrow u=v ; \quad u, v \in \stackrel{\stackrel{\rightharpoonup}{R}}{R}
$$

Requirement of ordering for explodeds of exploded real numbers:

$$
\begin{equation*}
u<^{(R)} v \Longleftrightarrow u<v ; \quad u, v \in \stackrel{\rightharpoonup}{R} \tag{18}
\end{equation*}
$$

Requirement of monotonity of super-super addition:

Requirement of monotonity of super-super multiplication:

## 2 The explosion of exploded real numbers

Let $u$ and arbitrary exploded real number. Using (1) we construe its exploded by definition

$$
\begin{equation*}
\stackrel{\stackrel{\rightharpoonup}{u}}{u}=(\stackrel{\stackrel{\rightharpoonup}{R e u}}{ }, \operatorname{Im} u), \quad u \in \stackrel{\stackrel{\rightharpoonup}{R}}{R} \tag{21}
\end{equation*}
$$

which satisfies (12). We can see, that is this model $\stackrel{\stackrel{4}{u}}{ }$ is an ordered pair, such that $\stackrel{\stackrel{\rightharpoonup}{\operatorname{Re} u}}{ } \in C$;

Expressing that $u$ is an exploded real number, we write $u=\stackrel{\rightharpoonup}{x} ; x \in R$. By (4) we have that $x=u_{1}$. Moreover by (1) we have

$$
\begin{gathered}
\stackrel{\operatorname{Re} u}{ }=\overline{((\operatorname{sgn} x) \cdot \operatorname{areath}\{|x|\})}= \\
=(\operatorname{sgn}((\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{|x|\})) \cdot \operatorname{area} \operatorname{th}\{|(\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{|x|\}|\}+ \\
+i(\operatorname{sgn}((\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{|x|\})) \cdot[|(\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{|x|\}|]= \\
=(\operatorname{sgn} x) \cdot \operatorname{areath}\{\operatorname{areath}\{|x|\}\}+i(\operatorname{sgn} x) \cdot[\operatorname{areath}\{|x|\}]
\end{gathered}
$$

Considering $\operatorname{Re} u$ as an ordered pair of real numbers we can write

$$
\stackrel{\rightharpoonup}{\operatorname{Re} u}=((\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{\operatorname{area} \operatorname{th}\{|x|\}\}, \quad(\operatorname{sgn} x) \cdot[\operatorname{area} \operatorname{th}\{|x|\}]) .
$$

Having that

$$
\operatorname{Im} u=(\operatorname{sgn} x) \cdot[|x|],
$$

we may consider $\stackrel{\stackrel{\rightharpoonup}{u}}{ }$ as pair

$$
\stackrel{\rightharpoonup}{u}=((t, n), m) \quad \text { with } \quad(t, n)=t+i n
$$

or simply as an ordered trio

$$
\begin{equation*}
\stackrel{\rightharpoonup}{u}(t, n, m) \in R^{3} \tag{22}
\end{equation*}
$$

of real numbers, such that

$$
\begin{align*}
t & =(\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{\operatorname{area} \operatorname{th}\{|x|\}\} \in R, \\
n & =(\operatorname{sgn} x) \cdot[\operatorname{area} \operatorname{th}\{|x|\}] \in Z \tag{23}
\end{align*}
$$

and

$$
\begin{equation*}
m=(\operatorname{sgn} x) \cdot[|x|] \in Z \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
m=(\operatorname{sgn} x) \cdot[|x|] \in Z \tag{25}
\end{equation*}
$$

1
$\frac{1}{H}$
Using (22), the equality " $=^{(R)} "$ is understable as the usual equality $"=R^{3} "$.
By (23), (24) and (25) we observe

$$
\begin{equation*}
t(n+m) \geq 0 ; \quad t \in R ; \quad n, m \in Z \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
n \cdot m \geq 0, \quad n, m \in Z \tag{27}
\end{equation*}
$$

Now, we prove the following theorem.
Theorem of completeness. If $(t, n, m) \in R^{3}$ with (26) and (27) then

$$
\begin{equation*}
n+\operatorname{th} t+i m \in \stackrel{\rightharpoonup}{R} \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\stackrel{\square}{n+\operatorname{th} t+i m}=(t, n, m) \tag{29}
\end{equation*}
$$

Proof. First, we show that

$$
\begin{equation*}
t \cdot n \geq 0, \quad t \in R ; n \in Z \tag{30}
\end{equation*}
$$

In the first case we assume that $m=1,2,3$, Hence (27) yields that $n=0,1,2,3, \ldots$ If $n=0$ then (30) is obvious. If $n=1,2,3$, then $0<n+m$ so, (26) yields, that $t \geq 0$ and (30) is fulfilled. In the second case we assume that $m=0$. In this case (26) gives (30), immediately. In the third case we assume that $m=-1,-2,-3$. Hence (27) yields that $n=0,-1,-2,-3, \ldots$. If $n=0$ then (30) is obvious. If $n=-1,-2,-3$, then $n+m<0$ so, (26) yields, that $t \leq 0$ and (30) is fulfilled. Collecting the three cases, (30) is proved. By (30) we have

$$
\operatorname{sgn}(n+\operatorname{th} t)=\left\{\begin{array}{cc}
\operatorname{sgn} n & \text { if }  \tag{31}\\
\operatorname{sgn} t & \text { if } \quad n= \pm 1, \pm 2, \pm 3, \ldots,
\end{array}\right.
$$

As $\operatorname{Im}(n+\operatorname{th} t+i m)=m \in N$ and by (31) and (27) we have that if $n \neq 0$ then $(\operatorname{Re}(n+\operatorname{th} t+$ $i m)) \cdot(\operatorname{Im}(n+\operatorname{th} t+i m))=(n+\operatorname{th} t) \cdot m \geq 0$. If $n=0$, then (26) gives that $t \cdot m \geq 0$ which with (31) yields $(\operatorname{Re}(n+\operatorname{th} t+i m)) \cdot(\operatorname{Im}(n+\operatorname{th} t+i m))=(\operatorname{th} t) \cdot m \geq 0$. Hence, by (2) we have (28).
Now we turn toward (29). Using (21) we obtain

$$
\begin{equation*}
\stackrel{\square}{n+\operatorname{th} t+i m}=(\stackrel{\sqrt{\operatorname{Re}(n+\operatorname{th} t+i m)}}{ }, \operatorname{Im}(n+\operatorname{th} t+i m))=(\sqrt{n+\operatorname{th} t} ; m) \tag{32}
\end{equation*}
$$

For our aim it is sufficient to prove

$$
\begin{equation*}
\overline{n+\operatorname{th} t}=t+i n \tag{33}
\end{equation*}
$$

As $n+\operatorname{th} t \in R$, we may apply definition (1). Using (31) and (30) we have

$$
|n+\operatorname{th} t|=\left\{\begin{array}{cc}
n+\text { th } t & \text { if } n=1,2,3, \ldots \\
|\operatorname{th} t| & \text { if } n=0, \\
-n-t h t & \text { if } n=-1,-2,-3, \ldots
\end{array}\right.
$$

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$$
[|n+\operatorname{th} t|]=\left\{\begin{array}{cc}
n & \text { if } n=1,2,3, \ldots \\
0 & \text { if } n=0, \\
-n & \text { if } n=-1,-2,-3, \ldots
\end{array}\right.
$$

and

$$
\{|n+\operatorname{th} t|\}=\left\{\begin{array}{cc}
\text { th } t \quad \text { if } n=1,2,3, \ldots \\
|\operatorname{th} t| & \text { if } n=0, \\
-\operatorname{th} t & \text { if } n=-1,-2,-3, \ldots
\end{array}\right.
$$

Finally, we consider three cases:
First case: $n=1,2,3, \ldots$ Using (31) again, we can write that

$$
\begin{gathered}
\overline{n+\operatorname{th} t}=(\operatorname{sgn}(n+\operatorname{th} t)) \cdot \operatorname{area} \operatorname{th}\{|n+\operatorname{th} t|\}+i(\operatorname{sgn}(n+\operatorname{th} t)) \cdot[|n+\operatorname{th} t|]= \\
=\operatorname{area} \operatorname{th}(\operatorname{th} t)+i n=t+i n
\end{gathered}
$$

Second case: $n=0$. Using (31) again, we can write that

$$
\begin{gathered}
\overline{n+\operatorname{th} t}=(\operatorname{sgn}(n+\operatorname{th} t) \cdot \operatorname{area} \operatorname{th}\{|n+\operatorname{th} t|\}+i(\operatorname{sgn}(n+\operatorname{th} t)) \cdot[|n+\operatorname{th} t|]= \\
=(\operatorname{sgn} t) \cdot \text { area } \operatorname{th}|\operatorname{th} t|=t+i n
\end{gathered}
$$

Third case: $n=-1,-2,-3, \ldots$
Using (31) again, we can write that

$$
\begin{gathered}
\overline{n+\operatorname{th} t}=(\operatorname{sgn}(n+\operatorname{th} t)) \cdot \operatorname{area} \operatorname{th}\{|n+\operatorname{th} t|\}+i(\operatorname{sgn}(n+\operatorname{th} t)) \cdot[|n+\operatorname{th} t|]= \\
=-\operatorname{area} \operatorname{th}(-t h t)-i(-n)=t+i n
\end{gathered}
$$

Having (33) with (32) the statement (29) is proved. The proof of Theorem of completeness is finished.
By the facts mentioned above we give the following model of $(\stackrel{\boxed{4}}{R})$

$$
\begin{equation*}
\overrightarrow{(\stackrel{\rightharpoonup}{R})}=\left\{\stackrel{\rightharpoonup}{u}=(t, n, m) \in R^{3}: t \cdot(n+m) \geq 0 ; t \in R ; n, m \in Z\right\} \tag{34}
\end{equation*}
$$

Considering an $x \in R$, (1) gives that $\stackrel{\stackrel{\rightharpoonup}{x}}{x}=\operatorname{area} \operatorname{th} x \in R(\subset \stackrel{\rightharpoonup}{R} \subset C)$. Moreover (23), (24) and
(25) gives that $\overline{(\vec{x})}=((\operatorname{sgn} x) \cdot \operatorname{areath}\{\operatorname{areath}|x|\},(\operatorname{sgn} x) \cdot[\operatorname{area} \operatorname{th}|x|], 0) \in \stackrel{\leftrightarrow}{R}$. If in $((34))$ we fix $m=0$ then (2) shows that the set $\left\{(t, n, 0) \in R^{3}: t \cdot n \geq 0 ; t \in R ; n \in Z\right\}$ with $u=t+i n$ is just the set $\stackrel{\rightharpoonup}{R}$. So, we have (11), that is the Postulate of extension is satisfied.

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Remark. The real number $x \in R$ is considered $x \equiv(1,0,0) \in(\stackrel{\rightharpoonup}{R})$, the complex number $u \in \stackrel{\rightharpoonup}{R}$ । $\xrightarrow[\square]{\square}$
$\qquad$ is considered $u \equiv(\operatorname{Re} u, \operatorname{Im} u, 0) \in(\stackrel{\rightharpoonup}{R})$. For example $0 \equiv(0,0,0) ; \stackrel{\rightharpoonup}{1} \equiv(0,1,0)$.
Theorem of unambiguity. If $u, v \in \stackrel{\rightharpoonup}{R}$ and $\stackrel{\stackrel{\rightharpoonup}{u}}{u} \stackrel{\stackrel{+}{4}}{(R)} \stackrel{\rightharpoonup}{v}$, then $u=v$.
Proof. By definition (21) we have that $(\overrightarrow{\operatorname{Re} u}, \operatorname{Im} u)=(\overrightarrow{\operatorname{Rev} v}, \operatorname{Im} v)$. By the definition of
$\qquad$
$"={ }^{(R)} "$

$$
\stackrel{\rightharpoonup}{\operatorname{Re} u}=\stackrel{\rightharpoonup}{\operatorname{Re} v} \quad \text { and } \quad \operatorname{Im} u=\operatorname{Im} v
$$

are obtained. Applying Th2 of [2] which says that if $x, y \in R$ and $\stackrel{\stackrel{\rightharpoonup}{x}}{x}=\stackrel{\rightharpoonup}{y}$ then $x=y$, we have that $\operatorname{Re} u=\operatorname{Re} v$. Hence, $u=\operatorname{Re} u+i \operatorname{Im} u=\operatorname{Re} v+i \operatorname{Im} v=v$, that is Theorem of unambiguity is proved.

Clearly, if $u, v \in \stackrel{\rightharpoonup}{R}$ and $u=v$ then $\stackrel{\stackrel{\rightharpoonup}{u}}{u} \begin{aligned} & \stackrel{( }{4} \\ &(R) \stackrel{\rightharpoonup}{v}\end{aligned}$. Using this fact together Theorem of unambiguity we have (13), that is Postulate of unambiguity is satisfied.

As we already have to relation (11) we investigate the meaning of the equality $u={ }^{(R)} v$ where $u, v \in \stackrel{\rightharpoonup}{R}$. Now, $u=\stackrel{\rightharpoonup}{x} ; x \in R$ and $v=\stackrel{\rightharpoonup}{y} ; y \in R$. By (21) $u=(\stackrel{\rightharpoonup}{\operatorname{Re} x}, \operatorname{Im} x)$ and


$$
\begin{aligned}
& \square \\
& "=(R), \\
& u=\stackrel{\rightharpoonup}{x}=\stackrel{\rightharpoonup}{\operatorname{Re} x}=\stackrel{\rightharpoonup}{\operatorname{Re} y}=\stackrel{\rightharpoonup}{y}=v
\end{aligned}
$$

is obtained. So, (17) shows that the Requirement of equality for explodeds of exploded real
number is fulfilled and we may use " $=$ " instead of " $=(R)$ ".
By Theorems completeness and unambiguity, (29) yields the definition of the compressed of the exploded of exploded real number

$$
\begin{equation*}
(t, n, m)=n+\operatorname{th} t+i m, \quad(t, n, m) \in(\overline{\stackrel{\rightharpoonup}{R}}) . \tag{35}
\end{equation*}
$$

Hence, we get the inversion identity

$$
\begin{equation*}
\stackrel{\stackrel{\text { ' }}{((t, n, m))}}{ }=(t, n, m), \quad(t, n, m) \in(\stackrel{\rightharpoonup}{R}) . \tag{36}
\end{equation*}
$$

We remark, that in the case $m=0,(t, n, 0)=t+i n(\in \stackrel{\rightharpoonup}{R})$ and $(t, n, 0)_{d}=n+\operatorname{th} t$. So, by (3), the inversion identity is reduced (4). Writing $u=n+\operatorname{th} t+i m$ with (26) and (27). Now (28) shows that $u \in \overleftrightarrow{R}$ and by (29) and (35) is given the the second inversion identity

$$
\begin{equation*}
(\stackrel{\rightharpoonup}{u})_{\Delta}=u, \quad u \in \stackrel{\rightharpoonup}{R} \tag{37}
\end{equation*}
$$

which is an extension of (5).
Now we turn toward the ordering of the set $(\stackrel{\boxed{R}}{ })$. Our definition is:

$$
\stackrel{\stackrel{\rightharpoonup}{u}}{u}<\stackrel{(R)}{\stackrel{\rightharpoonup}{v}} \Longleftrightarrow\left\{\begin{array}{l}
\operatorname{Im} u<\operatorname{Im} v  \tag{38}\\
\text { or } \\
\operatorname{Im} u=\operatorname{Im} \text { vand } \quad \stackrel{\stackrel{\rightharpoonup}{\operatorname{Re} u}<\stackrel{\rightharpoonup}{\operatorname{Re} v}}{ } \quad ; \quad u, v \in \stackrel{\leftrightarrow}{R}
\end{array}\right.
$$

Considering (1), (2), (8) we already have

$$
\stackrel{\stackrel{\rightharpoonup}{x}}{x}<\stackrel{\stackrel{\rightharpoonup}{y}}{ } \Longleftrightarrow\left\{\begin{array}{l}
\operatorname{Im} \stackrel{\stackrel{\rightharpoonup}{x}}{2}<\operatorname{Im} \stackrel{\stackrel{\rightharpoonup}{y}}{\text { or }}  \tag{39}\\
\operatorname{Im} \stackrel{\rightharpoonup}{x}=\operatorname{Im} \stackrel{\stackrel{\rightharpoonup}{y}}{ } \text { and } \operatorname{Re} \stackrel{\stackrel{\rightharpoonup}{x}}{x} \operatorname{Re} \stackrel{\stackrel{\rightharpoonup}{y}}{ } \quad ; \quad x, y \in R .
\end{array}\right.
$$

Applying (39) for $u=\stackrel{\rightharpoonup}{x} ; v=\stackrel{\rightharpoonup}{y} ; x, y \in R$ with

$$
\stackrel{\rightharpoonup}{\operatorname{Re} u}=((\operatorname{sgn} x) \cdot \operatorname{area} \operatorname{th}\{\operatorname{area} \operatorname{th}\{|x|\}\}+i(\operatorname{sgn} x) \cdot[\operatorname{area} \operatorname{th}\{|x|\}]
$$

and

$$
\stackrel{\rightharpoonup}{\operatorname{Re} v}=((\operatorname{sgn} y) \cdot \operatorname{area} \operatorname{th}\{\operatorname{area} \operatorname{th}\{|y|\}\}+i(\operatorname{sgn} y) \cdot[\operatorname{area} \operatorname{th}\{|y|\}],
$$

we obtain

$$
\begin{equation*}
\stackrel{\stackrel{\operatorname{Re} u}{\operatorname{Re} v}}{ } \quad \text { if and only if : } \tag{40}
\end{equation*}
$$

$$
(\operatorname{sgn} x) \cdot[\operatorname{area} \operatorname{th}\{|x|\}]<(\operatorname{sgn} y) \cdot[\operatorname{area} \operatorname{th}\{|y|\}]
$$

or

$$
\begin{gathered}
(\operatorname{sgn} x) \cdot[\operatorname{area} \operatorname{th}\{|x|\}]=(\operatorname{sgn} y) \cdot[\operatorname{area} \operatorname{th}\{|y|\}] \\
\text { and }
\end{gathered}
$$

$(\operatorname{sgn} x) \cdot \operatorname{areath}\{\operatorname{area} \operatorname{th}\{|x|\}\}<(\operatorname{sgn} y) \cdot \operatorname{areath}\{\operatorname{areath}\{|y|\}\}$.
Let be $\stackrel{\stackrel{\rightharpoonup}{u}}{u}=(t, n, m) \in(\stackrel{\rightharpoonup}{\square})$ with (23), (24), (25) and $\stackrel{\stackrel{\rightharpoonup}{v}}{ }=(\tau, \nu, \mu) \in(\stackrel{\stackrel{\rightharpoonup}{R})}{ }$ with

$$
\begin{align*}
\tau & =(\operatorname{sgn} y) \cdot \operatorname{areath}\{\operatorname{area} \operatorname{th}\{|y|\}\} \in R  \tag{41}\\
\nu & =(\operatorname{sgn} y) \cdot[\operatorname{area} \operatorname{th}\{|y|\}] \in Z  \tag{42}\\
\mu & =(\operatorname{sgn} y) \cdot[|y|] \in Z \tag{43}
\end{align*}
$$

Now, (35) gives the exploded numbers

$$
\begin{equation*}
u=n+\operatorname{th} t+i m \quad \text { and } \quad v=\nu+\operatorname{th} \tau+i \mu \tag{44}
\end{equation*}
$$

Considering (23), (24), (25), (41), (42), (43) and (44) definition (38) and

$$
(t, n, m)<\stackrel{(\stackrel{\rightharpoonup}{\square})}{ }(\tau, \nu, \mu) \Longleftrightarrow\left\{\begin{array}{l}
m<\mu  \tag{45}\\
\text { or } \\
m=\mu \text { and } n<\nu \\
\text { or } \\
m=\mu ; n=\nu \text { and } t<\tau
\end{array}\right.
$$

are equivalent definitions. We can see that the keywords of this ordering are: "under-above", "before-behind" and "left-right". Moreover, the set $\stackrel{\rightharpoonup}{R}$ is "very big" open interval in $\stackrel{\rightharpoonup}{(\stackrel{\rightharpoonup}{R})}$. Namely

Theorem of ordering. For any $u, v \in \stackrel{\rightharpoonup}{R}, \stackrel{\stackrel{\rightharpoonup}{u}}{<^{(R)}} \stackrel{\stackrel{\rightharpoonup}{v}}{v}$ if and only if $u<v$.
Proof. Let $u=\operatorname{Re} u+i \operatorname{Im} u ; v=\operatorname{Re} v+i \operatorname{Im} v \in \stackrel{\stackrel{\rightharpoonup}{R}}{ }$, that is, by (2)

$$
(\operatorname{Re} u) \cdot(\operatorname{Im} u) \geq 0 ; \quad \operatorname{Im} u \in Z
$$

and

$$
(\operatorname{Re} v) \cdot(\operatorname{Im} v) \geq 0 ; \quad \operatorname{Im} v \in Z
$$

are fulfilled.
Necessity. We prove that $\left(\stackrel{\square}{u} \stackrel{\rightharpoonup}{4}_{(R)}^{\stackrel{\rightharpoonup}{v}}\right) \Rightarrow(u<v)$. By definition (38) we distinguish two cases: Case I.: $\operatorname{Im} u<\operatorname{Im} \nu$. Now, by (8) $u<v$ is obtained, immediately.
Case II.: $\operatorname{Im} u=\operatorname{Im} v$ and $\overline{\operatorname{Re} u}<\overrightarrow{\operatorname{Rev} v}$. Applying Th13 of [2] which says that if $x, y \in R$ and $\stackrel{\rightharpoonup}{x}<\stackrel{\rightharpoonup}{y}$ then $x<y$, we have that $\operatorname{Re} u<\operatorname{Re} v$. Now, by (8) we have, that $u<v$.

Sufficiency. We prove that $(u<v) \Rightarrow\left(\underset{\sim}{\stackrel{\rightharpoonup}{u}} \begin{array}{|l}\overline{4} \\ \hline(R) \\ v\end{array}\right)$. Two cases are distinquished:

$$
\square
$$

Case I.: $\operatorname{Im} u<\operatorname{Im} v$. Now, by (38) $u<^{(R)} v$ is obtained, immediately.
Case II.: $\operatorname{Im} u=\operatorname{Im} v$ and $\operatorname{Re} u<\operatorname{Re} v$. Applying Th13 of [2] which says that if $x, y \in R$ and $x<y$ then $\stackrel{\rightharpoonup}{x}<\stackrel{\rightharpoonup}{y}$, we have that $\stackrel{\rightharpoonup}{\operatorname{Re} u}<\stackrel{\rightharpoonup}{\operatorname{Rev}}$. Now, by (38) we have, that $u<\stackrel{\stackrel{\rightharpoonup}{R})}{ } v$.

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Theorem of ordering says that (14) is valid, so the Postulate of ordering is satisfied. Having the relation (11) we investigate the meaning of the inequality $u<^{\stackrel{\rightharpoonup}{4}} v$ where $u, v \in \stackrel{\rightharpoonup}{R}$. Now,
 $\operatorname{Im} x=\operatorname{Im} y=0, x=\operatorname{Re} x$ and $y=\operatorname{Re} y$, the definition of (38) says that $\overrightarrow{\operatorname{Re} x}<\overrightarrow{\operatorname{Re} y}$. Hence

So, (18) shows that the Requirement of ordering for explodes of exploded real number is fulfilled and we may use " $<$ " instead of " $<\stackrel{\stackrel{\square}{\square}}{(R)}$ ".
In the set $(\stackrel{\boxed{4}}{R})$ we define the super-super addition and super-super multiplication

$$
\begin{equation*}
(t, n, m)-(\tau, \nu, \mu)=(\lambda, \rho, \kappa) ; \quad(t, n, m),(\tau, \nu, \mu) \in(\stackrel{\stackrel{\rightharpoonup}{R}}{\square} \tag{47}
\end{equation*}
$$

with

$$
\begin{align*}
\lambda= & (\operatorname{sgn}(m+\operatorname{th}(n+\operatorname{th} t)+\mu+\operatorname{th}(\nu+\operatorname{th} \tau))) \cdot \\
& \operatorname{areath}\{\operatorname{areath}\{|m+\operatorname{th}(n+\operatorname{th} t)+\mu+\operatorname{th}(\nu+\operatorname{th} \tau)|\}\}  \tag{48}\\
\rho= & (\operatorname{sgn}(m+\operatorname{th}(n+\operatorname{th} t)+\mu+\operatorname{th}(\nu+\operatorname{th} \tau))) \\
& {[\operatorname{areath}\{|m+\operatorname{th}(n+\operatorname{th} t)+\mu+\operatorname{th}(n+\operatorname{th} \tau)|\}] }  \tag{49}\\
\kappa= & (\operatorname{sgn}(m+\operatorname{th}(n+\operatorname{th} t)+\mu+\operatorname{th}(\nu+\operatorname{th} \tau))) . \\
& {[|m+\operatorname{th}(n+\operatorname{th} t)+\mu+\operatorname{th}(v+\operatorname{th} \tau)|] } \tag{50}
\end{align*}
$$

and

$$
\begin{equation*}
(t, n, m)-\varnothing-(\tau, \nu, \mu)=(\lambda, \rho, \kappa) ; \quad(t, n, m),(\tau, \nu, \mu) \in\left(\frac{\stackrel{\rightharpoonup}{R}}{\prime}\right) . \tag{51}
\end{equation*}
$$

with

$$
\begin{align*}
\lambda= & (\operatorname{sgn}((m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau)))) \cdot \\
& \operatorname{area} \operatorname{th}\{\operatorname{areath}\{\mid m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau)) \mid\}\}  \tag{52}\\
\rho= & (\operatorname{sgn}((m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau)))) \cdot \\
& {[\operatorname{areath}\{|(m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau))|\}] }  \tag{53}\\
\kappa= & (\operatorname{sgn}((m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau)))) . \\
& {[|(m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau))|], } \tag{54}
\end{align*}
$$

respectively.

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Theorem of super-super addition. If $u, v \in \stackrel{\rightharpoonup}{R}$ then $\stackrel{\rightharpoonup}{u}-\oplus-\stackrel{\rightharpoonup}{v}=\stackrel{\stackrel{\rightharpoonup}{~-~}-v}{ }$.
Proof. Denoting $\stackrel{\stackrel{\rightharpoonup}{u}}{u}=(t, n, m) \in(\stackrel{\rightharpoonup}{R})$ with (23), (24), (25) and $\stackrel{\stackrel{\rightharpoonup}{v}}{v}=(\tau, \nu, \mu) \in(\stackrel{\text { ' }}{\stackrel{\rightharpoonup}{R}})$ with (41), (42), (43) and using (44) $u=n+\operatorname{th} t+i m$ and $v=\nu+\operatorname{th} \tau+i \mu$ are obtained. Moreover, (3) shows, that $u=m+\operatorname{th}(n+\operatorname{th} t)$ and $v=\mu+\operatorname{th}(\nu+\operatorname{th} \tau)$. Denoting

$$
\sigma=(m+\operatorname{th}(n+\operatorname{th} t))+(\mu+\operatorname{th}(\nu+\operatorname{th} \tau))
$$

by (6), (1), (21), (48), (49) and (50) we can write

$$
\begin{aligned}
& =\overline{(\operatorname{sign} \sigma) \cdot \operatorname{areath}\{|\sigma|\}+i(\operatorname{sign} \sigma) \cdot[|\sigma|]}=(\overline{(\operatorname{sign} \sigma) \cdot \operatorname{area} \operatorname{th}\{|\sigma|\}},(\operatorname{sgn}[|\sigma|])= \\
& =((\operatorname{sgn} \sigma) \cdot \operatorname{areath}\{\operatorname{areath}\{|\sigma|\}\}+i(\operatorname{sgn} \sigma) \cdot[\operatorname{area} \operatorname{th}\{|\sigma|\}], \kappa)=(\lambda+i \rho, \kappa)=(\lambda, \rho, \kappa) .
\end{aligned}
$$

Casting a glance at (47), Theorem of super-super addition is proved. By (15) the Postulate of super-super addition is satisfied.
Theorem of super-super multiplication. If $u, v \in \stackrel{\rightharpoonup}{R}$ then $\stackrel{\rightharpoonup}{u}-\underset{\varnothing-\infty}{v}=\stackrel{\rightharpoonup}{u-\varnothing-v}$.
Proof. Denoting $\stackrel{\stackrel{\rightharpoonup}{u}}{u}=(t, n, m) \in(\stackrel{\rightharpoonup}{\stackrel{\rightharpoonup}{R}})$ with (23), (24), (25) and $\stackrel{\stackrel{\rightharpoonup}{v}}{ }=(\tau, \nu, \mu) \in(\stackrel{\rightharpoonup}{\stackrel{\rightharpoonup}{R}})$ with (41), (42), (43) and using (44) $u=n+\operatorname{th} t+i m$ and $v=\nu+\operatorname{th} \tau+i \mu$ are obtained. Moreover, (3) shows, that $u=m+\operatorname{th}(n+\operatorname{th} t)$ and $v=\mu+\operatorname{th}(\nu+\operatorname{th} \tau)$. Denoting

$$
\sigma=(m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau))
$$

by (7), (1), (21), (52), (53) and (54) we can write

$$
\begin{aligned}
& =\overline{(\operatorname{sign} \sigma) \cdot \operatorname{areath}\{|\sigma|\}+i(\operatorname{sign} \sigma) \cdot[|\sigma|]}=(\overline{(\operatorname{sign} \sigma) \cdot \operatorname{areath}\{|\sigma|\}}, \quad(\operatorname{sgn} \sigma) \cdot[|\sigma|)= \\
& =((\operatorname{sgn} \sigma) \text { area } \operatorname{th}\{\text { area } \operatorname{th}\{|\sigma|\}\}+i(\operatorname{sgn} \sigma) \cdot[\operatorname{area} \operatorname{th}\{|\sigma|\}], \kappa)=(\lambda+i \rho, \kappa)=(\lambda, \rho, \kappa) .
\end{aligned}
$$

Casting a glance at (51), Theorem of super-super multiplication is proved. By (16) the Postulate of super-super multiplication is satisfied.
 So, $\overline{\boxed{R}},-\Phi,-\Phi-)$ is a field. The unit-element of super-super addition is $\left.\begin{array}{l}\stackrel{4}{0} \\ 0\end{array}\right)=(0,0,0)$. This $\stackrel{\text { । }}{\stackrel{\square}{R}}$ means, that for any $(t, n, m) \in(\stackrel{\rightharpoonup}{R})$
 shows

$$
\begin{equation*}
(t, n, m)-\oplus-(-t,-n,-m)=\stackrel{\stackrel{\rightharpoonup}{\stackrel{\rightharpoonup}{4}}}{(0)}, \quad(t, n, m) \in(\stackrel{\text { । }}{(\stackrel{\rightharpoonup}{R})} . \tag{56}
\end{equation*}
$$

By (56) we introduce

$$
\begin{equation*}
-(t, n, m)=(-t,-n,-m), \quad(t, n, m) \in(\stackrel{\rightharpoonup}{\stackrel{\rightharpoonup}{R}}) . \tag{57}
\end{equation*}
$$

Moreover, the super-super extraction

$$
\begin{equation*}
(t, n, m)-母^{-}(\tau, \nu, \mu)=(t, n, m)-\not-(-(\tau, \nu, \mu)), \quad(t, n, m) ;(\tau, \nu, \mu) \in(\stackrel{\stackrel{\rightharpoonup}{R}}{\boldsymbol{\prime}} . \tag{58}
\end{equation*}
$$

1
$\square$
The unit-element of super-super multiplication is $(1)=(0,0,1)$. Really by (16), (34), (37) and (36)
is obtained.
It is easy to see that if $(t, n, m) \neq(0,0,0)$ and $(t, n, m) \in(\stackrel{\rightharpoonup}{R})$ then $m+\operatorname{th}(n+\operatorname{th} t) \neq 0$. Considering ( $\tau, \nu, \mu$ ) such that

$$
\begin{align*}
\tau & =\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right) \cdot \operatorname{area} \operatorname{th}\left\{\operatorname{area} \operatorname{th}\left\{\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right|\right\}\right\}  \tag{60}\\
\nu & =\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right) \cdot\left[\operatorname{areath}\left\{\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right|\right\}\right] \tag{61}
\end{align*}
$$

and

$$
\begin{equation*}
\mu=\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right) \cdot\left[\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right|\right] \tag{62}
\end{equation*}
$$

we have $(\tau, \nu, \mu) \in(\stackrel{\stackrel{\rightharpoonup}{R}}{\prime})$. Hence,

$$
\begin{gathered}
\mu+\operatorname{th}(\nu+\operatorname{th} \tau)=\mu+\operatorname{th}\left(\nu+\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right) \cdot\left\{\operatorname{area} \operatorname{th}\left\{\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right|\right\}\right\}\right)= \\
=\mu+\operatorname{th}\left(\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right) \operatorname{area} \operatorname{th}\left\{\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t}\right|\right\}\right)= \\
=\mu+\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right) \cdot\left\{\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right|\right\}=
\end{gathered}
$$

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$$
=\left(\operatorname{sgn} \frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right)\left|\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)}\right|=\frac{1}{m+\operatorname{th}(n+\operatorname{th} t)} .
$$

So, $(m+\operatorname{th}(n+\operatorname{th} t)) \cdot(\mu+\operatorname{th}(\nu+\operatorname{th} \tau))=1$ and by (51), (52), (53) and (54) we have

$$
\begin{equation*}
\left.(t, n, m)-\infty-(\tau, \nu, \mu)=\frac{\stackrel{4}{4}}{1}\right) . \tag{63}
\end{equation*}
$$

Collecting these items, we have
Reciprocal theorem for explodes of exploded real numbers. For any $(t, n, m) \in(\overrightarrow{\boxed{R}})$ which differs $\stackrel{1}{\square}$
from $\overline{\binom{0}{0}}$ there exists one and only one reciprocal partner $(\tau, \nu, \mu) \in \overline{(\stackrel{\rightharpoonup}{R})}$, with $(60),(61)$ and (62) such that (63) is valid.
$\square$ $\xrightarrow{\square}$
Clearly, $(\stackrel{4}{1})$ is own reciprocal partner. Similarly, $(\stackrel{\rightharpoonup}{-1})$ is own reciprocal partner.
Using the reciprocal theorem for explodeds of exploded real numbers by (51) we define the super-super division:
with

$$
\begin{align*}
& \lambda=\left(\operatorname{sgn} \frac{m+\operatorname{th}(n+\operatorname{th} t)}{\mu+\operatorname{th}(\nu+\operatorname{th} \tau)}\right) \cdot \operatorname{areath}\left\{\operatorname{areath}\left\{\left|\frac{m+\operatorname{th}(n+\operatorname{th} t)}{\mu+\operatorname{th}(\nu+\operatorname{th} \tau)}\right|\right\}\right\}  \tag{65}\\
& \rho=\left(\operatorname{sgn} \frac{m+\operatorname{th}(n+\operatorname{th} t)}{\mu+\operatorname{th}(\nu+\operatorname{th} \tau)}\right) \cdot\left[\operatorname{areath}\left\{\left|\frac{m+\operatorname{th}(n+\operatorname{th} t)}{\mu+\operatorname{th}(\nu+\operatorname{th} \tau)}\right|\right\}\right]  \tag{66}\\
& \kappa=\left(\operatorname{sgn} \frac{m+\operatorname{th}(n+\operatorname{th} t)}{\mu+\operatorname{th}(\nu+\operatorname{th} \tau)}\right) \cdot\left[\left|\frac{m+\operatorname{th}(n+\operatorname{th} t)}{\mu+\operatorname{th}(\nu+\operatorname{th} \tau)}\right|\right] . \tag{67}
\end{align*}
$$

In the special case $(t, n, m)=(\underset{1}{1}),(m=1, n=0, t=0)(64),(65),(66),(67),(60),(61)$ and (62) show that

$$
\overline{\binom{1}{1}-\odot-(\tau, \nu, \mu)}
$$

is the reciprocal partner of $(\tau, \nu, \mu) \neq(0)$.
Finally, we turn towards the requirements of monotonities.
Theorem of monotonity of super-super addition. If $(t, n, m),(\tau, \nu, \mu),(\lambda, \rho, \kappa) \in(\stackrel{\rightharpoonup}{R})$ and

$$
\begin{equation*}
(t, n, m)<(\tau, \nu, \mu) \tag{68}
\end{equation*}
$$

then

$$
\begin{equation*}
(t, n, m)-(\lambda, \rho, \kappa)<\tau, \nu, \mu)-\oplus-(\lambda, \rho, \kappa) \tag{69}
\end{equation*}
$$

is valid too.
Proof. Denoting $\stackrel{\rightharpoonup}{u}=(t, n, m), \stackrel{\rightharpoonup}{v}=(\tau, \nu, \mu), \stackrel{\rightharpoonup}{w}=(\lambda, \rho, \kappa)$, by (37) we have that $u=(t, n, m)$, $v=(\tau, \nu, \mu), w={ }_{( }(\lambda, \rho, \kappa)$. Condition (68) says that $\stackrel{\rightharpoonup}{u}<\stackrel{\rightharpoonup}{v}$. Hence (14) gives, that $u<v$. (See the Theorem of ordering.) By (9) $u-\infty<v-\infty$ - $w$ is obtained. Applying the Theorem of ordering again, we have that $\overline{u-\Varangle-w}<\overrightarrow{v-\Varangle-w}$. Hence (15) gives (69). (See Theorem of super-super addition.)
Theorem of monotonity of super-super multiplication. If $(t, n, m),(\tau, \nu, \mu),(\lambda, \rho, \kappa) \in(\stackrel{\stackrel{\rightharpoonup}{R}}{\prime})$ and $(2,48)$ fulfills, moreover $0<(\lambda, \rho, \kappa)$ then

$$
\begin{equation*}
(t, n, m)-(\lambda, \rho, \kappa)<(\tau, \nu, \mu)-\varnothing-(\lambda, \rho, \kappa) \tag{70}
\end{equation*}
$$

is valid, too.
Proof. Denoting $\stackrel{\stackrel{4}{u}}{u}=(t, n, m), \stackrel{\rightharpoonup}{v}=(\tau, \nu, \mu), \stackrel{\rightharpoonup}{w}=(\lambda, \rho, \kappa)$, by (37) we have that $u=(t, n, m)$, $v=(\tau, \nu, \mu), w=(\lambda, \rho, \kappa)$. Condition (68) says that $\stackrel{\rightharpoonup}{u}<\stackrel{\rightharpoonup}{v}$. Hence (14) gives, that $u<v$. (See the Theorem of ordering.) Similarly, we have $0<w$. By (10) $u-\nsim-w<v-\nsim-w$ is obtained. Applying the Theorem of ordering again, we have that $u-\not x-w<v-\infty-w$. Hence (15) gives (70). (See Theorem of super-super multiplication.)

The construction of ordered field $((\overline{\boxed{R}}),-\varnothing-,-\varnothing-)$ is finished.

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# GOLDEN SECTION QUASIGROUPS, FINITE EXAMPLES 

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#### Abstract

G S\)-quasigroups (golden section quasigroups, a particular class of idempotent medial quasigroups) were introduced by V. Volenec in [11], and their relation to geometric configurations was examined. The identity $(x \cdot(x y \cdot z)) \cdot z=y$ characterizing the class might appear to be useful in cryptography or coding theory. For this purpose, we investigate existence of finite examples; next step should be analyzing of their non-associativity. In [1] examples of $G S$-quasigroups in low orders ( $4,5,9,11,19,29$ ) were found by "paper and pencil" from finite fields by the method suggested in [11], [3]. With the computer aid, we found examples over $\mathbb{Z}_{p}$ up to prime order $p=991$.


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## 1 Introduction

Recall some terminology and notation. If $(Q, \cdot)$ is a groupoid we say that an element $q \in Q$ is idempotent if $q \cdot q=q$. The groupoid is idempotent if all its elements are idempotent, i.e. the identity $x \cdot x=x$ holds, and is said to be medial, or entropic if the following identity holds:

$$
\begin{equation*}
(x \cdot y) \cdot(u \cdot v)=(x \cdot u) \cdot(y \cdot v) . \tag{1}
\end{equation*}
$$

To save brackets we suppose that juxtaposition is preferred to composition where the product is explicitly written, i.e. we write mediality as $x y \cdot u v=x u \cdot y v$ etc. Denote by $L_{u}, R_{u}: Q \rightarrow Q$, $L_{u}: x \mapsto u x, R_{u}: x \mapsto x u$ the left and right translation by $u \in Q$ in $(Q, \cdot)$, respectively.

A groupoid $(Q, *)$ is dual to the groupoid $(Q, \cdot)$ if the operations are related by $a * b=b a$ for all $a, b \in Q$.

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Under the isotopy of a groupoid $(Q, \cdot)$ onto $\left(Q^{\prime}, .^{\prime}\right)$ we mean a triplet of bijections $(\alpha, \beta, \gamma): Q \rightarrow Q^{\prime}$ such that $\alpha(a)^{\prime} \beta(b)=\gamma(a \cdot b)$ for all $a, b \in Q$, or equivalently, $a^{\prime} \cdot \cdot^{\prime} b^{\prime}=$ $\gamma\left(\alpha^{-1}\left(a^{\prime}\right) \cdot \beta^{-1}\left(b^{\prime}\right)\right)$ for all $a^{\prime}, b^{\prime} \in Q^{\prime},[6]$. In the case of coinciding bijections $\alpha=\beta=\gamma: Q \rightarrow Q^{\prime}$, we speak about isomorphism.

Quasigroups can be characterized in two ways. Either equationally, as algebras with three binary operation $\cdot, /$ and $\backslash$ (called multplication, right and left division) which are connected by the identities

$$
x y / y=y \backslash y x=(x / y) y=y(y \backslash x)=x,
$$

or as qroupoids in which the equations $a y=b, x a=b$ admit unique solution, denoted $a \backslash b$ or $b / a$, respectively, for which the above identities can be verified.

Let us follow the second view-point here. Recall that a group can be introduced as a quasigroup which is at the same time a semigroup [5] (i.e. associative). While any group isotopic to the given group is isomorphic with it, for quasigroups, isotopism plays the central role.

Under a pointed groupoid or pointed quasigroup we mean the algebraic structure together with a distinguished element from the underlying set (support); we use the notation $(Q, \cdot ; q)$ etc. In the case of a pointed group, we agree to distinguish just the identity element.

If $\mathcal{G}$ is a group let $\operatorname{Aut}(\mathcal{G})$ denote its automorphism group.

## 2 Pointed medial quasigroups

First let us recall and modify some techniques from the original Toyoda's paper [8] where the following was proved: A necessary and sufficient condition that a set $G$ forms an abelian set is that the product $x \cdot y$ of $G$ is of the form

$$
x \cdot y=\Gamma_{1}(x) * \Gamma_{1}(y) \text { for any elements } x \text { and } y
$$

where $y=\Gamma_{1}(x)$ and $y=\Gamma_{2}(x)$ denote the automorphic transformations of the abelian group $G$ with respect to the product $x * y$ and moreover these operators are mutually permutable. Note that the original results were proved for the so-called abelian sets. According to [8], an abelian set is in fact a medial quasigroup with at least one idempotent element, although the up-to-date quasigroup terminology was not used in the original paper; if an abelian set is idempotent it is called a mean set in [8]. Also for means sets and commutative means sets, analogous theorems were proved in [8].

Lemma 2.1 In a medial quasigroup $(Q, \cdot)$ the following cross-rule is satisfied:

$$
\begin{equation*}
\text { if } u a=b v \text { and } z a=b w \text { then } u w=z v . \tag{2}
\end{equation*}
$$

Proof. Due to the assumptions and mediality, $u w \cdot a b=u a \cdot w b=b v \cdot w b=b w \cdot v b=z a \cdot v b=$ $z v \cdot a b$, and we use right cancellation to get $u w=z v$.

If ( $Q, \cdot \cdot$ ) is a quasigroup and $q \in Q$ a fixed element, let us introduce a "conjugation" $Q \rightarrow Q$, $a \mapsto a^{\prime}$, with respect to $q$ as a unary operation $a^{\prime}=L_{q}^{-1} R_{q}(a)=q \backslash(a q)$; that is, $a^{\prime}$ is the unique element satisfying $a q=q a^{\prime}$. If we apply the cross-rule to $a q=q a^{\prime}$ and $b q=q b^{\prime}$ we get:

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Lemma 2.2 If $(Q, \cdot)$ is a medial quasigroup, $q \in Q$ a fixed element and $a^{\prime}, b^{\prime}$ are conjugates of $a, b$ with respect to $q$ then $a b^{\prime}=b a^{\prime}$.

Lemma 2.3 Let $(Q, \cdot ; q)$ be a pointed medial quasigroup. Then the binary operation $+_{q}$ defined on $Q$ by

$$
\begin{equation*}
x+{ }_{q} y=R_{q}^{-1}(x) \cdot L_{q}^{-1}(y)=(x / q) \cdot(q \backslash y) \quad \text { for } x, y \in Q \tag{3}
\end{equation*}
$$

is associative and commutative.

Proof. Let $a, b, c \in Q$. Due to unique solvability of equations, there are $d_{1}, d_{2}$ in $Q$ such that $(a / q) \cdot(b / q)^{\prime}=d_{1} q=q d_{1}^{\prime}=(b / q) \cdot(a / q)^{\prime},(b / q) \cdot(c / q)^{\prime}=d_{2} q=q d_{2}^{\prime}=(b / q) \cdot(c / q)^{\prime}$. Using Lemma 2.1 for $d_{1} q=(b / q) \cdot(a / q)^{\prime}$ and $d_{2} q=(b / q) \cdot(c / q)^{\prime}$ we obtain $d_{1}(c / q)^{\prime}=d_{2}(a / q)^{\prime}=(a / q) \cdot d_{2}^{\prime}$. Now

$$
\begin{aligned}
& \left.\left(a+_{q} b\right)+{ }_{q} c=((a / q)) \cdot(b / q)^{\prime}\right)+_{q} c \\
& =d_{1} \cdot(c / q)^{\prime}=(a / q) \cdot d_{2}^{\prime}=a+_{q}\left((b / q) \cdot(c / q)^{\prime}\right)=a+_{q}\left(b+_{q} c\right)
\end{aligned}
$$

If we use $a=(a / q) \cdot q=q \cdot(a / q)^{\prime}$ and similarly for $b$, we get

$$
a+{ }_{q} b=((a / q) \cdot q)+{ }_{q}\left(q \cdot(b / q)^{\prime}\right)=(a / q) \cdot\left(q \cdot(b / q)^{\prime}\right)
$$

From Lemma 2.2, $a+{ }_{q} b=(b / q) \cdot(a / q)^{\prime}=b+{ }_{q} a$.

## 3 Modification of the Toyoda's Theorem

Theorem 3.1 Let $(Q, \cdot ; q)$ be a pointed medial quasigroup with $q$ idempotent. Define a binary operation on $Q$ by (3). Then $\left(Q,+_{q}\right)$ is a commutative group with the identity element $q$.

Proof. By Lemma $2.3,\left(Q,+_{q}\right)$ is a semigroup. Check that $\left(Q,+_{q}\right)$ is a quasigroup: $\left(R_{q}, L_{q}, \operatorname{id}_{Q}\right)$ is a principal isotopy of $(Q, \cdot)$ onto $\left(Q,+_{q}\right)$ since we can write $(3)$ equivalently as

$$
x y=R_{q}(x)+{ }_{q} L_{q}(y)=x q+{ }_{q} q y \quad \text { for all } x, y \in Q
$$

Moreover, for all $x \in Q, x q=x q+{ }_{q} q q=x q+{ }_{q} q$ holds, and $q x=q q+{ }_{q} q x=q+{ }_{q} q x$, hence $q$ is the identity element (and $\left(Q,+_{q}\right)$ is an $L P$-isotope). Commutativity was already checked.

Lemma 3.2 Let $(Q, \cdot ; q)$ be a pointed medial quasigroup with $q$ idempotent. Then for any $x, y, z, w \in Q$,

$$
x y+{ }_{q} z w=\left(x+{ }_{q} z\right) \cdot\left(y+{ }_{q} w\right)
$$

Proof. Let us write $x=a q=q a^{\prime}$ (i.e. $a=x / q$ ) and similarly ( $b=y / q$ etc.) $y=b q=q b^{\prime}$, $z=c q=q c^{\prime}, w=d q=q d^{\prime}$. Then due to mediality, $x y=(a q) \cdot(b q)=(a \cdot b) \cdot(q \cdot q)=a b \cdot q$, and similarly $z w=c q \cdot d q=q q \cdot c^{\prime} d^{\prime}=q \cdot c^{\prime} d^{\prime}$. Finally, $x y+_{q} z w=a b \cdot c^{\prime} d^{\prime}=a c^{\prime} \cdot b d^{\prime}=$ $\left(a q+{ }_{q} q c^{\prime}\right) \cdot\left(b q+{ }_{q} q d^{\prime}\right)=\left(x+_{q} z\right) \cdot\left(y+_{q} w\right)$ which proves the entropicity, "mixed mediality".

Lemma 3.3 If $\mathcal{G}=\left(Q,+_{q} ; q\right)$ is a commutative group which arises from a pointed medial quasigroup $(Q, \cdot ; q)$ and its idempotent element $q$ by the construction (3) then the translations $R_{q}$ and $L_{q}$ are commuting automorphisms of the group $\mathcal{G}$.

Proof. First check that $R_{q}, L_{q} \in \operatorname{Aut}(\mathcal{G})$. We have $R_{q}(x)+{ }_{q} R_{q}(y)=x q+{ }_{q} y q=\left(x+{ }_{q} y\right)$. $\left(q+{ }_{q} q\right)=\left(x+_{q} y\right) \cdot q=R_{q}\left(x+{ }_{q} y\right)$, analogously for $L_{q}$. For any $x \in Q, R_{q} L_{q}(x)=q x \cdot q q=$ $q q \cdot x q=L_{q} R_{q}(x)$, hence $R_{q}$ and $L_{q}$ commute.

Theorem 3.4 Let $\mathcal{Q}=(Q, \cdot ; e)$ be a pointed medial quasigroup with e idempotent. Then there is a commutative group $\mathcal{G}=(Q,+)$ and commuting automorphisms $\alpha, \beta \in \operatorname{Aut}(\mathcal{G})$ such that

$$
x \cdot y=\alpha(x)+\beta(y) \quad \text { for all } x \in Q .
$$

Proof. It is sufficient to take $\alpha=R_{e}, \beta=L_{e}$ and $+=+_{e}$ ( $e$ is then the identity of the group).

Theorem 3.5 Let $\mathcal{G}=(Q,+; e)$ be a commutative group and let $\alpha, \beta \in \operatorname{Aut}(\mathcal{G})$ be commuting automorphisms of the group. Let $\mathcal{Q}=(Q, \cdot)$ be a groupoid with the operation

$$
\begin{equation*}
x \cdot y=\alpha(x)+\beta(y) \quad \text { for all } x \in Q \tag{4}
\end{equation*}
$$

Then $\mathcal{Q}$ is a medial quasigroup with the idempotent element $e$ and the automorphisms are translations of the quasigroup, namely, $\alpha=R_{e}$ and $\beta=L_{e}$.

Proof. Let the assumptions be satisfied. Then ( $\alpha, \beta, \mathrm{id}$ ) is a principal isotopy of the groupoid $\mathcal{Q}$ onto the group $\mathcal{G}$, hence $\mathcal{Q}$ is a quasigroup. Moreover, since $\mathcal{G}$ is the $L P$-isotope (with the identity element $e$ ) we can write $\alpha=R_{v}, \beta=R_{u}$ for convenient elements $u, v \in Q$. As $\alpha, \beta$ are group automorphisms we get $\alpha(e)=e v=e, \beta(e)=u e=e$, and $e \cdot e=\alpha(e)+\beta(e)=e v+u e=$ $e+e=e$ which proves idempotency of $e$. Moreover, using (left and right) cancellation in $\mathcal{Q}$ we get $v=e=u$. So we have $x y=x e+e y$. It remains to check mediality. Using the defining relation (4) and the properties of automorphisms, $a b \cdot c d=(a e+e b) \cdot(c e+e d)=((a e+e b) \cdot e)+(e \cdot$ $(c e+e d))=(a e \cdot e+e b \cdot e)+(e \cdot c e+e \cdot e d)$. On the other hand, $a c \cdot b d=(a e \cdot e)+(e \cdot c e)+(e \cdot b e)+(e \cdot d e)$. Now let us apply the equality $e \cdot x e=L_{e}\left(R_{e} x\right)=R_{e}\left(L_{e} x\right)=e x \cdot e$ for $x=c$ and $x=b$ to the right hand side. We obtain the same expression as above, hence mediality is checked.

We get as a consequence:

Corollary 3.6 A pointed groupoid $(Q, \cdot ; ;)$ is a medial quasigroup with an idempotent element $e$ if and only if there is a commutative group $\mathcal{G}=(Q,+; e)$ and a pair of commuting automorphisms $\alpha, \beta \in \operatorname{Aut}(\mathcal{G})$ such that

$$
x \cdot y=\alpha(x)+\beta(y) .
$$

If this is the case, then $\alpha=R_{e}$ and $\beta=L_{e}$.

Let $(Q,+; e)$ be a group with the identity element $e$ and $f: Q \rightarrow Q$. It is a convention to call the map $f$ linear if $f(x+y)=f(x)+f(y)$, and affine if $f(x+y)=f(x)+f(y)-f(e)$ for all $x, y \in Q$. Let us adopt here the following convention. If $\mathcal{G}=(Q,+; e)$ is a commutative group and $\varphi \in \operatorname{Aut}(\mathcal{G})$ is a fixed non-trivial ${ }^{1}$ automorphism of $\mathcal{G}$, and "." a binary operation on $Q$ introduced by

$$
\begin{equation*}
x \cdot y=x+\varphi(y-x) \quad \text { for } x, y \in Q \tag{5}
\end{equation*}
$$

we say that the groupoid $(Q, \cdot)$ is linear over the commutative group $\mathcal{G}$ under the automorphism $\varphi$.

Note that if $\varphi=\operatorname{id}_{Q}$ then $x y=y$, and $(Q, \cdot)$ is a doubly homogeneous groupoid, [7], but not a quasigroup.

Theorem 3.7 Let $\mathcal{G}=(Q,+; e)$ be a commutative group, $\varphi \in \operatorname{Aut}(\mathcal{G})$ a non-trivial automorphism of $\mathcal{G}$. Let $\mathcal{Q}=(Q, \cdot)$ be a groupoid linear over $\mathcal{G}$ with the automorphism $\varphi$. Then $\mathcal{Q}$ is an idempotent medial quasigroup and $\varphi=L_{e}$.

Proof. We check easily that also the map $\psi=\operatorname{id}_{Q}-\varphi$ is the group automorphism. Indeed, $\psi(x+y)=x+y-\varphi(x+y)=(x-\varphi(x))+(y-\varphi(y))=\psi(x)+\psi(y)$. The automorphisms commute, $\psi \varphi(x)=\varphi(x)-\varphi^{2}(x)=\varphi \psi(x)$. The rest follows from Theorem 3.5 for the choice $\psi=\alpha=R_{e}$ and $\varphi=\beta=L_{e}$.

Theorem 3.8 Let $(Q, \cdot)$ be a non-trivial ${ }^{2}$ idempotent medial quasigroup. Then for any choice of a fixed element $e \in Q$ there is a commutative group $\mathcal{G}=(Q,+; e)$ such that $x y=x e+e y$ such that the translations $R_{e}, L_{e}$ are commuting automorphisms of $\mathcal{G}$ and for all $x \in Q, x+e \cdot x=x$ holds. Moreover, $(Q, \cdot)$ is linear over $\mathcal{G}$ with the automorphism $\varphi=L_{e}$. Various choices of $e \in Q$ give mutually isomorphic groups $\left(Q,++_{e} ; e\right)$.

Proof. If $(Q, \cdot)$ satisfies the assumption and $e \in Q$ is fixed we get $x=x x=x e+_{e} e x$ and $x y=x e+_{e} e y=x e+_{e} e x-{ }_{e} e x+_{e} e y=x+{ }_{e} L_{e}\left(y-{ }_{e} x\right)$ for all $x, y \in Q$, where $L_{e} \neq \mathrm{id}_{Q}{ }^{3}$. Hence $(Q, \cdot)$ is linear over $\mathcal{G}$ with the automorphism $\varphi=L_{e}$. From Lemma 3.3, $L_{e}, R_{e}$ are commuting automorphisms of the group. Since each of the groups $\left(Q,+_{e}\right), e \in Q$, is isotopic to $(Q, \cdot)$ they are necessarily pairwise isotopic, and therefore isomorphic.

## 4 Golden section quasigroups

Let $t$ denote the term $t(x, y, z)=(x \cdot y) \cdot z$ where "." is a binary operation. In the class of groupoids, consider the identity

$$
t(x, t(x, y, z), z)=y, \quad \text { left golden section identity. }
$$

[^3]
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In what follows let us investigate also its dual, right golden section identity $\hat{t}(x, t(x, y, z), z)=y$, where $\hat{t}(x, y, z)=x \cdot(y \cdot z)$.

A groupoid $(Q, \cdot)$ is said to be a golden section groupoid, [3] if it is idempotent and satisfies the following two conditions (left and right golden section identity):

$$
\begin{align*}
& (x \cdot(x y \cdot z)) \cdot z=y,  \tag{6}\\
& x \cdot((x \cdot y z) \cdot z)=y . \tag{7}
\end{align*}
$$

Obviously, $(Q, \cdot)$ satisfies (6), or (7), respectively, if and only if its dual groupoid satisfies the dual identity.

In a cancellative groupoid $\mathcal{Q}=(Q, \cdot),(6)$ and (7) are equivalent, [11]. In fact, putting $b c$ for $b$, (6) reads $(a \cdot((a \cdot b c) \cdot c)) \cdot c=b c$. Accordingly, (7) follows by right cancellation. Similarly for the converse implication.

Lemma 4.1 Any groupoid satisfying both (6) and (7) is a medial quasigroup.
Proof. Indeed, (6) guarantees solvability in $\mathcal{Q}$ of equations of the form $x c=b$ for $b, c \in Q$. To verify uniqueness, let $a x_{1}=a x_{2}$. Then, according to (6), $x_{1}=\left(a \cdot\left(a x_{1} \cdot a\right)\right) \cdot a=\left(a \cdot\left(a x_{2}\right.\right.$. $a)) \cdot a=x_{2}$. Hence $x_{1}=x_{2}$, and $\mathcal{Q}$ is left cancellative. Similarly, solvability of equations of the form $c y=b$ and right cancellation are consequences of (7). Let us check the mediality: $a c \cdot(a b \cdot c d) d=a(a b \cdot(a b \cdot c d) d) \cdot(a b \cdot c d) d=b=a c \cdot(a c \cdot b d) d$, and it is sufficient to use cancellation.

Recall that mediality together with idempotency imply elasticity $x y \cdot x=x \cdot y x$ as well as left and right distributivities, $x \cdot y z=x y \cdot x z, x y \cdot z=x z \cdot y z$.

It follows from the discussion above that we can accept a definition of a golden section quasigroup, briefly a $G S$-quasigroup, as an idempotent quasigroup satisfying (6) (or equivalently, (7)). ${ }^{4}$ Let us recall examples that bring a motivation and justify the terminology used, [11]:

Example 4.2 Let $(F,+, \cdot)$ be a field in which the equation

$$
\begin{equation*}
x^{2}-x-1=0 \tag{8}
\end{equation*}
$$

has a solution $q \in F$, and let $\circ$ be a binary operation on $F$ defined by

$$
\begin{equation*}
a \circ b=(1-q) a+q b=a+q(b-a) \quad \text { for all } a, b \in F . \tag{9}
\end{equation*}
$$

The mapping $\varphi=L_{q}: F \rightarrow F, \varphi(x)=L_{q}(x)=q \cdot x$ is an (additive) automorphism of the additive group of the field, $\varphi=L_{q} \in \operatorname{Aut}(F,+)$, the equality (9) may be written as $a \circ b=a+\varphi(b-a)$, i.e. $(F, \circ)$ is linear over $(F,+)$ with the automorphism $\varphi$, and

$$
\begin{equation*}
\varphi^{2}-\varphi-\mathrm{id}_{Q}=0 \tag{10}
\end{equation*}
$$

holds (since $q$ is a root), $\left(\varphi^{2}-\varphi-\mathrm{id}\right)(x)=\left(q^{2}-q-1\right) \cdot x=0$. The Theorem 4.5 below guarantees that $(F, \circ)$ is a $G S$-quasigroup.

[^4]Example 4.3 (A particular case, the motivation from the Gaussian plane)
Consider the field of complex numbers $(\mathbb{C},+, \cdot)$. Take one of the roots of the equation $x^{2}-x-1=$ $0, t=\frac{1}{2}(1 \pm \sqrt{5})$. Then the corresponding groupoid ( $\mathbb{C}, \circ$ ) is a $G S$-quasigroup. In the Gaussian plane, the equality (9) with $a \neq b$ may be written as $\frac{a \circ b-a}{b-a}=q$, which means that the point $a \circ b$ divides the segment between $a$ and $b$ in the ratio $q$. If $q=\frac{1}{2}(1+\sqrt{5})$ then the golden section of the pair $(a, a \circ b)$ is $b$, if $q=\frac{1}{2}(1-\sqrt{5})$ then the golden section of the pair $(b, a \circ b)$ is $a$.

We can specialize the Toyoda-like Theorem 3.7 for $G S$-quasigroups as follows.
Theorem 4.4 For any $G S$-quasigroup $\mathcal{Q}=(Q, \cdot)$ there is a commutative group $\mathcal{G}=(Q,+; e)$ and an automorphism $\varphi \in \operatorname{Aut}(\mathcal{G})$ such that $(Q, \cdot)$ is linear over the group $\mathcal{G}$ with the (nontrivial) automorphism $\varphi$, i.e. (5) holds, and the following is satisfied:

$$
\begin{equation*}
\varphi^{2}-\varphi-\operatorname{id}_{Q}=e \tag{11}
\end{equation*}
$$

The following theorem justifies our construction of examples ([3], [11]).
Theorem 4.5 Let $\mathcal{Q}=(Q, \cdot)$ be a groupoid linear over a commutative group $\mathcal{G}=(Q,+; e)$ with an automorphism $\varphi \in \operatorname{Aut}(\mathcal{G})$. If $\varphi$ satisfies $\varphi^{2}-\varphi-\mathrm{id}_{Q}=e$ then $\mathcal{Q}$ is a $G S$-quasigroup.

Let us construct finite $G S$-quasigroups using the method suggested above in Example 4.2 and Theorem 4.5 from an abelian group of a finite field and a convenient element $q \neq 0,1$ satisfying $q^{2}=q+1$. Note that such examples are rather specific ones (namely, the constructed quasigroups are doubly homogeneous in the sence of Stein, [7], [3]), the full automorphism group of the structure acts doubly transitively on the underlying set, and should not exploite the class of finite examples of $G S$-quasigroups.

Obviously, there is no such element for $G F(2)$, and for $G F(3)$ either (indeed, $a=1+1$ is the only element distinct from 0,1 , but $a^{2}=1 \neq 0=a+1$ ).

Example 4.6 In $G F\left(2^{2}\right)$, both elements $a, b$ distinct from 0,1 solve the equation (8). The corresponding quasigroups are dual to each other, and isomorphic under $\alpha=(a b)$ (one of the operations, $\circ$, is given below).

|  |  | - $012 a b c d e f$ |
| :---: | :---: | :---: |
|  |  | $00 b f 1 c d 2 a e$ |
| $\bigcirc 01 a b$ | $*_{3} \mid 01234$ | $1 d 1 c e 2 a f 0 b$ |
| 0 0 O $b 1 a$ | (0)03142 | $2 a e 2 b f 0 c d 1$ |
|  | 131420 | $a c d 1 a e 2 b f 0$ |
|  | 214203 | $b 2 a e 0 b f 1 c d$ |
| b     <br> 1 $a$ $a$ 0  | 342031 | $c f 0 b d 1 c e 2 a$ |
| ${ }^{\text {b }} 1$ | $4 \mid 20314$ | $d e 2 a f 0 b d 1 c$ |
|  |  | $e b f 0 c d 1 a e 2$ |
|  |  | $f 1 c d 2 a e 0 b f$ |

Example 4.7 In $G F(5)$, the elements of which are labeled by $0,1, \ldots, 4$ and viewed as the rest classes mod 5 , the equation (8) has the unique double root $q=3$, the corresponding multiplication $*_{3}$ is given above, the quasigroup is self-dual. It seems that it is the only selfdual example of prime order we have met yet.

Example 4.8 In $G F\left(3^{2}\right)$, the elements are $F=\{0,1,2, a, 1+a, 2+a, 1+2 a, 2+2 a\}$. Denote $b=1+a, c=2+a, d=2 a=a+a, e=1+2 a, f=2+2 a$. The equation (8) has distinct roots $b, d$. Multiplication $\bullet$ corresponding to $q=b$ is given above, the quasigroup is self-dual.

The polynomial $x^{2}-(x+1)$ is irreducible over $\mathbb{F}_{p}$ for the prime modulus $p \in\{7,13,17,23\}$. Also, there are no roots of (8) in $G F(8)$.

On the other hand, in $\mathbb{Z}_{11}$, we get a pair $q=4, q=8$ of distinct roots, in $\mathbb{Z}_{19}$, (8) has distinct roots $q=5, q=15$; in $\mathbb{Z}_{29}$, the solutions are 6 and 24 . Accordingly, in orders 11, 19, 29 , we get the corresponding pairs of $G S$-quasigroups, dual to each other, [1].

Recently, J. Doležalová (UP Olomouc) tried to find roots of $x^{2}-(x+1)$ by a computer search, and constructed prime-order examples up to order 991 (by means of Maple).

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# ON A CLASS OF CURVATURE PRESERVING ALMOST GEODESIC MAPPINGS OF MANIFOLDS WITH AFFINE CONNECTION 

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#### Abstract

In this paper we pay attention to a particular case of almost geodesic mappings of the first type between (differentiable) manifolds with affine connection. We use here classical tensor methods and the apparatus of partial differential equations. We prove that under the mappings under consideration, the invariant geometric object is just the (Riemannian) curvature tensor of the connection. We present the basic equations of the class of mappings under consideration in an equivalent form of the Cauchy system in covariant derivatives.


Key words and phrases. almost geodesic mappings, invariant geometric object, manifolds with affine connection.
Mathematics Subject Classification. 53B05.

## 1 Introduction

Many monographs and papers are devoted to the theory of geodesic and holomorphically projective mappings, see [1]-[23]. We continue here a research project on geodesic and almost geodesic mappings of spaces with affine connection, or pseudo-Rimannian spaces, respectively.

In this paper we pay attention to a particular case of almost geodesic mappings of the first type between (differentiable) manifolds endowed affine connection. We prove that under such maps, the invariant geometric object is just the (Riemannian) curvature tensor of the connection. We present the basic equations of the class of mappings under consideration in an equivalent form of the Cauchy system in covariant derivatives.

As the main tool, we use here classical tensor methods and the apparatus of partial differential equations.

## 2 Almost geodesic mappings of manifolds with affine connection

Let us recall the basic concepts of the theory of almost geodesic mappings of manifolds with affine connection introduced in $[16,17,18]$. A (differentiable) curve defined in a manifold with affine connection $A_{n}$ is called almost geodesic if there is a (differentiable) two-dimensional parallel distribution along the curve such that the tangent vectors of the curve, being parallely transported along the curve, still belong to the distribution.

A diffeomorphism $f: A_{n} \rightarrow \bar{A}_{n}$ of manifolds with affine connection is called almost geodesic if all geodesics in $A_{n}$ are mapped onto almost geodesic curves of $\bar{A}_{n}$.

A map of $A_{n}$ onto $\bar{A}_{n}$ is almost geodesic if and only if in a common coordinate system $\left(x^{1}, \ldots, x^{n}\right)$ (with respect to the diffeomorphism $f,[15$, p. 85]), the deformation tensor of the connections ([15, p. 86]) $P_{i j}^{h}(x)=\bar{\Gamma}_{i j}^{h}(x)-\Gamma_{i j}^{h}(x)$ satisfies

$$
A_{\alpha \beta \gamma}^{h} \lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} \equiv a P_{\alpha \beta}^{h} \lambda^{\alpha} \lambda^{\beta}+b \lambda^{h}
$$

where $A_{i j k}^{h} \equiv P_{i j, k}^{h}+P_{i j}^{\alpha} P_{\alpha k}^{h}, \Gamma_{i j}^{h}\left(\bar{\Gamma}_{i j}^{h}\right.$, respectively) are components of the connection in the manifold $A_{n}\left(\bar{A}_{n}\right.$, respectively $), \lambda^{h}$ is an arbitrary vector, and $a, b$ are some functions of the variables $x^{h}, \lambda^{h}$. Here and in what follows, "," denotes the covariant derivative with respect to the connection of $A_{n}$.
N.S. Sinyukov distinguished in $[12,16,17,18]$ three types of almost geodesic mappings denoted by $\pi_{1}, \pi_{2}$ and $\pi_{3}$. We proved in [1, 12] that for dimensions $n>5$, there are no others. Almost geodesic mappings (in short, AGM) of type $\pi_{1}$ are characterized by the following conditions for the deformation tensor

$$
A_{(i j k)}^{h}=\delta_{(i}^{h} a_{j k)}+b_{(i} P_{j k)}^{h}
$$

where $a_{i j}$ is a symmetric tensor, $b_{i}$ is a covector, $\delta_{i}^{h}$ is the Kronecker tensor, and ( $i j k$ ) means symmetrization (without division) with respect to the listed indices.

## 3 A particular subclass of the first type AGM

Let then following conditions are satisfied under the diffeomorphism of manifolds with affine connection:

$$
\begin{equation*}
P_{i j, k}^{h}=-P_{i j}^{\alpha} P_{\alpha k}^{h}+\delta_{(k}^{h} a_{i j)} \tag{1}
\end{equation*}
$$

Such mappings belong, as a particular case, to AGM of the first type. The Riemannian tensors $R_{i j k}^{h}$ and $\bar{R}_{i j k}^{h}$ of the manifolds $A_{n}$ and $\bar{A}_{n}$, respectively, are related by [17]

$$
\begin{equation*}
\bar{R}_{i j k}^{h}=R_{i j k}^{h}+P_{i[k, j]}^{h}+P_{i[k}^{\alpha} P_{j] \alpha}^{h} \tag{2}
\end{equation*}
$$

where $[k j]$ denotes alternation with respect to the distinguished indices. According to (1) and (2), the following holds:

Theorem 3.1 The Riemannian tensor $R_{i j k}^{h}$ is an invariant object of manifolds with affine connection under almost geodesic mappings satisfying (1).

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Since the Riemmanian tensor vanishes in affine spaces, we deduce that the affine spaces form a class closed under AGM satisfying (1):
Theorem 3.2 If an affine space $A_{n}$ admits an almost geodesic mapping onto $\bar{A}_{n}$ that satisfies (1), then $\bar{A}_{n}$ is an affine space.

Regarding (1) as a system of Cauchy type with respect to the components of the deformation tensor $P_{i j}^{h}$ we find the corresponding integrability conditions. For this purpose, let us calculate covariant derivatives of (1) with respect to $x^{m}$, and let us alternate in $k$ and $m$. Accounting the Ricci identity we get

$$
\begin{equation*}
\delta_{i}^{h} a_{j[k, m]}+\delta_{j}^{h} a_{i[k, m]}+\delta_{[k \mid}^{h} a_{i j, \mid m])}=-P_{i j}^{\alpha} R_{\alpha k m}^{h}+P_{\alpha(j}^{h} R_{i) k m}^{\alpha}+a_{j[m} P_{k] i}^{h}+a_{i[m} P_{k] j}^{h} \tag{3}
\end{equation*}
$$

Now contracting the integrability conditions in $h$ and $m$ we get

$$
\begin{equation*}
a_{j k, i}+a_{i k, j}-(n+1) a_{i j, k}=-P_{i j}^{\alpha} R_{\alpha k}+P_{\alpha j}^{\beta} R_{i k \beta}^{\alpha}+P_{\alpha i}^{\beta} R_{j k \beta}^{\alpha}+a_{j \alpha} P_{k i}^{\alpha}-a_{j k} P_{\alpha i}^{\alpha}+a_{i \alpha} P_{j k}^{\alpha}-a_{i k} P_{j \alpha}^{\alpha} . \tag{4}
\end{equation*}
$$

Further, alternating (4) over $k$ and $j$ we obtain

$$
\begin{align*}
a_{i j, k}= & a_{i k, j}+\frac{1}{n+2}\left(P_{i j}^{\alpha} R_{\alpha k}-P_{i k}^{\alpha} R_{\alpha j}-P_{\alpha j}^{\beta} R_{i k \beta}^{\alpha}-P_{\alpha i}^{\beta} R_{j k \beta}^{\alpha}\right.  \tag{5}\\
& \left.+P_{\alpha k}^{\beta} R_{i j \beta}^{\alpha}+P_{\alpha i}^{\beta} R_{k j \beta}^{\alpha}-a_{j \alpha} P_{k i}^{\alpha}+a_{k \alpha} P_{i j}^{\alpha}+a_{i k} P_{j \alpha}^{\alpha}-a_{i j} P_{k \alpha}^{\alpha}\right)
\end{align*}
$$

In (5), let us interchange the indices $k$ and $i$,

$$
\begin{align*}
a_{k j, i}= & a_{i k, j}+\frac{1}{n+2}\left(P_{k j}^{\alpha} R_{\alpha i}-P_{k i}^{\alpha} R_{\alpha j}-P_{\alpha j}^{\beta} R_{k i \beta}^{\alpha}-P_{\alpha k}^{\beta} R_{j i \beta}^{\alpha}+P_{\alpha i}^{\beta} R_{k j \beta}^{\alpha}+P_{\alpha k}^{\beta} R_{i j \beta}^{\alpha}\right.  \tag{6}\\
& \left.-a_{j \alpha} P_{i k}^{\alpha}+a_{i \alpha} P_{k j}^{\alpha}+a_{k i} P_{j \alpha}^{\alpha}-a_{k j} P_{i \alpha}^{\alpha}\right)
\end{align*}
$$

Plugging (5) and (6) to (4) we find

$$
\begin{align*}
a_{i k, j}= & \frac{1}{(n-1)(n+2)}\left[n\left(P_{i k}^{\alpha} R_{\alpha j}-P_{\alpha(k}^{\beta} R_{i) j \beta}^{\alpha}\right)+R_{\alpha(k} P_{i) j}^{\alpha}-P_{\alpha j}^{\beta} R_{(i k) \beta}^{\alpha}-P_{\alpha(i}^{\beta} R_{|j| k) \beta}^{\alpha}\right.  \tag{7}\\
& +(n+1)\left(a_{j(i} P_{k) \alpha}^{\alpha}-a_{\alpha(i} P_{k) j}^{\alpha}+2\left(a_{i k} P_{j \alpha}^{\alpha}-a_{j \alpha} P_{i k}^{\alpha}\right)\right] .
\end{align*}
$$

Obviously, the equations (1) and (7) in the given space $A_{n}$ respresent a Cauchy system in the functions $P_{i j}^{h}(x)$ and $a_{i j}(x)$ which, naturally, must satisfy also the following system of conditions of an algebraic character

$$
\begin{equation*}
P_{i j}^{h}(x)=P_{j i}^{h}(x), \quad a_{i j}(x)=a_{j i}(x) \tag{8}
\end{equation*}
$$

Hence we have proved:
Theorem 3.3 $A$ manifold with affine connection $A_{n}$ admits almost geodesic mappings, satisfying the equation (1), onto a manifold with affine connection $\bar{A}_{n}$ if and only if there exists, in $A_{n}$, a solution of the mixed system of Cauchy type (1), (7) and (8) in the functions $P_{i j}^{h}$ and $a_{i j}$.

It is proved that the number of relevant parameters on which the solution of a system under consideration depends has the upper boundary

$$
r \leq \frac{1}{2} n(n+1)^{2} .
$$

## 4 An example of a particular subclass of the first type AGM

It the tensor $a_{i j}$ vanishes identically the equation (1) reads

$$
\begin{equation*}
P_{i j, k}^{h}=-P_{i j}^{\alpha} P_{\alpha k}^{h} . \tag{9}
\end{equation*}
$$

The equations (9) are completely integrable in a manifolds with affine connection. That is, the system is solvable for any initial conditions $P_{i j}^{h}\left(x_{0}\right)$. If we choose initial values satisfying $P_{i j}^{h}\left(X_{0}\right) \neq \delta_{(i}^{h} \psi_{j)}\left(x_{0}\right)$ then the obtained solution determines an almost geodesic map of the first type of an affine space $A_{n}$ onto an affine space $\bar{A}_{n}$ that is not a geodesic one. Hence we have as a consequence

Theorem 4.1 There exists an almost geodesic map of the first type of the affine space onto itself under which all straight lines are mapped onto plane curves not all of which are straight lines.

Let $\left(x^{1}, \ldots, x^{n}\right)$ and $\left(\bar{x}^{1}, \ldots, \bar{x}^{n}\right)$ be affine coordinates in affine spaces $A_{n}$ and $\bar{A}_{n}$, respectively. We give here a particular example of an almost geodesic map of the first type of a flat space $A_{n}$ onto a flat space $\bar{A}_{n}$ as follows. Pointwise, the map is given in coordinates by

$$
\begin{equation*}
\bar{x}^{h}=\frac{1}{2} C_{\alpha}^{h}\left(x^{\alpha}-C^{\alpha}\right)^{2}+x_{0}^{h} \tag{10}
\end{equation*}
$$

where $C_{i}^{h}, C^{h}, x_{0}^{h}$ are constants such that $x^{h} \neq C^{h}, \operatorname{det}\left(C_{i}^{h}\right) \neq 0$. It can be checked directly that the only non-zero components of the deformation tensor are

$$
P_{i i}^{i}=\frac{1}{x^{i}-C^{i}}, \quad i=1, \ldots, n
$$

It can be verified that the tensor $P_{i j}^{h}$ with such components satisfies (9). At the same time, we realize that the map just constructed belongs neither to the type $\pi_{2}$ nor $\pi_{3}$. Under this mapping, straight lines of the space $A_{n}$, given by parametrizations $x^{h}=a^{h}+b^{h} t$ where $t$ is a parameter, are mapped onto parabolas in $\bar{A}_{n}$ given by the equations

$$
\bar{x}^{h}=D^{h}+E^{h} t+F^{h} t^{2}
$$

where $D^{h}=\frac{1}{2} C_{\alpha}^{h}\left(a^{\alpha}-C^{\alpha}\right)^{2}, E^{h}=\frac{1}{2} C_{\alpha}^{h}\left(a^{\alpha}-C^{\alpha}\right) b^{\alpha}$, and $F^{h}=\frac{1}{2} C_{\alpha}^{h}\left(b^{\alpha}\right)^{2}$. The only exceptions come for those straight lines for which the vectors $E^{h}$ and $F^{h}$ happen to be collinear: if this is the case the image of such a line is a straight line again.

Finally let us note that the equations (10) generate a system of almost geodesic maps of type $\pi_{1}$ of a flat space if we consider the coefficients $C_{i}^{h}, C^{h}$ and $x_{0}^{h}$ to be continuous parameters.

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# SYMBOLIC ALGORITHM FOR COMPUTING APPROXIMATE PARAMETERIZATIONS OVER RATIONALS 

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#### Abstract

It is well known that an irreducible algebraic curve is rational if and only if its genus is zero. In this paper, we provide a simple symbolic algorithm to parameterize approximately affine rational plane curves by means of linear systems of adjoint curves and one chosen rational point with the guaranty that the coefficients of the obtained parameterization are rational. The designed technique is suitable especially for curves not containing rational points and for curves for which it is too difficult to find these points.


Key words and phrases. Rational curves, simple points, polynomials over rationals, exact and approximate parameterization algorithm.
Mathematics Subject Classification. Primary 60A05, 08A72; Secondary 28E10.

## 1 Introduction

The introduction of real algebraic curves and surfaces e.g. into industrial applications based on CAD representation depends on efficient algorithms for computing rational parametric curve and surface representations. However, parameterizing general algebraic surfaces is a difficult and challenging problem and moreover in many cases no such rational parameterization exists. Thus computational techniques for approximate rational parameterization of curves and surfaces have to be studied, developed and implemented.

In this paper we consider the approximate parametrization problem for affine plane algebraic curves given by the defining polynomial of degree $d$ over rationals. The problem reads as follows: we are given a rational curve and we want to find not only its rational parametrization but a parameterization possessing solely rational coefficients. Nevertheless, the rationality of a curve does not guarantee the existence of rational points on it. So we have to use a certain approximate technique and parameterize a suitable curve near to the given one. This requirement originated

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from practical applications which need algorithms implemented in exact arithmetic, i.e., if the input is given by rational numbers, then the used technique produces output given again by numbers from this field.

The symbolic approximate algorithm designed in this paper uses a modification of the exact algorithm for the rational parameterization presented in [4]. We construct a near curve with the same size and shape, having the same singularities as the original curve and going through one chosen rational point. Then we parameterize the new curve by the exact rational parameterization algorithm and compute its exact parameterization. The functionality of the designed algorithm is proved on several examples.

## 2 Preliminaries

We start with short recalling the theory of rational curves and the so called Sturm's sequence. More details can be found e.g. in $[5,6,7,8,12,15,16]$.

### 2.1 Rational curves

Let $K$ be an algebraically closed field of characteristic zero. The affine or projective space of the dimension $n$ over $K$ will be denoted by $\mathbb{A}^{n}$ or $\mathbb{P}^{n}$. An affine plane algebraic curve $\mathcal{C}_{a}$ in $\mathbb{A}^{2}$ is the set of zeros of a square-free defining polynomial $f(x, y) \in K[x, y]$. The algebraic degree $d$ of $f$ is called the degree of $\mathcal{C}_{a}$. If $f$ is irreducible then $\mathcal{C}_{a}$ is called an irreducible curve.

Analogously, a projective plane algebraic curve $\mathcal{C}$ in $\mathbb{P}^{2}$ is the set of zeros of a homogenous defining polynomial $F(x, y, z) \in K[x, y, z]$. Clearly, for the defining polynomial $f(x, y)$ of some affine algebraic curve $\mathcal{C}_{a}$ we can construct the associated homogenous polynomial $F(x, y, z)$ describing the associated projective curve $\mathcal{C}$ by multiplying each term of degree $k$ by $z^{d-k}$.

Let $\mathcal{C}$ be a curve in $\mathbb{P}^{2}$ defined by the homogenous polynomial $F(x, y, z)$ and let $P$ be a point on $\mathcal{C}$. This point is called simple or regular on $\mathcal{C}$ iff at least one of the partial derivatives $\frac{\partial F}{\partial x}(P), \frac{\partial F}{\partial y}(P), \frac{\partial F}{\partial z}(P)$ does not vanish at $P$.

If $P$ is not simple then it is a singular point (or a singularity) on $\mathcal{C}$. Its multiplicity on $\mathcal{C}$ is defined as a number $m_{P}$ such that for all $i+j+k<m_{P}$ the partial derivatives

$$
\begin{equation*}
\frac{\partial^{i+j+k} F}{\partial x^{i} \partial y^{j} \partial z^{k}}(P) \tag{1}
\end{equation*}
$$

vanish at $P$ but at least one of the partial derivatives of order $m_{P}$ does not vanish at $P$. All projective singular points of $\mathcal{C}$ can be computed by solving the system of equations $\partial F / \partial x=0$, $\partial F / \partial y=0, \partial F / \partial z=0$. In particular, an $m_{P}$-fold singular point $P$ on $\mathcal{C}$ is called ordinary iff all the tangent lines of $\mathcal{C}$ at $P$ are distinct. Otherwise, i.e., if at least two of them are coincident, the singularity is called non-ordinary.

One of the most important invariants associated with a singular point $P$ is its delta invariant. It is a number of double points concentrated at this point given by $\delta_{P}=\frac{1}{2}\left(\mu_{P}+r_{P}-1\right)$, where $\mu_{P}$ is the Milnor number and $r_{P}$ is the branching number of $P \in \mathcal{C}$, cf. [5].

Definition 2.1 Let $\mathcal{C} \in \mathbb{P}^{2}$ be an irreducible curve of degree $d$ with singular points $P_{1}, \ldots, P_{n}$
possessing the delta invariants $\delta_{P_{1}}, \ldots, \delta_{P_{n}}$, respectively. Then the genus of $\mathcal{C}$ is defined as

$$
\begin{equation*}
\operatorname{genus}(\mathcal{C})=\frac{1}{2}(d-1)(d-2)-\sum_{i=1}^{n} \delta_{P_{i}} . \tag{2}
\end{equation*}
$$

The genus of the affine curve $\mathcal{C}_{a}$ is equal to the genus of its associated projective curve $\mathcal{C}$.
The main difficulty in computing genera of algebraic curves consists in determining delta invariants of all singular points. Nevertheless, the situation becomes considerably simpler in the case of ordinary singularities for which it holds $\delta_{P}=m_{P}\left(m_{P}-1\right) / 2$. In the case of non-ordinary singularities, it is convenient to determine firstly the so called neighbouring tree.

The neighbouring tree of any singularity consists of its neighbouring points which are computed by a birational transformation $\mathcal{T}$ (a composition of suitable projective transformations and quadratic transformations $(x: y: z) \mapsto(y z: x z: x y)$ - see $[4,16]$ for more details $)$. In particular, the neighbouring point of any simple point is again a simple point and the first neighbourhood of an ordinary $m$-fold point contains $m$ simple points. The union of the neighbouring trees of all singular points of $\mathcal{C}$ is called the neighbouring graph of $\mathcal{C}$. Hence, we have:

Corollary 2.2 Let $\mathcal{C}$ be an irreducible algebraic curve of degree $d, \mathcal{N}$ its neighbourhood graph. We denote $m_{P}$ the multiplicities of points $P \in \mathcal{N}$. Then it holds

$$
\begin{equation*}
\operatorname{genus}(\mathcal{C})=\frac{1}{2}\left((d-1)(d-2)-\sum_{P \in \mathcal{N}} m_{P}\left(m_{P}-1\right)\right) \tag{3}
\end{equation*}
$$

An irreducible affine curve $\mathcal{C}_{a}$ is called rational if there exist rational functions $\phi(t), \chi(t) \in$ $K(t)$ such that (i) for almost all $t_{0} \in K,\left(\phi\left(t_{0}\right), \chi\left(t_{0}\right)\right)$ is a point on $\mathcal{C}_{a}$, and (ii) for almost all $\left(x_{0}, y_{0}\right)$ on $\mathcal{C}_{a}$ there exists $t_{0} \in K$ such that $\left(x_{0}, y_{0}\right)=\left(\phi\left(t_{0}\right), \chi\left(t_{0}\right)\right)$. Then, $(\phi, \chi)(t)$ is called a rational parameterization of $\mathcal{C}_{a}$. Similarly we speak about a rational parameterization $(\phi, \chi, \psi)(t)$ of a projective curve $\mathcal{C}$, where $\phi(t), \chi(t), \psi(t)$ are now polynomials from $K[t]$.

Theorem 2.3 An algebraic curve $\mathcal{C}$ is rationally parameterizable if and only if its genus is equal to zero.

### 2.2 Intersection points of two curves

Consider the problem of intersection of two algebraic curves $\mathcal{C}$, defined implicitly by $f(x, y)=0$, and $\mathcal{K}$, given parametrically by $(x(t), y(t))$. Substituting the rational parameterization of $\mathcal{K}$ into the defining polynomial of $\mathcal{C}$ we obtain the polynomial $f(t)$. For determining whether $f(t)$ has a real root in an interval $(a, b)$ we can use Sturm's theorem, see e.g. [6].

Let $f$ be a polynomial from $K[t]$. We define a polynomial $g$ possessing the same roots as the polynomial $f$ but not having multiple roots

$$
\begin{equation*}
g(t)=\frac{f(t)}{\operatorname{gcd}\left(f(t), f^{\prime}(t)\right)} \tag{4}
\end{equation*}
$$

where $f^{\prime}$ denotes the first derivative of $f$ with respect to $t$. Next, we construct polynomials $P_{i}$ as follows: $P_{0}=f, P_{1}=f^{\prime}$ and $P_{i+1}$ is the negative of the remainder of the Euclidean division of $P_{i-1}$ by $P_{i}$. We stop just before we get 0 . The sequence $P_{0}, P_{1}, \ldots, P_{n}$ is called the Sturm sequence of $g$. Next, we denote $v_{f}(a)$ the number of sign changes in the sequence $P_{0}(a), P_{1}(a), \ldots, P_{n}(a)$ for any $a \in \mathbb{R}$, not a root of $g$. Then in holds ([6, p. 5]):

Theorem 2.4 Let $a<b$ in $\mathbb{R}$, neither a nor $b$ being a root of $g$. The number of roots of $g$ in the interval $(a, b)$ is equal to $v_{g}(a)-v_{g}(b)$.

## 3 Exact parameterization algorithm

In this section, we recall the algorithm for computing exact rational parameterizations of algebraic curves, which is based on the theory from [15] and then studied in more detail and modified in $[3,4,11,12,13,14]$. In the next section we will use this classical algorithm for finding an approximate parameterization of a given rational curve.

Definition 3.1 Let $\mathcal{C}$ be a curve of degree $d$. We consider a linear system of curves $L^{n}$ of degree $n$ such that:
(1) Every curve from the system $L^{n}$ is of the degree $n \in\{d-2, d-1, d\}$.
(2) Every $m$-fold singular point of $\mathcal{C}$ is $(m-1)$-fold point $L^{n}$.
(3) Every s-fold neighbouring point of $\mathcal{C}$ is an $(s-1)$-fold neighbouring point of each curve from $L^{n}$.
(4) All curves from $L^{n}$ do not have common component with $\mathcal{C}$.

Then, $L^{n}$ is called a system of adjoint curves of degree $n$ to the given curve $\mathcal{C}$. In addition, the system is called a 1-parameter system of adjoint curves of degree $n$ to $\mathcal{C}$ (denoted by $L^{n}(t)$ ) if also the following condition is fulfilled:
(5) There exist $n d-(d-1)(d-2)-1$ simple points of $\mathcal{C}$ being simple points of each curve from $L^{n}$.

The coefficients of $L^{n}(t)$ are polynomials in one variable $t$. Almost every curve from this system intersects $\mathcal{C}$ in one additional point and for almost every simple point on $\mathcal{C}$, which is not among fixed points (condition (5)), there exists a curve from $L^{n}(t)$, which is passing through this point. The technique for parameterizing rational curves with arbitrary singularities is summarized in Algorithm 1, cf. [12] and references therein for further details.

## 4 Improvement of the exact parameterization method

One of the main computational drawbacks of Algorithm 1 is the necessity of determining a high number of simple points on $\mathcal{C}$. Generally, we have to determine $n d-(d-1)(d-2)-1$ simple points to parameterize a given curve of degree $d$ using a 1-parameter system of adjoint curves of degree $n \in\{d-2, d-1, d\}$. In [12], this limitation was significantly overcome by proving that it is enough to find only one, two, or three point(s) for a 1-parameter system of adjoint curves of degree $d, d-1$, or $d-2$, respectively.

```
Algorithm 1 Exact parameterization of irreducible plane curves
    1. INPUT: A curve \(\mathcal{C}\) defined by a polynomial \(F(x, y, z)\).
    2. Find all singular points \(P_{1}, \ldots, P_{s} \in \mathcal{C}\) having multiplicities \(m_{P_{1}}, \ldots, m_{P_{n}}\).
    3. Find the neighbouring graph \(\mathcal{N}\) of \(\mathcal{C}\) with respect to a sequence of transformations \(\mathcal{T}\).
    4. If genus \((\mathcal{C})=0\) then \(\mathcal{C}\) is rational. Otherwise a rational parameterization does not exist.
    5. Choose \(n \in\{d-2, d-1, d\}\) and construct the system of adjoint curves \(L^{n}\) to \(\mathcal{C}\).
    6. Find \(n d-(d-1)(d-2)-1\) simple points and guarantee that the curves from \(L^{n}\) are
        passing through those simple points - hence, obtain 1-parameter system \(L^{n}(t)\).
    7. Compute the coordinates of the intersection point \(R=(x(t), y(t))\) of \(\mathcal{C}_{a}\) and \(L_{a}^{n}(t)\), where
        \(\mathcal{C}_{a}\) and \(L_{a}^{n}(t)\) are the affine versions of \(\mathcal{C}\) and \(L^{n}(t)\), respectively.
    8. OUTPUT: Pair of functions \((x(t), y(t))\) represents the rational parameterization of \(\mathcal{C}_{a}\).
```

Recently, it has been proved in [4] that only one simple point for determining a 1-parameter system of adjoint curves of any degree $n \in\{d-2, d-1, d\}$ is always sufficient. This novel approach is based on the construction of a special family of $n d-(d-1)(d-2)-1$ points by using only one particular simple point on $\mathcal{C}$. In this section, we shortly repeat this method we refer to [4] for more details.

Consider a simple point $P=\left(p_{1}: p_{2}: p_{3}\right)$ on an algebraic curve $\mathcal{C}$ of degree $d$ given by a polynomial $F$. We guarantee that all curves from the system of adjoint curves $L^{n}$ (of degree $n \in\{d-2, d-1, d\})$ are passing through $P-$ such a system is denoted by $L_{*}^{n}$. Then, we choose one particular curve $\mathcal{M}$ from $L_{*}^{n}$, given by $H(x, y, z)$, and compute a system of simple intersections of $\mathcal{M}_{a}$ and $\mathcal{C}_{a}$. Such system contains $n d-(d-1)(d-2)$ simple points for $p_{3} \neq 0$ and $n d-(d-1)(d-2)-1$ for $p_{3}=0$ (by Bezout's theorem and the rationality of $\mathcal{C}$ ).

First, we construct a polynomial $A(x)$ whose roots are exactly the $x$-coordinates of simple intersection points of $\mathcal{C}_{a}$ and $\mathcal{M}_{a}$. Next, we introduce the polynomial $g$ to guarantee that the intersection of $\mathcal{C}_{a}$ and $\mathcal{M}_{a}$ does not contain the point $P$

$$
g(x)= \begin{cases}\frac{A(x)}{x-p_{1} / p_{3}}, & \text { for } p_{3} \neq 0 \\ A(x), & \text { for } p_{3}=0\end{cases}
$$

Now, we want to find the corresponding $y$-coordinates. By using some elimination technique (e.g. Gröbner basis) we arrive at a family of $n d-(d-1)(d-2)-1$ simple points on $\mathcal{C}_{a}$

$$
\begin{equation*}
\mathcal{F}=\left\{\left(\alpha: \frac{a(\alpha)}{b(\alpha)}: 1\right)_{g(\alpha)=0}\right\}, \tag{5}
\end{equation*}
$$

where $a(\alpha), b(\alpha), g(\alpha) \in K[\alpha]$. Finally, using Theorem 4.1 (whose proof can be be found in [12]) we ensure that every point from $\mathcal{F}$ is lying on each curve from $L^{n}$. Hence we arrive at $L^{n}(t)$.

Theorem 4.1 Let $\mathcal{F}=\left\{\left(m_{1}(\alpha): m_{2}(\alpha): m_{3}(\alpha)\right)_{p(\alpha)=0}\right\}$ be a family of conjugate points, where $m_{1}, m_{2}, m_{3}, p \in K[x]$ and $F(x, y, z) \in K[x, y, z]$. Then $F(x, y, z)$ vanishes at all points of the set $\mathcal{F}$ if and only if $p(\alpha)$ divides $F\left(m_{1}(\alpha), m_{2}(\alpha), m_{3}(\alpha)\right)$.

## 5 Approximate parameterization algorithm

In [4] we have shown that only one simple point is sufficient for parameterizing rational curves - see Section 3. Thus, the only remaining problem is to find a certain simple point. Of course, the most suitable simple points are those with rational coordinates. However, such points do not have to exist or it might be very difficult to find them, cf. [13, 14].

In what follows we consider $K=\mathbb{Q}$ only (i.e., we will work only with polynomials over rationals). Sometimes one wants have a parameterization with only rational coefficients although a given curve does not contain rational points (or it is not easy to find them). In such cases, it is useful to apply an approximation method, cf. [1, 2, 9, 10]. In this section we present a simple method how a rational simple point close enough to a given curve can be found and consequently used for the approximate parameterization process.

Let $\mathcal{C}$ be an irreducible rational curve of degree $d$. The main idea of our approach is to choose one simple rational point $P$ close enough to $\mathcal{C}$ and construct a system of curves $M^{d}$ going through all singular points of $\mathcal{C}$ with appropriate multiplicities (including the neighbouring ones). Next, we obtain the system of curves $\widetilde{M^{d}}$ by forcing all curves from $M^{d}$ to go through $P$ and choose a particular curve $\mathcal{K}$ from $\widetilde{M}^{d}$ such that $\mathcal{K}$ would be as close as possible to $\mathcal{C}$. Finally, we parameterize $\mathcal{K}$ using the chosen rational point $P$, cf. Sections 2 and 3.

First, we introduce a technique for finding a rational point close enough to $\mathcal{C}$. We start with bounding $\mathcal{C}$. If $\mathcal{C}$ is a closed curve, we select the minimum $a$ and the maximum $b$ from the roots of the polynomial $g_{1}(x)$ and and the minimum $c$ and the maximum $d$ from the roots of the polynomial $g_{2}(y)$, where

$$
\begin{equation*}
g_{1}(x)=\operatorname{Res}_{y}\left(f(x, y), \frac{\partial f}{\partial y}(x, y)\right), \quad g_{2}(y)=\operatorname{Res}_{x}\left(f(x, y), \frac{\partial f}{\partial x}(x, y)\right) \tag{6}
\end{equation*}
$$

Then, $\mathcal{C}$ is contained in the box $\langle a+\varepsilon, b+\varepsilon\rangle \times\langle c+\varepsilon, d+\varepsilon\rangle$, for any $\varepsilon>0$. If $\mathcal{C}$ is not closed (in a particular direction), we can bound it by some chosen (arbitrary but fixed) constant.

Now, we have determined an area of our interest and continue with subdividing it. Each box is divided into four smaller equal boxes. In what follows we use only those "subboxes" which have at least one side intersecting $\mathcal{C}$ (this can be easily managed by Sturm's theorem, see Section 2.2.) - the five iteration steps of this process are demonstrated in Figure 1. When the subdivision stops we obtain a certain discrete approximation of $C$, i.e., an area of our interest. We consider all rational points in this area with the denominator less than a chosen particular positive integer constant. Next, we measure the distances of all such rational points from $\mathcal{C}$. In particular, for a point $P$ the distance is computed as (see [9])

$$
\begin{equation*}
\frac{f^{2}(P)}{\|\nabla f(P)\|^{2}} \tag{7}
\end{equation*}
$$

where $f$ is the defining polynomial of $\mathcal{C}$. Finally, we choose the rational point $P$ with a minimal distance from $\mathcal{C}$.

We construct a system of curves $M^{d}$ with the same singular points as $\mathcal{C}$ (including the neighbouring ones). Next, we arrive at the system $\widetilde{M}^{d}$ by forcing all curves from $M^{d}$ to go through the chosen rational simple point $P$. However, this is not always possible because in


Figure 1: The subdivision process demonstrated on cardioid.
some cases the particular curve is already determined only by its singularities (i.e., the system $M^{d}$ contains only one curve). For instance this occurs for rational curves of the degree bigger than 5 possessing only double points (e.g. the asteroid).

If other cases the system $\widetilde{M}^{d}$ depends on some parameters, say $\lambda_{1}, \ldots, \lambda_{k}$. Its defining polynomial has the form

$$
\begin{equation*}
h\left(x, y, \lambda_{1}, \ldots, \lambda_{k}\right)=\sum_{i=1}^{k} \lambda_{i} h_{i}(x, y) . \tag{8}
\end{equation*}
$$

For the polynomial $f$ we can write

$$
\begin{equation*}
f(x, y)=\sum_{i=1}^{k} c_{i} h_{i}(x, y) \tag{9}
\end{equation*}
$$

Then, we determine $\lambda_{1}, \ldots, \lambda_{k}$ by minimizing the objective function

$$
\begin{equation*}
\Phi=\sum_{i=1}^{k} w_{i}\left|c_{i}-\lambda_{i}\right| \tag{10}
\end{equation*}
$$

where $\lambda_{i}$ are the coefficients of (8), $c_{i}$ are the coefficients of (9), and $w_{i}$ denotes a certain weight. We suggest the weight $w_{i}$ to be selected according to the total degree of the term $h_{i}$ in $h$ with the coefficient $\lambda_{i}$. Thus, we have the particular curve $\mathcal{K}$ from $\widetilde{M}^{d}$, close enough to

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$\mathcal{C}$, and one simple rational point $P$ lying on it. Hence, we can parameterize $\mathcal{K}$ by a method for the exact rational parameterization requiring only one input simple point. We obtain an exact parameterization of $\mathcal{K}$ which is simultaneously an approximate parameterization of $\mathcal{C}$. The final step is computing the deviance of $\mathcal{C}$ and $\mathcal{K}$ using the following integral

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \frac{f^{2}((x(t), y(t)))}{\|\nabla f((x(t), y(t)))\|^{2}} \tag{11}
\end{equation*}
$$

where $f$ is the defining polynomial of $\mathcal{C}$ and $(x(t), y(t))$ is the computed parameterization of $\mathcal{K}$.

## 6 Computed examples

Example 6.1 We would like to parameterize an ellipse $\mathcal{C}$ with the equation

$$
\begin{equation*}
f(x, y)=2 x^{2}+7 y^{2}+3 y-5=0 \tag{12}
\end{equation*}
$$

This curve does not contain any rational point (by Legendre's theorem - cf. [17]). In order to obtain a parameterization of $\mathcal{C}$ with rational coefficients we have to use an approximate technique. First, we find a rational point, closest to $\mathcal{C}$ with the denominator less than 10, i.e., the point $P=\left(-\frac{4}{3}, \frac{2}{7}\right)$. Now, we construct a system of curves $\mathcal{K}$ of degree 2 going through $P$

$$
\begin{aligned}
h\left(x, y, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}\right)= & 441 x^{2} \lambda_{2}+441 x y \lambda_{4}+441 x \lambda_{1}+441 y^{2} \lambda_{5}+441 y \lambda_{3}+588 \lambda_{1} \\
& -784 \lambda_{2}-126 \lambda_{3}+168 \lambda_{4}-36 \lambda_{5} .
\end{aligned}
$$

To obtain the final form of $h$ we have to minimize a function

$$
\begin{aligned}
F= & 8820\left|\lambda_{1}\right|+30\left|441 \lambda_{2}-2\right|+20\left|441 \lambda_{3}-3\right|+13230\left|\lambda_{4}\right| \\
& +10\left|588 \lambda_{1}-784 \lambda_{2}-126 \lambda_{3}+168 \lambda_{4}-36 \lambda_{5}+5\right|+30\left|441 \lambda_{5}-7\right| .
\end{aligned}
$$

The solution is $\lambda_{1}=0, \lambda_{2}=\frac{2}{441}, \lambda_{3}=\frac{1}{147}, \lambda_{4}=0$ and $\lambda_{5}=\frac{1}{63}$. Thus, $\mathcal{K}$ in its final form is defined by

$$
\begin{equation*}
h(x, y)=126 x^{2}+441 y^{2}+189 y-314 . \tag{13}
\end{equation*}
$$

By parameterizing $\mathcal{K}$ with the help of the point $P$ we arrive at

$$
\begin{equation*}
\mathcal{K}(t)=\left(\frac{-28 t^{2}+21 t+8}{3\left(7 t^{2}+2\right)}, \frac{-105 t^{2}-112 t+12}{147 t^{2}+42}\right), \quad t \in \mathbb{R} . \tag{14}
\end{equation*}
$$

In this case, the total deviation of $\mathcal{C}_{a}$ and $\mathcal{K}_{a}$ (computed by (11)) is less then $2.8 \cdot 10^{-4}$.
Example 6.2 We parameterize the curve $\mathcal{C}$ with the defining polynomial

$$
\begin{equation*}
f(x, y)=-3 x^{4}+6 x^{3}-6 x^{2} y^{2}-x^{2} y-5 x^{2}+6 x y^{2}-5 x y-3 y^{4}-y^{3}+9 y^{2} . \tag{15}
\end{equation*}
$$

$\mathcal{C}$ contains three double points: $P_{1}=(0: 0: 1), P_{2}=(i: 1: 0)$ and $P_{3}=(-i: 1: 0)$. We determine a rational point which is closest to $\mathcal{C}$ and has the denominator less than 10, i.e., we

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obtain $P=\left(-\frac{3}{8},-\frac{4}{3}\right)$. Now, we construct the system $\mathcal{K}$ of degree 4 having three double points $P_{1}, P_{2}$ and $P_{3}$ and one simple rational point $P$.

$$
\begin{aligned}
h\left(x, y, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}\right)= & 46656 x^{4} \lambda_{5}+46656 x^{3} \lambda_{3}+93312 x^{2} y^{2} \lambda_{5}+46656 x^{2} y \lambda_{4} \\
& -165888 x^{2} \lambda_{1}-589824 x^{2} \lambda_{2}+238680 x^{2} \lambda_{3}+848640 x^{2} \lambda_{4} \\
& -1221025 x^{2} \lambda_{5}+46656 x y^{2} \lambda_{3}+46656 x y \lambda_{1}+46656 y^{4} \lambda_{5} \\
& +46656 y^{3} \lambda_{4}+46656 y^{2} \lambda_{2} .
\end{aligned}
$$

Next, we minimize a function

$$
\begin{aligned}
F= & 3\left|46656 \lambda_{1}+5\right|+3\left|46656 \lambda_{2}-9\right|+8\left|46656 \lambda_{3}-6\right|+8\left|46656 \lambda_{4}+1\right| \\
& +3\left|-165888 \lambda_{1}-589824 \lambda_{2}+238680 \lambda_{3}+848640 \lambda_{4}-1221025 \lambda_{5}+5\right| \\
& +10\left|46656 \lambda_{5}+3\right|+5\left|93312 \lambda_{5}+6\right|
\end{aligned}
$$

The solution is $\lambda_{1}=-\frac{5}{46656}, \lambda_{2}=\frac{589915}{3057647616}, \lambda_{3}=\frac{1}{7776}, \lambda_{4}=-\frac{1}{46656}, \lambda_{5}=-\frac{1}{15552}$ and thus the defining polynomial of the curve $\mathcal{K}$ has the form

$$
\begin{aligned}
h(x, y)= & 196608 x^{4}-393216 x^{3}+393216 x^{2} y^{2}+65536 x^{2} y+327680 x^{2} \\
& -393216 x y^{2}+327680 x y+196608 y^{4}+65536 y^{3}-589915 y^{2} .
\end{aligned}
$$

Finally, the exact parameterization of $\mathcal{K}$ being an approximate parameterization of $\mathcal{C}$ obtained by our method has the form

$$
\begin{aligned}
& x(t)=\frac{415044 t^{4}-3106611 t^{3}-13856475 t^{2}-6152075 t+10635625}{954720 t^{4}+2272752 t^{3}+9720438 t^{2}+28466940 t+24691350} \\
& y(t)=\frac{64\left(11529 t^{4}+41058 t^{3}+38640 t^{2}-3150 t-27625\right)}{3\left(159120 t^{4}+378792 t^{3}+1620073 t^{2}+4744490 t+4115225\right)}
\end{aligned}
$$

where $t \in \mathbb{R}$. The total deviation of $\mathcal{C}_{a}$ and $\mathcal{K}_{a}$ (computed by (11)) is in this case less then $4.02 \cdot 10^{-5}$.

Example 6.3 We consider the cardioid $\mathcal{C}$ defined by the polynomial $f(x, y)$

$$
\begin{equation*}
f(x, y)=x^{4}+10 x^{3}+2 x^{2} y^{2}+10 x y^{2}+y^{4}-25 y^{2} . \tag{16}
\end{equation*}
$$

This curve has again three double points: $P_{1}=(0: 0: 1), P_{2}=(i: 1: 0)$ and $P_{3}=(-i: 1: 0)$. By the subdivision process we have found the rational point $P=(-10,0)$ lying exactly on $\mathcal{C}$. Hence, $\mathcal{K}=\mathcal{C}$ and we obtain an exact parametric expression

$$
\begin{equation*}
\mathcal{K}(t)=\mathcal{C}(t)=\left(\frac{250\left(t^{2}-25\right)}{\left(t^{2}+25\right)^{2}},-\frac{2500 t}{\left(t^{2}+25\right)^{2}}\right), \quad t \in \mathbb{R} \tag{17}
\end{equation*}
$$

## 7 Conclusion

In this paper, we presented a simple and quick algorithm for approximate parameterization of rational curves with rational coefficients. Our method is universal for all curves with zero genus which are not determined only by its singularities. The designed technique is suitable especially for curves which do not contain any rational points or for curves for which it is difficult to find these rational points.

The method uses a modification of the classical exact algorithm for the rational parameterization presented in [4] based on only one simple point. Thus, the approximate algorithm also needs only one rational point close enough to a given curve. We construct a new curve with the same size and shape, having the same singularities as the original curve but passing through a (suitably) chosen rational point. Then we parameterize the new curve by the exact rational parameterization algorithm and obtain its exact parameterization which is simultaneously an approximate parameterization of the original curve. The functionality of the designed algorithm was presented on several examples.

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# VISIBILITY CRITERION FOR FOUR-DIMENSIONAL GEOMETRIC SOLIDS 

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#### Abstract

Mathematics have many methods and structures for working in $n$-dimensional space. We know formal properties of geometric solids and we know theirs special properties. But we want to know how four-dimensional geometric solids are visible in threedimensional projection. The present paper immediately follows the paper [1]. We use geometric algebra approach for representation of geometric solids and we apply criterion of visibility for convex polyhedra in four-dimensional space.


Key words and phrases. visibility, geometric algebra
Mathematics Subject Classification. Primary 15A66, 65D17; Secondary 68U99.

## 1 Introduction

Many papers about geometric algebra are constrained to three-dimensional space only. Paper [2] contains basic information about relations in three-dimensional space. We try to look into the four-dimensional space from the geometric algebra point of view and we derive basic relations and properties.

## 2 Geometric Algebra

Consider vector space over the field of real numbers $V(\mathbb{R})$ with three special operations.

- The inner product $a \cdot b$, also known as the dot or scalar product, of two vectors $a$ and $b$, is a scalar with magnitude $|a||b| \cos \phi$, where $|a|$ and $|b|$ are the lengths of $a$ and $b$, and $\phi$ is the angle between them. Here $|a| \equiv(a \cdot a)^{\frac{1}{2}}$, so that the expression for $a \cdot b$ is effectively an algebraic definition of $\cos \phi$.


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- The outer product $a \wedge b$ has magnitude $|a||b| \sin \phi$ and it has the following properties anticommutative, linearity and distributivity

$$
\begin{aligned}
\forall a, b, c \in V(\mathbb{R}), \alpha & \in \mathbb{R} \\
a \wedge b & =-b \wedge a \\
(\alpha a) \wedge b & =\alpha(a \wedge b) \\
\alpha(a \wedge b) & =(a \wedge b) \alpha \\
a \wedge(b+c) & =a \wedge b+a \wedge c \\
a \wedge a & =0
\end{aligned}
$$

Outer product is anticommutative so $e_{i j}=-e_{j i}, i, j \in \mathbb{N} \Rightarrow e_{i i}=0$ and from this property we obtain $e_{i} \wedge e_{i}=0$

- The geometric product is defined by $a b=a \cdot b+a \wedge b$ [3].

We will not use the geometric product in our paper.
The term $k$-blade is introduced e.g. in [2] as

$$
\begin{align*}
e_{i j} & =e_{i} \wedge e_{j}, j>i \in \mathbb{N} & & \text { 2-blades (bivector), } \\
e_{i j k} & =e_{i} \wedge e_{j} \wedge e_{k}, k>j>i \in \mathbb{N} & & \text { 3-blades (trivector). } \tag{1}
\end{align*}
$$

Using the operation $\wedge$ we construct 3-blades (trivector) represented by

$$
e_{i j k}=e_{i} \wedge e_{j} \wedge e_{k}
$$

We assume associativity in this construction. It follows that the result of the operation $\wedge$ is an element of $V(\mathbb{R})$. We also obtain that $e_{i j}$ cannot be represented as a linear combination of $e_{i}, e_{j}$.

$$
\begin{align*}
e_{i j} & =\alpha e_{i}+\beta e_{j} \\
e_{i j} \wedge e_{i} & =e_{i} \wedge e_{j} \wedge e_{i}=e_{j} \wedge e_{i} \wedge e_{i}=0 \\
\alpha e_{i i}+\beta e_{j i} & =0 \Rightarrow \beta=0  \tag{2}\\
e_{i j} \wedge e_{j} & =e_{i} \wedge e_{j} \wedge e_{j}=0 \\
\alpha e_{i}+\beta e_{j j} & =0 \Rightarrow \alpha=0
\end{align*}
$$

We can use four-dimensional vector space over the field of real numbers with operations of geometric algebra $\mathcal{G}_{4}$. For outer product $a \wedge b$, where $a=a_{1} e_{1}+a_{2} e_{2}+a_{3} e_{3}+a_{4} e_{4}$, $b=b_{1} e_{1}+b_{2} e_{2}+b_{3} e_{3}+b_{4} e_{4}$ we obtain

$$
\begin{equation*}
a \wedge b=\left(a_{1} e_{1}+a_{2} e_{2}+a_{3} e_{3}+a_{4} e_{4}\right) \wedge\left(b_{1} e_{1}+b_{2} e_{2}+b_{3} e_{3}+b_{4} e_{4}\right) \tag{3}
\end{equation*}
$$

Using distributive and associative rules and (1), we may rewrite (3) to obtain

$$
\begin{align*}
a \wedge b & =\left(a_{1} b_{2}-a_{2} b_{1}\right) e_{12}+\left(a_{1} b_{3}-a_{3} b_{1}\right) e_{13}+\left(a_{1} b_{4}-a_{4} b_{1}\right) e_{14} \\
& +\left(a_{2} b_{3}-a_{3} b_{2}\right) e_{23}+\left(a_{2} b_{4}-a_{4} b_{2}\right) e_{24}+\left(a_{3} b_{4}-a_{4} b_{3}\right) e_{34} \tag{4}
\end{align*}
$$

$$
\begin{align*}
a \wedge b= & e_{12}
\end{align*}\left|\begin{array}{ll}
a_{1} & a_{2} \\
b_{1} & b_{2}
\end{array}\right|+e_{13}\left|\begin{array}{ll}
a_{1} & a_{3}  \tag{5}\\
b_{1} & b_{3}
\end{array}\right|+e_{14}\left|\begin{array}{ll}
a_{1} & a_{4} \\
b_{1} & b_{4}
\end{array}\right|
$$

The outer product of three vectors $a, b, c$ in four-dimensional space is defined by

$$
\begin{align*}
a \wedge b \wedge c=(a \wedge b) \wedge c & =(a \wedge b) \wedge c_{1} e_{1}+(a \wedge b) \wedge c_{2} e_{2}+(a \wedge b) \wedge c_{3} e_{3}+(a \wedge b) \wedge c_{4} e_{4}  \tag{6}\\
a \wedge b \wedge c= & e_{231} c_{1} \quad\left|\begin{array}{ll}
a_{2} & a_{3} \\
b_{2} & b_{3}
\end{array}\right|+e_{241} c_{1}\left|\begin{array}{ll}
a_{2} & a_{4} \\
b_{2} & b_{4}
\end{array}\right|+e_{341} c_{1}\left|\begin{array}{ll}
a_{3} & a_{4} \\
b_{3} & b_{4}
\end{array}\right| \\
& +e_{132} c_{2}\left|\begin{array}{ll}
a_{1} & a_{3} \\
b_{1} & b_{3}
\end{array}\right|+e_{142} c_{2}\left|\begin{array}{ll}
a_{1} & a_{4} \\
b_{1} & b_{4}
\end{array}\right|+e_{342} c_{2}\left|\begin{array}{ll}
a_{3} & a_{4} \\
b_{3} & b_{4}
\end{array}\right| \\
& +e_{123} c_{3}\left|\begin{array}{ll}
a_{1} & a_{2} \\
b_{1} & b_{2}
\end{array}\right|+e_{143} c_{3}\left|\begin{array}{ll}
a_{1} & a_{4} \\
b_{1} & b_{4}
\end{array}\right|+e_{243} c_{3}\left|\begin{array}{ll}
a_{2} & a_{4} \\
b_{2} & b_{4}
\end{array}\right|  \tag{7}\\
& +e_{124} c_{4}\left|\begin{array}{ll}
a_{1} & a_{2} \\
b_{1} & b_{2}
\end{array}\right|+e_{134} c_{4}\left|\begin{array}{ll}
a_{1} & a_{3} \\
b_{1} & b_{3}
\end{array}\right|+e_{234} c_{4}\left|\begin{array}{ll}
a_{2} & a_{3} \\
b_{2} & b_{3}
\end{array}\right|
\end{align*}
$$

$$
\begin{align*}
a \wedge b \wedge c & =\left(\left(a_{1} b_{2} c_{3}+c_{1} a_{2} b_{3}+c_{2} a_{1} b_{3}\right)-\left(a_{2} b_{1} c_{3}+c_{1} a_{2} b_{3}+c_{2} a_{1} b_{3}\right)\right) e_{123} \\
& +\left(\left(a_{2} b_{4} c_{1}+c_{2} a_{4} b_{1}+a_{1} b_{2} c_{4}\right)-\left(a_{4} b_{2} c_{1}+a_{1} b_{4} c_{2}+a_{2} b_{1} c_{4}\right)\right) e_{124} \\
& +\left(\left(a_{3} b_{4} c_{1}+a_{4} b_{1} c_{3}+a_{1} b_{3} c_{4}\right)-\left(a_{4} b_{3} c_{1}+a_{1} b_{4} c_{3}+a_{3} b_{1} c_{4}\right)\right) e_{134} \\
& +\left(\left(a_{3} b_{4} c_{2}+a_{4} b_{2} c_{3}+a_{2} b_{3} c_{4}\right)-\left(a_{4} b_{3} c_{2}+a_{2} b_{4} c_{3}+a_{3} b_{2} c_{4}\right)\right) e_{234} \tag{8}
\end{align*}
$$

We can formulate now as

$$
a \wedge b \wedge c=\left|\begin{array}{cccc}
e_{234} & e_{143} & e_{124} & e_{132}  \tag{9}\\
a_{1} & a_{2} & a_{3} & a_{4} \\
b_{1} & b_{2} & b_{3} & b_{4} \\
c_{1} & c_{2} & c_{3} & c_{4}
\end{array}\right|
$$

We use this result to introduce right-orientation in four-dimensional space which is analogous to the right-hand rule in three-dimensional space, see Fig.(1).


Figure 1: Right-hand orientation

## 3 Visibility

The aim of visibility algorithms is to found objects and their parts, which are visible from a particular place. Every algorithm is determined for special representation of geometric solids. We have two basic kinds of visibility algorithms:

## - Raster algorithm

Main idea of this kind of algorithm is that the nearest face for each pixel found. The disadvantage of this algorithm is that the result image has constant size.

## - Vector algorithm

The output of this kind of algorithm is a set of geometric elements (e.g. point, line), which represent visible or invisible parts of displayed solids. Some algorithms returns only line (Hidden Line Elimination) or plane (Hidden Surface Elimination). There are some disadvantages, if the choice of visible edge is wrong. We can affect all solids.

We use visibility criterion for convex polyhedra, which it is based on BVCA [1].
Theorem 3.1 2D-face is visible $\Leftrightarrow$ face is intersection of visible and invisible ( $n-1$ )-D projections of body.

We apply only special version this theorem, because we use only four-dimensional space.
Theorem 3.2 Face is visible $\Leftrightarrow$ face is intersection of visible and invisible 3D-projections of body.

Theorems 3.1 and 3.2 were presented in paper [1]. Here we will find projection of a body from 4 D to 3 D and in the next step we choose the view vector so that it marks visible or invisible 3D bodies. We use geometric algebra and their features.

## 4 Geometric Solids

We want to use visibility criterion for convex polyhedra, but we need to know the difference between external and internal side of geometric solids. Consider four-dimensional vector space over the field of real numbers with operations of geometric algebra $\mathcal{G}_{4}$. This space is generated by the set of blades [2]

$$
\mathcal{G}_{4}=\left\{1 ; e_{1} ; e_{2} ; e_{3} ; e_{4} ; e_{12} ; e_{13} ; e_{14} ; e_{23} ; e_{24} ; e_{34} ; e_{123} ; e_{124} ; e_{134} ; e_{234} ; e_{1234}\right\} .
$$

The orientation of ordered quadruplet of base vectors $\mathcal{G}_{4}$ is based on (9), so

$$
\begin{align*}
e_{132} & =e_{1} \wedge e_{3} \wedge-e_{2} \\
e_{134} & =e_{1} \wedge e_{3} \wedge e_{4}  \tag{10}\\
e_{142} & =e_{1} \wedge e_{4} \wedge-e_{2} \\
e_{234} & =e_{2} \wedge e_{3} \wedge e_{4}
\end{align*}
$$

We can see advantage of geometric algebra, because we can use negative vector as invisible edge of solution.

First we present a projection of tesseract (4D cube). This representation is simple, because we use basis vectors $\left(e_{1} ; e_{2} ; e_{3} ; e_{4}\right)$. We can create trivector by right-oriented vector, because directed area is a cube. It means that its volume is equal to the volume of a cube. It is basic feature of trivectors in geometry algebra. But we must split 4D cube in two parts: top and bottom section. The result of 3 D projection for general point $0=[0,0,0,0]$ are cubes $C_{0 i} ; i \in\{1 . .4\}$ and cubes $C_{1 i} ; i \in\{1 . .4\}$ are for general point $1=[1,1,1,1]$.

Typical property of 4D cubes is different rotation of top and bottom sections. 3D projection preserves this property, because individual cubes are point symmetrical. For example trivector $e_{132}$ represents the cube $C_{14}$ and their point symmetrical cube is $C_{04}$ with trivector $-e_{132}=$ $-e_{1} \wedge e_{3} \wedge e_{2}=e_{1} \wedge e_{2} \wedge e_{3}$.


Figure 2: 3D projection of the oriented 4D cube

Let us choose the view vector so that $C_{04}, C_{02}$ are visible and $C_{03}, C_{01}$ are invisible. Cubes $C_{1 i}$ is mutually opposite to cubes $C_{0 i}$. Intersection in geometric algebra is defined on some blades $A, B$,

$$
A \cap B=A^{*} \cdot B
$$

where $A^{*}$ is the normal vector to the dual of bivector $A$. If we apply Theorem 3.2 , the final projection will have 12 -contour faces. So, result of the 3 D projection of the 4 D cube is dodacehedron, see Fig.(3).


Figure 3: Result of the 3D projection of the oriented 4D cube

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Consider 4D simplex, defined by the vectors $a_{1} ; a_{2} ; a_{3} ; a_{4}$ in space $\mathcal{G}_{4}$, such that trivectors generate volume of tetrahedron. If we create 3D projections of this body, we obtain tetrahedrons $S_{1}, S_{2}, S_{3}, S_{4}$ with orientation (9).


Figure 4: 3D projection of the oriented pentachoron
Let us choose the view trivector so that $S_{4}, S_{2}$ are visible and $S_{3}, S_{1}$ are invisible. We apply Theorem 3.2, final projection will have 6 -contour faces. So, the final 3D projection of the pentachoron is a pyramid Fig.(5).


Figure 5: General final 3D projection of the pentachoron

## 5 Conclusions

Geometric algebra facilitates representation of solids. We can found visible contour of this solid, by the help of projection of 4D solid to three-dimensional space and by application of visibility criterion.

## 6 Acknowledgements

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## Journal of Applied Matbematics

## A SUPER-ELLIPSE

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## 1. Introduction

The concept of exploded real numbers and the super-operations form the basis of our calculations.
The postulates and requirements of the concept of exploded real numbers were given in [1]. We may satisfy them in the following way:
The exploded of the $u \in R$ is given by

$$
\bar{u}=(\operatorname{sgn} u)\left(\frac{1}{2} \ln \frac{1+\{|u|\}}{1-\{|u|\}}+i[|u|]\right),
$$

where $[x]$ is the greatest integer number which is less than or equal $x \in R$ and $\{x\}=x-[x]$. So, the set of exploded numbers $\bar{R}$ is a proper subset of complex numbers. This model of exploded numbers was introduced by Szalay in [3].
If $u$ is an element of the open interval $(-1,1)$ then:

$$
\begin{equation*}
\bar{u}=\operatorname{areath} u=\frac{1}{2} \cdot \ln \frac{1+u}{1-u} \tag{1.1}
\end{equation*}
$$

Of course, any real number $x$ is exploded real number, too, given by the formula:

$$
\begin{equation*}
x=\overline{\operatorname{th} x} \quad x \in R \quad \operatorname{th} x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \tag{1.2}
\end{equation*}
$$

For any exploded real number $u$ we define its compressed $\underline{u}$ by the first inversion identity:

$$
\begin{equation*}
\overline{(\underline{u})}=u \quad u \in \bar{R} \tag{1.3}
\end{equation*}
$$

Denoting $x=\underline{u}$, (1.3) shows that $\bar{x}=u$, and we have the second inversion identity:

$$
\begin{equation*}
(\bar{x})=x \quad x \in R \tag{1.4}
\end{equation*}
$$

Using the above mentioned identities, the (1.2) gives:

$$
\begin{equation*}
\underline{x}=\text { th } x, \quad \text { for any } x \in R \tag{1.5}
\end{equation*}
$$

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The set of super-operation used to treat the problem of the super-ellipse are:

$$
\begin{array}{lll}
u+^{s} v=\underline{\underline{u}+\underline{v}} & u, v \in \bar{R} & \text { (super-addition), } \\
u-^{s} v=\underline{\underline{u}-v} & u, v \in \bar{R} & \text { (super-substraction), } \\
u 0^{s} v=\overline{\underline{u} \cdot \underline{v}} u, v \in \bar{R} & \text { (super-multiplication), } \\
\left.u l^{s} v=\overline{\left(\frac{\underline{u}}{\underline{v}}\right.}\right) u, v \in \bar{R} & \text { (super-division), } \\
\sqrt{u}^{s}=\overline{\sqrt{\underline{u}}} \quad u \in \bar{R} & \text { (super-square root). } \tag{1.10}
\end{array}
$$

An ordered algebraic structure for the set $\bar{R}$ by the super-operations was given in [3].

We also use the following operations based on (1.6) and (1.8), introduced by Szalay in [1], for any elements of $\overline{R^{3}}, U=\left(u_{1}, u_{2}, u_{3}\right)$ and $V=\left(v_{1}, v_{2}, v_{3}\right)$, and $c \in \bar{R}$ :

$$
\begin{array}{ll}
U+^{s} V=\left(u_{1}+{ }^{s} v_{1}, u_{2}+{ }^{s} v_{2}, u_{3}+{ }^{s} v_{3}\right) & \text { (super-addition) } \\
U \circ^{s} V=u_{1} \circ^{s} v_{1}+{ }^{s} u_{2} \circ^{s} v_{2}+{ }^{s} u_{3} \circ^{s} v_{3} & \text { (super-inner product) } \\
c \circ^{s} U=\left(c \circ^{s} u_{1}, c \circ^{s} u_{2}, c \circ^{s} u_{3}\right) & \text { (super-multiplication) } \\
U-^{s} V=U+^{s}\left(\overline{-1} \circ^{s} V\right) & \text { (super-difference) } \tag{1.14}
\end{array}
$$

The familiar three dimensional space

$$
\begin{equation*}
R^{3}=\{(u, v, w):=\overline{-1}<u<\overline{1} ; \overline{-1}<v<\overline{1} ; \overline{-1}<w<\overline{1}\}, \tag{1.15}
\end{equation*}
$$

with the rectangular coordinate system $u, v, w$ is an open cube in the exploded three dimensional space

$$
\begin{equation*}
\overline{R^{3}}=\{(u, v, w): u, v, w \in \bar{R}\} \tag{1.16}
\end{equation*}
$$

Considering a set $H \subseteq \overline{R^{3}}$, the subset

$$
\begin{equation*}
H_{b o x}=H \cap R^{3} \tag{1.17}
\end{equation*}
$$

is called the box-phenomenon of $H$. It is possible that a box-phenomenon is empty. Clearly, $\overline{R_{b o x}^{3}}=R^{3}$. Moreover, if $H \subseteq R^{3}$ then $H_{b o x}=H$.
The definition of the super-norm of super-Euklidean space $\overline{R^{k}}$ was introduced by Szalay in [1], using the concept of the super-inner product. By the definition, for any $U \in \overline{R^{3}}$, its super-norm is

$$
\begin{equation*}
\|U\|_{\overline{R^{3}}}=\sqrt[s]{U \circ^{s} U} \tag{1.18}
\end{equation*}
$$

and yields the identity

$$
\begin{equation*}
\|U\|_{\overline{R^{3}}}=\overline{\|\underline{U}\|_{R^{3}}} U \in \overline{R^{3}} \tag{1.19}
\end{equation*}
$$

## 2. The concept of the super-plane, super-cylinder and super-cone

The exploded set of a plane of the space $R^{3}$ is called a super-plane.
It is known that a plane of the space $R^{3}$ is characterized by the equation

$$
\begin{equation*}
\left(X-X_{0}\right) \cdot N=0 \tag{2.1}
\end{equation*}
$$

where $X_{0}, N \in R^{3}$ such that $\|N\|_{R^{3}}=1$. Denoting by $S$ the set of $X$ given by the equation (2.1) and considering the property of the super-operations (1.11)-(1.14) we have for the points of superplane $S$

$$
\begin{equation*}
\left(\bar{X}-{ }^{s} \overline{X_{0}}\right) o^{s} \bar{N}=0 \tag{2.2}
\end{equation*}
$$

So, denoting $\bar{X}=U, \overline{X_{0}}=U_{0}$ and $\bar{N}=M$ we have the equation of super-plane

$$
\begin{equation*}
\left(U-{ }^{s} U_{0}\right)_{0}{ }^{s} M=0 \tag{2.3}
\end{equation*}
$$

Where by the definition of super-norm ( see [1] ) we have that

$$
\|M\|_{\overline{R^{3}}}=\overline{1}
$$

Applying (1) with $X_{0}=(0,0,0)$ and $N=\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$ we have the plane

$$
\begin{equation*}
S=\left\{X=(x, y, z) \in R^{3}: x, y \in R \text { and } z=x+y\right\} \tag{2.4}
\end{equation*}
$$

Using $u=\bar{x}, v=\bar{y}, w=\bar{z}$ and applying the super-operations 1.6-1.8, definition (2.2) say that

$$
\begin{equation*}
\bar{S}=\left\{U=(u, v, w) \in \overline{R^{3}}: u, v \in \bar{R} \text { and } w=u+^{s} v\right\} \tag{2.5}
\end{equation*}
$$

is a super-plane.
Having a parameter $\mu$ such that $0<\mu<\overline{1}$ we can define the super-cylinder surface in the following way:

$$
\begin{equation*}
\bar{\Omega}=\left\{(u, v, w) \in \overline{R^{3}}: \sqrt{\left(u \circ^{s} u\right)+^{s}\left(v \circ^{s} v\right)^{s}}=\mu ; w \in \bar{R}\right\} \tag{2.6}
\end{equation*}
$$

We can see $|u| \leq \mu ;|v| \leq \mu$, so $u$ and $v$ are real numbers. This implies that $\bar{\Omega} \subset R^{3}$ if $w$ is real number too, this implies that $w<\overline{1}$.

The concept of the super-cone was given in [2].
We can define the super-cone surface, using a second parameter $\gamma \in \bar{R}$

$$
\begin{equation*}
\bar{\Lambda}=\left\{(u, v, w) \in \overline{R^{3}}: w=\left(\overline{1}-\sqrt{\left(u \circ^{s} u\right)+^{s}\left(v \circ^{s} v\right)^{s}} /^{s} \gamma\right) ; \gamma \in \bar{R} ; 0<\gamma\right\} \tag{2.7}
\end{equation*}
$$

## 3. The concept of the super-ellipse

Definition 1. For any pair $P_{1}\left(u_{1}, v_{1}, w_{1}\right), P_{2}\left(u_{2}, v_{2}, w_{2}\right) \in \overline{R^{3}}$, their super-distance is the exploded number

$$
\begin{equation*}
d_{\overline{R^{3}}}\left(P_{1}, P_{2}\right)=\overline{\left.d_{R^{3}} \underline{\left(\underline{P_{1}}, \underline{P_{2}}\right.}\right)} \tag{3.1}
\end{equation*}
$$

where $\underline{P_{1}}\left(\underline{u_{1}}, \underline{v_{1}}, \underline{w_{1}}\right), \underline{P_{2}}\left(\underline{u_{2}}, \underline{v_{2}}, \underline{w_{2}}\right) \in R^{3}$ and $d_{R^{3}}\left(\underline{P_{1}}, \underline{P_{2}}\right)$ is the Euklidean distance of $\underline{P_{1}}$ and $\underline{P_{2}}$.

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Definition 2. If $\sigma$ is a plane in $\overline{R^{3}}$ and $F_{l}\left(u_{l}, v_{l}, w_{l}\right), F_{r}\left(u_{r}, v_{r}, w_{r}\right) \in \sigma$, and $a \in \bar{R}$, satisfying the inequation $d_{\overline{R^{3}}}\left(F_{l}, F_{r}\right)<\overline{2}^{\circ} \circ^{s} a$, the super-ellipse is the set of $P \in \sigma$ points, which satisfy the equation

$$
\begin{equation*}
d_{\overline{R^{3}}}\left(P, F_{l}\right)+^{s} d_{\overline{R^{3}}}\left(P, F_{r}\right)=\overline{2} \circ^{s} a \tag{3.2}
\end{equation*}
$$

Theorem 1. For any point $P \in \sigma$ if and only if $\underline{P} \in \underline{\sigma}$,
and

$$
\begin{equation*}
d_{\overline{R^{3}}}\left(P, F_{l}\right)+d_{\overline{R^{3}}}\left(P, F_{r}\right)=\overline{2} \circ^{s} a \text { if and only if } d_{R^{3}}\left(\underline{P}, \underline{F_{l}}\right)+d_{R^{3}}\left(\underline{P}, \underline{F_{r}}\right)=2 \cdot \underline{a} \tag{3.4}
\end{equation*}
$$

is valid.
Proof. Applying Definition 1., the concept of super-operations and the familiar properties of traditional distance, the (3.3) is trivial. For the (3.4) by (3.1) and (3.2), we can write

$$
\begin{aligned}
& d_{\overline{R^{3}}}\left(P, F_{l}\right)+{ }^{s} d_{\overline{R^{3}}}\left(P, F_{r}\right)=\overline{2} o^{s} a \Leftrightarrow \overline{d_{R^{3}}}\left(\underline{P}, \underline{F_{l}}\right)+s \overline{d_{R^{3}}}\left(\underline{P}, \underline{F_{r}}\right)=\overline{2} o^{s} a \Leftrightarrow \\
& \Leftrightarrow \overline{d_{R^{3}}\left(\underline{P}, \underline{F_{l}}\right)+d_{R^{3}}\left(\underline{P}, \underline{F_{r}}\right)=\overline{2 \cdot \underline{a}} \Leftrightarrow d_{R^{3}}\left(\underline{P}, \underline{F_{l}}\right)+d_{R^{3}}\left(\underline{P}, \underline{F_{r}}\right)=2 \cdot \underline{a}, ~}
\end{aligned}
$$

Theorem 1. means that the super-ellipse is the exploded of the traditional ellipse.

## 4. The section between the super-plane and a super-cylinder surface

Now we investigate the super-ellipse, as the section between the super-plane defined by (2.5) and the super-cylinder surface defined by (2.6).
Now we fix $\mu=0.8$. Using (1.1), (1.6), (1.8) and (1.10) we have that the super-cylinder surface $\bar{\Omega}$ has the equation

$$
\begin{equation*}
\operatorname{areath}\left(\sqrt{(t h u)^{2}+(t h v)^{2}}\right)=0.8 \quad w \in \bar{R} \tag{4.1}
\end{equation*}
$$

Using $x=\underline{u}, y=\underline{v}$ and $z=\underline{w}$ we have the equation of a traditional cylinder-surface:

$$
\begin{equation*}
\sqrt{x^{2}+y^{2}}=\operatorname{th}(0.8), \quad z \in R \tag{4.2}
\end{equation*}
$$

We can say that the traditional cylinder surface defined by

$$
\begin{equation*}
\Omega=\left\{(x, y, z) \in R^{3}: \sqrt{x^{2}+y^{2}}=\text { th } 0.8 ; z \in R\right\} \tag{4.3}
\end{equation*}
$$

is the compresed of the super-cylinder surface $\bar{\Omega}$, and on the other hand, $\bar{\Omega}$ is the exploded of $\Omega$. The section between $\bar{\Omega}$ and $\bar{S}$ given by the equations (2.5) and (2.6) respectively is a super-ellipse, and the section between $\Omega$ and $S$ is a traditional ellipse. Using Theorem 1 we can see that the super-ellipse mentined above is the exploded of the traditional ellipse.
For the begining we calculate the coordinates of the points of the traditional ellipse, and their exploded ones are the points of the super-ellipse.
The end points of the axes are $\underline{P_{1}}, \underline{P_{2}}$ and $\underline{P_{3}}, \underline{P_{4}}$, respectively. The end points of the big axis $P_{1}$ and $P_{2}$ satisfy the equation $x=y$ and applying the equations (2.4) and (4.3) we have:

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$$
\begin{equation*}
\underline{P_{1}}\left(\frac{\operatorname{th} 0.8}{\sqrt{2}} ; \frac{\operatorname{th} 0.8}{\sqrt{2}} ; \sqrt{2} \cdot \operatorname{th} 0.8\right) \quad \underline{P_{2}}\left(-\frac{t h 0.8}{\sqrt{2}} ;-\frac{t h 0.8}{\sqrt{2}} ;-\sqrt{2} \cdot \operatorname{th} 0.8\right) \tag{4.4}
\end{equation*}
$$

The length of the big axis is the Euklidean distance:

$$
\begin{equation*}
\left.d_{\overline{R^{3}}} \underline{\left(P_{1}\right.}, \underline{P_{2}}\right)=2 \cdot \sqrt{3} \cdot \operatorname{th} 0.8=2 \cdot \underline{a} \tag{4.5}
\end{equation*}
$$

The centre of the super-ellipse is the point $O(0 ; 0 ; 0)$, and the straight line, which holds the $\underline{P}_{3}$ and $\underline{P_{4}}$ has the parametric equation:

$$
\begin{equation*}
x=t \quad z=0 \quad y=-t \tag{4.6}
\end{equation*}
$$

Considering $\underline{P_{3}}$ and $\underline{P_{4}}$ are the common points of the cylindre surface defined by (4.3) and the plane defined by (2.4), we can find the $\underline{P_{3}}$ and $\underline{P_{4}}$ coordinates:

$$
\begin{equation*}
\underline{P_{3}}\left(\frac{t h 0.8}{\sqrt{2}} ;-\frac{t h 0.8}{\sqrt{2}} ; 0\right) \quad \underline{P_{4}}\left(-\frac{t h 0.8}{\sqrt{2}} ; \frac{t h 0.8}{\sqrt{2}} ; 0\right) \tag{4.7}
\end{equation*}
$$

and the Euklidean distance:

$$
\begin{equation*}
\left.d_{\overline{R^{3}}} \underline{\left(P_{3}\right.}, \underline{P_{4}}\right)=2 \cdot \operatorname{th} 0.8=2 \cdot \underline{b} \tag{4.8}
\end{equation*}
$$

The linear eccentricity of the ellipse is:

$$
\begin{equation*}
\underline{c}=\sqrt{\underline{a}^{2}-\underline{b}^{2}}=\operatorname{th} 0.8 \cdot \sqrt{2} \tag{4.9}
\end{equation*}
$$

We calculate the coordinates of focus-points $\underline{F_{r}}$ and $\underline{F_{l}}$, using $\overrightarrow{O \underline{F_{r}}}=\vec{e} \cdot \underline{c}$ and $\overrightarrow{O F_{l}}=-\vec{e} \cdot \underline{c}$, where $\vec{e}\left(\frac{1}{\sqrt{6}} ; \frac{1}{\sqrt{6}} ; \frac{2}{\sqrt{6}}\right)$ is the unit vector pointing in the big axis direction.
Hence $\underline{F_{r}}$ and $\underline{F_{l}}$ have the coordinates:

$$
\begin{equation*}
\underline{F_{r}}\left(\frac{t h 0.8}{\sqrt{3}} ; \frac{t h 0.8}{\sqrt{3}} ; 2 \cdot \frac{t h 0.8}{\sqrt{3}}\right) \quad \underline{F_{l}}\left(-\frac{t h 0.8}{\sqrt{3}} ;-\frac{t h 0.8}{\sqrt{3}} ;-2 \cdot \frac{t h 0.8}{\sqrt{3}}\right) \tag{4.10}
\end{equation*}
$$

The points $P_{1}, P_{2}, P_{3}, P_{4}, F_{r}$ and $F_{l}$ are the exploded of $\underline{P_{1}}, \underline{P_{2}}, \underline{P_{3}}, \underline{P_{4}}, \underline{F_{r}}$ and $\underline{F_{l}}$ respectively, and their coordinates are:

$$
\begin{align*}
& P_{1}\left(\frac{\overline{\operatorname{th} 0.8}}{\sqrt{2}} ; \frac{\overline{\operatorname{th} 0.8}}{\sqrt{2}} ; \overline{\sqrt{2}} \cdot \operatorname{th} 0.8\right) \\
& P_{2}\left(-\frac{\overline{t h 0.8}}{\sqrt{2}} ;-\frac{\overline{t h 0.8}}{\sqrt{2}} ;-\overline{\sqrt{2}} \cdot \operatorname{th} 0.8\right) \\
& P_{3}\left(\frac{\overline{t h 0.8}}{\sqrt{2}} ;-\frac{\overline{t h 0.8}}{\sqrt{2}} ; 0\right)  \tag{4.11}\\
& P_{4}\left(-\frac{\overline{t h 0.8}}{\sqrt{2}} ; \frac{\overline{t h 0.8}}{\sqrt{2}} ; 0\right) \\
& F_{r}\left(\overline{\frac{t h 0.8}{\sqrt{3}}} ; \frac{\overline{t h 0.8}}{\sqrt{3}} ; \overline{2 \cdot \frac{t h 0.8}{\sqrt{3}}}\right) \\
& F_{r}\left(-\frac{\overline{t h 0.8}}{\sqrt{3}} ;-\frac{\overline{t h 0.8}}{\sqrt{3}} ;-\overline{-2 \cdot \frac{t h 0.8}{\sqrt{3}}}\right)
\end{align*}
$$

An important question is the super-ellipse is a plane-curve or a space-curve? It is obvious, that if we choose four points of the super-ellipse and they are not on the same plane, the super-ellipse is a space-curve. Easy to see that four points are on the same plane, the determinant

$$
\operatorname{det}\left(P_{1}, P_{2}, P_{3}, P_{4}\right)=\left|\begin{array}{llll}
x_{1} & y_{1} & z_{1} & 1  \tag{4.12}\\
x_{2} & y_{2} & z_{2} & 1 \\
x_{3} & y_{3} & z_{3} & 1 \\
x_{4} & y_{4} & z_{4} & 1
\end{array}\right|=0
$$

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where $P_{i}\left(x_{i} ; y_{i} ; z_{i}\right) \quad i=1 . .4$ are the chosen points.
In the case of the super-cylinder the end-points of the axis of the super-ellipse are on the same plane, but the super-ellipse is a space-curve. To prove this we choose the points satisfying the condition $w=u=0.8 ; v=0\left(P_{5}\right.$ point $), v=w=0.8 ; u=0\left(P_{6}\right.$ point $)$ or the condition $u=w=-0.8 ; v=0\left(P_{7}\right.$ point $)$. The determinant has the following values:

$$
\begin{align*}
& \operatorname{det}\left(P_{1}, P_{2}, P_{3}, P_{4}\right)=0 \\
& \operatorname{det}\left(P_{1}, P_{2}, P_{5}, P_{6}\right)=0.9104818948 \\
& \operatorname{det}\left(P_{3}, P_{4}, P_{5}, P_{6}\right)=0  \tag{4.13}\\
& \operatorname{det}\left(P_{3}, P_{1}, P_{5}, P_{6}\right)=-0.4552409472 \\
& \operatorname{det}\left(P_{1}, P_{7}, P_{5}, P_{6}\right)=0.9104818956
\end{align*}
$$

The results shows that in the case of the super-cylinder surface the super-ellipse is symmetrical to the $O(0 ; 0 ; 0)$ point, but its points are not on the same plane, therefore the super-ellipse is a spacecurve.
The form of the super-ellipse can be seen in Figure 1 and Figure 2.


Figure 1.


Figure 2.

The base of the super-cylindre surface is a super-circle and it can be seen in Figure 3.


Figure 3.
An other question is how to fix the parameter $\mu$, as

$$
\begin{equation*}
E_{b o x}=E \tag{4.14}
\end{equation*}
$$

where $E \subseteq \overline{R^{3}}$ is the super-ellipse, and $E_{b o x}=E \cap R^{3}$ is the box phenomenon of $E$. We can see that the third coordinate of $P_{1}$ is $\operatorname{areath}(\sqrt{2} \cdot t h \mu)$ and the condition (4.14) is satisfyed if

$$
\begin{equation*}
\sqrt{2} \cdot \operatorname{th} \mu<1 \tag{4.15}
\end{equation*}
$$

and therefore:

$$
\begin{equation*}
\mu<\operatorname{areath}\left(\frac{1}{\sqrt{2}}\right) \approx 0.881 \tag{4.16}
\end{equation*}
$$

## 5. The section between the super - plane and a super-cone surface

Similarly we can treat the problem of the super-ellipse in the case of a super-cone surface. The section between a super-plane and a super-cone surface can be seen in Figure 4. and Figure 5.


Figure 4.


Figure 5.

Now we consider a section between a traditional cone surface defined by

$$
\begin{equation*}
\Lambda=\left\{(x, y, z) \in R^{3}: 0 \leq z \leq 1-\sqrt{x^{2}+y^{2}} / c ; 0<c ; c \in R ;(x, y) \in C\right\} \tag{5.1}
\end{equation*}
$$

and a traditional plane, defined by the equation (2.4).
If

$$
\begin{equation*}
\alpha<\varphi \tag{5.2}
\end{equation*}
$$

the section-line is a traditional ellipse.
We fix $\gamma=0.3$ and we introduce the notation $c=\operatorname{th}(0.3)$ and therefore the condition (5.2) is satisfied. The $\underline{P}$ points, which satisfy the equations

$$
z \leq 1-\sqrt{x^{2}+y^{2}} / c \quad \text { and } \quad z=x+y
$$

are situated on a traditional ellipse. These points form the compressed of the super-ellipse or we can say that the exploded of the ellipse mentioned above is a super-ellipse.

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The end points of the axes are $\underline{P_{1}}, \underline{P_{2}}$ and $\underline{P_{3}}, \underline{P_{4}}$, respectively. The end points of the big axis $P_{1}$ and $P_{2}$ satisfy the equation $x=y$ and similarly with the super-cylinder calculations, we have:

$$
\begin{align*}
& \underline{P_{1}}\left(\frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)+\sqrt{2}}, \frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)+\sqrt{2}}, 2 \cdot \frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)+\sqrt{2}}\right)  \tag{5.3}\\
& \underline{P_{2}}\left(\frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)-\sqrt{2}}, \frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)-\sqrt{2}}, 2 \cdot \frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)-\sqrt{2}}\right) \tag{5.4}
\end{align*}
$$

The length of the big axis is the Euklidean distance

$$
\begin{equation*}
\left.d_{R^{3}} \underline{\left(P_{1}\right.}, \underline{P_{2}}\right)=\operatorname{th}(0.3) \cdot \frac{2 \cdot \sqrt{3}}{1-2 \cdot(\operatorname{th}(0.3))^{2}}=2 \cdot \underline{a} \tag{5.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\underline{a}=\operatorname{th}(0.3) \cdot \frac{\sqrt{3}}{1-2 \cdot(\operatorname{th}(0.3))^{2}} \tag{5.6}
\end{equation*}
$$

The centre-point $\underline{C}$ of the ellipse has the coordinates:

$$
\begin{equation*}
\underline{C}\left(\frac{t h^{2}(0.3)}{2 \cdot t h^{2}(0.3)-1} ; \frac{t h^{2}(0.3)}{2 \cdot t h^{2}(0.3)-1} ; \frac{2 \cdot t h^{2}(0.3)}{2 \cdot t h^{2}(0.3)-1}\right) \tag{5.7}
\end{equation*}
$$

We can find the end-points $\underline{P_{3}}$ and $\underline{P_{4}}$ as the common points of the straight line $f$ which holds the $\operatorname{axis} \underline{P_{3}} \underline{P_{4}}$, and the cone-surface defined by (5.1). The equation of the line $f$ has the parametric form:

$$
\begin{align*}
& x=\frac{t h^{2}(0.3)}{2 \cdot t h^{2}(0.3)-1}+t \\
& y=\frac{t h^{2}(0.3)}{2 \cdot t h^{2}(0.3)-1}-t  \tag{5.8}\\
& z=2 \cdot \frac{t h^{2}(0.3)}{2 \cdot \operatorname{th}^{2}(0.3)-1}
\end{align*}
$$

We introduce the notation:

$$
\begin{equation*}
k=\frac{t h^{2}(0.3)}{2 \cdot t h^{2}(0.3)-1} \tag{5.9}
\end{equation*}
$$

Using the parametric equation (5.8) and the equation of the traditional plane we find

$$
t= \pm \sqrt{-\frac{k}{2}}
$$

And the coordinates of the $\underline{P_{3}}$ and $\underline{P_{4}}$ :

$$
\begin{equation*}
\underline{P_{3}}\left(k+\sqrt{-\frac{k}{2}} ; k-\sqrt{-\frac{k}{2}} ; 2 \cdot k\right) \quad \underline{P_{4}}\left(k-\sqrt{-\frac{k}{2}} ; k+\sqrt{-\frac{k}{2}} ; 2 \cdot k\right) \tag{5.11}
\end{equation*}
$$

The length of the axis $\underline{P_{3}} \underline{P_{4}}$ is the Euklidean distance:

$$
\begin{equation*}
\left.d_{R^{3}} \underline{\left(P_{1}\right.}, \underline{P_{2}}\right)=2 \cdot \sqrt{-k}=2 \cdot \underline{b} \tag{5.12}
\end{equation*}
$$

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Using $\underline{a}=\operatorname{th}(0.3) \cdot \frac{\sqrt{3}}{1-2 \cdot(\operatorname{th}(0.3))^{2}}=-\frac{k \cdot \sqrt{3}}{\operatorname{th}(0.3)}$ and $\underline{b}=\sqrt{-k}$ we can find the linear eccentricity of the ellipse :

$$
\underline{c}=\sqrt{\underline{a}^{2}-\underline{b}^{2}}=\frac{|k|}{\operatorname{th}(0.3)} \cdot \sqrt{3-(\operatorname{th}(0.3))^{2}}
$$

An interesting question is the calculus of the focus points $F_{r}$ and $F_{l}$ of the super-ellipse as the exploded points of $\underline{F_{r}}$ and $\underline{F_{l}}$, respectively. Easy to see that $\overrightarrow{C \underline{F_{r}}}=\underline{c} \cdot \overrightarrow{e_{v}}$ and $\overrightarrow{C \underline{F_{l}}}=-\underline{c} \cdot \overrightarrow{e_{v}}$ where $\overrightarrow{e_{v}}\left(\frac{1}{\sqrt{6}} ; \frac{1}{\sqrt{6}} ; \frac{2}{\sqrt{6}}\right)$ is the unit vector pointing in the direction of the big axis. Applying $\underline{C}(k, k, 2 \cdot k)$ we can find the focus-points coordinates:

$$
\underline{F_{r}}\left(k+\frac{c}{\sqrt{6}} ; k+\frac{\underline{c}}{\sqrt{6}} ; 2 \cdot k+\frac{2 \cdot \underline{c}}{\sqrt{6}}\right) \quad \underline{F_{l}}\left(k-\frac{\underline{c}}{\sqrt{6}} ; k-\frac{c}{\sqrt{6}} ; 2 \cdot k-\frac{2 \cdot \underline{c}}{\sqrt{6}}\right)
$$

The points $P_{1}, P_{2}, P_{3}, P_{4}, F_{r}$ and $F_{l}$ have the coordinates:

$$
\begin{align*}
& P_{1}\left(\overline{\frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)+\sqrt{2}}}, \overline{\frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)+\sqrt{2}}}, \overline{2 \cdot \frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)+\sqrt{2}}}\right) \\
& P_{2}\left(\overline{\frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)-\sqrt{2}}}, \frac{\frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)-\sqrt{2}}}{2 \cdot \frac{\operatorname{th}(0.3)}{2 \cdot \operatorname{th}(0.3)-\sqrt{2}}}\right) \\
& P_{3}\left(\overline{k+\sqrt{-\frac{k}{2}}} ; k-\sqrt{-\frac{k}{2}} ; \overline{2 \cdot k}\right) \quad P_{4}\left(\overline{k-\sqrt{-\frac{k}{2}}} ; k+\sqrt{-\frac{k}{2}} ; \overline{2 \cdot k}\right) \\
& F_{r}\left(\overline{k+\frac{c}{\sqrt{6}}} ; \overline{k+\frac{c}{\sqrt{6}}} ; 2 \cdot k+\frac{2 \cdot \frac{c}{\sqrt{6}}}{\sqrt{6}}\right) \quad \quad F_{l}\left(\overline{k-\frac{c}{\sqrt{6}}} ; k-\frac{c}{\sqrt{6}} ; 2 \cdot k-\frac{2 \cdot \frac{c}{\sqrt{6}}}{\sqrt{6}}\right)
\end{align*}
$$

In the case of the super-cone surface, the determinant mentioned in (4.12) has the following values:

$$
\begin{align*}
& \operatorname{det}\left(P_{1}, P_{2}, P_{3}, P_{4}\right)=-0.03524785707 \\
& \operatorname{det}\left(P_{5}, P_{2}, P_{3}, P_{4}\right)=-0.01687359172 \\
& \operatorname{det}\left(P_{5}, P_{1}, P_{3}, P_{4}\right)=0.00190693914  \tag{5.16}\\
& \operatorname{det}\left(P_{1}, P_{5}, P_{6}, P_{2}\right)=-0.01478905776 \\
& \operatorname{det}\left(P_{3}, P_{5}, P_{6}, P_{4}\right)=0
\end{align*}
$$

where $P_{5}$ and $P_{6}$ are the points of the super-ellipse situated on the $x O y$ plane, having the coordinates:

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The results show the only points situated on the same plane are $P_{3}, P_{4}, P_{5}$ and $P_{6}$. Otherwise we can see the super-ellipse is not a plane-curve in the traditional sense, most of it, the straight lines which hold the end-points of its axes are not situated on the same plane.

In the case of the super-cone surface we can answer the question how to fix the parameter $\gamma$, as $E_{b o x}=E$. We can see that the third coordinate of $\underline{P_{2}}$ is $2 \cdot \frac{t h \gamma}{2 \cdot t h \gamma-\sqrt{2}}$, and has to satisfy the condition:

$$
\begin{equation*}
2 \cdot \frac{t h \gamma}{2 \cdot t h \gamma-\sqrt{2}}>-1 \tag{5.18}
\end{equation*}
$$

and therefore:

$$
\gamma<\text { areath }\left(\frac{\sqrt{2}}{4}\right) \approx 0.369
$$

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# SPECIAL VECTOR FIELDS ON SPACES WITH AFFINE CONNECTION 

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#### Abstract

In this paper we study special vector fields on spaces with affine connection, which satisfy $\nabla \boldsymbol{\varphi}=\varrho \boldsymbol{I d}+\sigma \boldsymbol{A}+\omega \boldsymbol{\varphi}$. These vector fields are denoted as $\boldsymbol{\varphi}(\boldsymbol{A})$-vector field. We investigate the properties of these fields. We give an example of a special case. We also present a result about $\boldsymbol{\varphi}(\boldsymbol{A})$ - vector fields and geodesic mappings. These vector fields are preserved under geodesic mappings. Key words and phrases. space with affine connection, special vector field, geodesic mapping. Mathematics Subject Classification. Primary 53B05, 53B30.


## 1 Introduction

Assume a symmetric affine connection $\nabla$ on an $n$ dimensional manifold and moreover an affine structure $\boldsymbol{A}$ in the form of a tensor of type $\binom{1}{1}$. Such manifolds are called spaces with affine connection, denoted by $A_{n}$.

Definition 1.1 A $\boldsymbol{\varphi}(\boldsymbol{A})$-vector field is a vector field $\varphi$ on $A_{n}$ which satisfies the condition

$$
\begin{equation*}
\nabla \boldsymbol{\varphi}=\rho \mathbf{I} \boldsymbol{d}+\sigma \boldsymbol{A}+\omega \boldsymbol{\varphi} \tag{1}
\end{equation*}
$$

where $\boldsymbol{I d}$ is the identity operator, $\rho, \sigma$ are functions, and $\omega$ is a linear operator on $A_{n}$. If $\omega \equiv 0$ then $\varphi$ is canonical.
$\boldsymbol{\varphi}(\boldsymbol{A})$ - vector fields represent a generalisation of the $\boldsymbol{\varphi}(\boldsymbol{R i c})$ - vector fields introduced and studied in detail in $[3,4,5]$.

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We note that $\boldsymbol{\varphi}(\boldsymbol{I d})$ - vector fields are the well known torse-forming vector fields [11, 14, 15, $19,20,23,25]$, and canonical $\boldsymbol{\varphi}(\boldsymbol{I d})$-vector fields are the concircular (especially - recurrent, convergent vector fields) $[9,11,14,15,19,20,22,23,24]$.

Many geometrical problems are connected with the existence of the vector fields (1). For example, there are conformal mappings onto Einstein spaces $[1,12,14,18]$. In this case tensor $\boldsymbol{A}$ is the Ricci operator, i.e. the Ricci tensor of type (1,1).

Equation (1) can be written in the following form

$$
\nabla_{X} \boldsymbol{\varphi}=\rho X+\sigma \boldsymbol{A}(X)+\omega(X) \boldsymbol{\varphi}
$$

for all tangent vectors $X \in T A_{n}$.
In a local coordinate system $x=\left(x^{1}, \ldots, x^{n}\right)$ we can express this as

$$
\begin{equation*}
\varphi_{, i}^{h}=\rho \delta_{i}^{h}+\sigma A_{i}^{h}+\varphi^{h} \omega_{j}, \tag{2}
\end{equation*}
$$

where $\varphi^{h}, \delta_{i}^{h}, A_{i}^{h}, \omega_{i}$ are components of $\boldsymbol{\varphi}, \boldsymbol{I d}, \boldsymbol{A}, \omega$, and the comma "," denotes the covariant derivative of $\nabla$.

## 2 On canonical $\varphi(A)$ - vector fields

It is known that a vector field $\tilde{\boldsymbol{\varphi}}$ is collinear to $\boldsymbol{\varphi}$ if $\tilde{\boldsymbol{\varphi}}=\alpha \cdot \boldsymbol{\varphi}$, where $\alpha$ is a function on $A_{n}$.
Evidently, if $\boldsymbol{\varphi}$ is a $\boldsymbol{\varphi}(\boldsymbol{A})$ - vector field on $A_{n}$, then $\tilde{\boldsymbol{\varphi}}$ is $\tilde{\boldsymbol{\varphi}}(\boldsymbol{A})$ - vector field on $A_{n}$, too. In this case for $\alpha \neq 0$ holds

$$
\nabla \tilde{\boldsymbol{\varphi}}=(\alpha \rho) \boldsymbol{I} \boldsymbol{d}+(\alpha \sigma) \boldsymbol{A}+\tilde{\boldsymbol{\varphi}}(\omega+\nabla \alpha / \alpha) .
$$

It is clear that if $\omega+\nabla \alpha / \alpha=0$, then $\tilde{\boldsymbol{\varphi}}$ is canonical.
If the linear form $\omega$ is gradient-like (locally gradient-like) then a function $\alpha$ exists so that $\tilde{\varphi}$ is canonical (respective locally canonical).

In the following we study canonical $\boldsymbol{\varphi}(\boldsymbol{A})$ - vector fields, which satisfy

$$
\begin{equation*}
\nabla \varphi=\rho \boldsymbol{I} \boldsymbol{d}+\sigma \boldsymbol{A} \tag{3}
\end{equation*}
$$

Theorem 2.1 Let $A_{n}$ be a space with an affine connection $\nabla$ and let $\boldsymbol{A}$ be an affine structure on $A_{n}$ for which

$$
\begin{equation*}
\operatorname{Rank}\|\boldsymbol{A}-\alpha \boldsymbol{I d}\|>1 \tag{4}
\end{equation*}
$$

Canonical $\boldsymbol{\varphi}(\boldsymbol{A})$-vector fields on $A_{n}$ depend on no more then $n+2$ real parameters.
This theorem follows from the next one.
Theorem 2.2 Let $A_{n}$ be a space with an affine connection $\nabla$ and let $\boldsymbol{A}$ be an affine structure on $A_{n}$ for which (4) holds.

The canonical $\boldsymbol{\varphi}(\boldsymbol{A})$-vector fields on $A_{n}$ are generated in a coordinate neighbourhood $U(x) \subset$ $A_{n}$ by the set of functions $\varphi^{h}(x), \rho(x)$ and $\sigma(x)$ which satisfy the following system of linear equations of Cauchy type

$$
\begin{align*}
\varphi_{, i}^{h} & =\rho \delta_{i}^{h}+\sigma A_{i}^{h}  \tag{5}\\
\rho_{, i} & =\sigma \stackrel{1}{T}_{i}+\varphi^{\alpha} \stackrel{2}{T}_{\alpha i}  \tag{6}\\
\sigma_{, i} & =\sigma \stackrel{3}{T}_{i}+\varphi^{\alpha} \stackrel{4}{T}_{\alpha i}, \tag{7}
\end{align*}
$$

where $\stackrel{a}{T}$ are tensor fields which are defined on $A_{n}$ and depend only on the connection $\nabla$ and the affinor structure $\boldsymbol{A}$.

Proof. In the coordinate neighbourhood $U(x) \subset A_{n}$ equation (3) is written as (5). It is known that the Ricci identities for a vector field $\varphi$ have the form

$$
\begin{equation*}
\varphi_{, j k}^{h}-\varphi_{, k j}^{h}=-\varphi^{\alpha} R_{\alpha j k}^{h}, \tag{8}
\end{equation*}
$$

where $R_{i j k}^{h}$ are components of the curvature tensor on $A_{n}$.
The integrability conditions of (5) follow from (8) and have the form

$$
\begin{equation*}
-\varphi^{\alpha} R_{\alpha j k}^{h}=\delta_{j}^{h} \rho_{k}-\delta_{k}^{h} \rho_{j}+A_{j}^{h} \sigma_{k}-A_{k}^{h} \sigma_{j}+A_{j k}^{h} \sigma \tag{9}
\end{equation*}
$$

where $\rho_{i}=\rho_{, i}, \sigma_{i}=\sigma_{, i}$, and $A_{j k}^{h}=A_{j, k}^{h}-A_{k, j}^{h}$.
By analyzing (9) under the condition (4) we can make sure (6) and (7).
As we can easily see, it follows from the theory of partial differential equations that in a space $A_{n}$ the equations (5), (6), and (7) can have a unique solution for arbitrary $n+2$ initial conditions

$$
\varphi^{h}\left(x_{0}\right)=\stackrel{0}{\varphi}^{h}, \quad \rho\left(x_{0}\right)=\stackrel{0}{\rho}, \quad \sigma\left(x_{0}\right)=\stackrel{0}{\sigma},
$$

which are satisfied in the point $x_{0} \in A_{n}$.

## 3 A simple example of a $\varphi($ Ric $)$ - vector field.

Special $\boldsymbol{\varphi}(\boldsymbol{A})$ - vector fields with $\rho=0, \omega=0$ and $\mathbf{A}=\boldsymbol{R i c}$ were introduced in $[3,4]$ and denoted as $\boldsymbol{\varphi}(\boldsymbol{R i c})$ - vector fields. Our example is a non-isotropic generalization of an equidistant space, motivated by the Kasner vacuum metric in general relativity [10]. For simplicity we have restricted ourselves to a $2+1$ dimensional Riemannian space with diagonal metric in the coordinates $x^{1}, x^{2}, x^{3}$,

$$
\begin{equation*}
\mathrm{d} s^{2}=-\left(\mathrm{d} x^{1}\right)^{2}+f\left(x^{1}\right)\left(\mathrm{d} x^{2}\right)^{2}+g\left(x^{1}\right)\left(\mathrm{d} x^{3}\right)^{2} \tag{10}
\end{equation*}
$$

where $f$ and $g$ are $C^{2}$ functions of only the first coordinate.
We assume a presumptive $\boldsymbol{\varphi}($ Ric $)$ - vector field in the $x^{1}$ direction,

$$
\varphi^{i}=\left(\varphi^{1}\left(x^{1}\right), 0,0\right),
$$

depending only on the coordinate $x^{1}$, too.
After some calculations based on $\nabla \boldsymbol{\varphi}=\boldsymbol{R i c}$, we obtain a one-parameter family of solutions for the metric components

$$
\begin{equation*}
f\left(x^{1}\right)=\left(x^{1}\right)^{2 \cos \theta}, \quad g\left(x^{1}\right)=\left(x^{1}\right)^{2 \sin \theta} \tag{11}
\end{equation*}
$$

with the parameter $\theta$ conveniently restricted by $\theta \in\langle 0,2 \pi\rangle$.
The non-vanishing components of the Ricci tensor are

$$
\begin{aligned}
& R_{11}=(1-\cos \theta-\sin \theta)\left(x^{1}\right)^{-2}, \quad R_{22}=\cos \theta(1-\cos \theta-\sin \theta)\left(x^{1}\right)^{-2(1-\cos \theta)}, \\
& R_{33}=(1-\cos \theta)(\sin \theta-1-\cos \theta)\left(x^{1}\right)^{-2(1-\sin \theta)}
\end{aligned}
$$

and the scalar curvature is

$$
\begin{equation*}
R=-2(1-\cos \theta)(1-\sin \theta)\left(x^{1}\right)^{-2} . \tag{12}
\end{equation*}
$$

In this space the $\varphi(\boldsymbol{R i c})$ - vector field is given by the component

$$
\begin{equation*}
\varphi^{1}=(1-\cos \theta-\sin \theta)\left(x^{1}\right)^{-1} . \tag{13}
\end{equation*}
$$

The Riemannian space with the above metric (10) with components (11) provides a nontrivial example of a space with a $\boldsymbol{\varphi}(\boldsymbol{R i c})$ - vector field, which is neither equidistant, nor an Einstein space, nor a space with constant curvature. The functional form (11) of the components is of Kasner type, but, of course, the metric is not a vacuum solution.

The vector field $\varphi \neq 0$ if and only if $\theta \neq 0, \pi / 2$. As special cases for $\theta=\pi / 4,5 \pi / 4$ the spaces are equidistant. From the expression (12) we see that in the generic case the metric (10), (11) displays a curvature singularity at $x_{1}=0$, like the Kasner metric.

## $4 \varphi(A)$-vector fields and geodesic mappings

We recall that the diffeomorphism $f$ between spaces with afinne connection $A_{n}$ and $\bar{A}_{n}$ is called a geodesic mapping of $A_{n}$ onto $\bar{A}_{n}$ if $f$ maps any geodesic curve in $A_{n}$ onto a geodesic curve in $\bar{A}_{n}$, see $[2,6,16,17,18,22]$.

Let $f: A_{n} \rightarrow \bar{A}_{n}$ be a geodesic mapping. We suppose that $A_{n}=\left(M_{n}, \nabla\right)$ and $\bar{A}_{n}=\left(M_{n}, \bar{\nabla}\right)$, so $f$ is a mapping from $M$ to $M$. Then for the afinne connections $\nabla$ and $\bar{\nabla}$ holds the Levi-Civita equation

$$
\begin{equation*}
\bar{\nabla}_{X} X=\nabla_{X} X+2 \psi(X) \cdot X \tag{14}
\end{equation*}
$$

for all tangent vectors $X, \psi$ is a 1 -form on the manifolds $M_{n}$.
Equation (14) is equivalent to the form

$$
\begin{equation*}
\bar{\nabla}_{X} Y=\nabla_{X} Y+\psi(X) \cdot Y+\psi(Y) \cdot X \tag{15}
\end{equation*}
$$

It is well known that under geodesic mappings Killing vectors, and also concircular vector fields are preserved.

Similar properties hold also for $\boldsymbol{\varphi}(\boldsymbol{A})$ - vector fields:

Theorem 4.1 Assume the vector field $\boldsymbol{\varphi}$ is a $\boldsymbol{\varphi}(\boldsymbol{A})$-vector field on the manifold with affine connection $A_{n}=\left(M_{n}, \nabla\right)$. The mapping $f: A_{n} \rightarrow \bar{A}_{n}=\left(M_{n}, \bar{\nabla}\right)$ is a geodesic mapping between $A_{n}$ and $\bar{A}_{n}$.

Then $\boldsymbol{\varphi}$ is a $\boldsymbol{\varphi}(\boldsymbol{A})$-vector field with respect to the connection $\bar{\nabla}$.
Proof. Evidently by application of equation (15) we get

$$
\begin{equation*}
\bar{\nabla}_{X} \boldsymbol{\varphi}=\nabla_{X} \boldsymbol{\varphi}+\psi(X) \cdot \boldsymbol{\varphi}+\psi(\boldsymbol{\varphi}) \cdot X \tag{16}
\end{equation*}
$$

After substitution from the formula (1) we get the relation

$$
\begin{equation*}
\bar{\nabla}_{X} \boldsymbol{\varphi}=(\rho+\psi(\boldsymbol{\varphi})) \cdot X+\sigma A+(\omega(X)+\psi(X)) \cdot \boldsymbol{\varphi} . \tag{17}
\end{equation*}
$$

This equation can briefly be written in the shape:

$$
\bar{\nabla} \boldsymbol{\varphi}=\bar{\rho} \boldsymbol{I} \boldsymbol{d}+\bar{\sigma} \boldsymbol{A}+\bar{\omega} \boldsymbol{\varphi}
$$

where $\bar{\rho}=\rho+\psi(\boldsymbol{\varphi}), \bar{\sigma}=\sigma$, and $\bar{\omega}=\omega+\psi$.

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# HOLOMORPHICALLY PROJECTIVE MAPPINGS PRESERVING THE EINSTEIN TENSOR 

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#### Abstract

In this paper there are discussed the holomorphically projective mappings which preserved the Einstein tensor. We proved that the tensor of $h$-concircular curvature is invariant under Einstein tensor-preserving holomorphically projective mappings. Key words and phrases. holomorphically projective mapping, Kähler space, Einstein tensor. Mathematics Subject Classification. Primary 53B30, 53B35.


## 1 Introduction

From the very beginning, the theory of holomorphically projective mappings attracted attention by a wide scale of possibilities for applications, not only in geometry itself, but also as a useful tool of modeling various processes in mechanics and physics [1]-[24].

If we distinguish some class of mappings between spaces from a fixed class, a natural questions arises, what objects and properties of spaces are preserved, invariant, under all mappings under consideration.

As far as invariant objects under holomorphically projective mappings are concerned, let us mention generalized Thomas' parameters and the tensor of holomorphically projective curvature. To mention some invariant properties, note that the class of spaces of constant curvature and the class of Einstein spaces are closed under holomorphically projective mappings.

In this paper, we examine nontrivial holomorphically projective mappings of pseudo-Kähler spaces preserving the Einstein tensor. We prove that the tensor of $h$-concircular curvature is an invariant of holomorphically projective mappings. Further, we examine some geometric properties of such spaces.

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## 2 Basic concepts

A (pseudo-) Riemannian space $K_{n}$ is called a Kähler space if it is endowed, besides a metric tensor $g$, with an affinor structure $F$ satisfying the following relations [16, 18, 19, 20]

$$
F^{2}=-\operatorname{Id}, \quad g(X, F X)=0, \quad \nabla F=0
$$

Here $X$ are all tangent vectors of $T K_{n}$ and $\nabla$ is a connection of $K_{n}$. The structure $F$ is a complex structure.

It is known, that the diffeomorphism $f$ between Kähler spaces $K_{n}$ and $\bar{K}_{n}$ is called a holomorphically projective mapping, if $f$ maps any analytical planar curve of $K_{n}$ onto a analytical planar curve of $\bar{K}_{n}$. Due to the diffeomorphism $f$, we can suppose that $\bar{M}=M$, where $M$ is "common" manifolds on which the metrics $g$ and $\bar{g}$ and the complex structure $F$ on $K_{n}$ and $\bar{K}_{n}$ are defined.

A holomorphically projective mapping $f$ from $K_{n}$ onto $\bar{K}_{n}$ preserves the structures and is characterized by the following condition

$$
\begin{equation*}
(\bar{\nabla}-\nabla)(X, Y)=\psi(X) Y+\psi(Y) X-\psi(F X) F Y-\psi(F Y) F X \tag{1}
\end{equation*}
$$

for any vector fields $X, Y$, where $\bar{\nabla}$ and $\nabla$ are affine connections of $K_{n}$ and $\bar{K}_{n} \psi$ is a linear form.

The mapping from $K_{n}$ onto $\bar{K}_{n}$ is holomorphically projective if the equations hold

$$
\begin{align*}
\nabla_{Z} \bar{g}(X, Y)= & 2 \psi(Z) \bar{g}(X, Y)+\psi(X) \bar{g}(Y, Z)+\psi(Y) \bar{g}(X, Z) \\
& +\bar{\psi}(X) \bar{F}(Y, Z)+\bar{\psi}(Y) \bar{F}(X, Z) \tag{2}
\end{align*}
$$

where $\nabla$ is Levi-Civita connection of $K_{n}, \psi$ is a linear form and $X, Y, Z$ are tangent vectors, $\bar{\psi}(X)=e \psi(F X), \bar{F}(X, Z)=\bar{g}(X, F Z)$. If $\psi=0$, then a holomorphically projective mapping is called trivial or affine. The equations (2) we rewrite in local coordinates:

$$
\begin{equation*}
\bar{g}_{i j, k}=2 \psi_{k} \bar{g}_{i j}+\psi_{i} \bar{g}_{j k}+\psi_{j} \bar{g}_{i k}-\bar{\psi}_{i} \bar{F}_{j k}-\bar{\psi}_{j} \bar{F}_{i k}, \tag{3}
\end{equation*}
$$

where $\bar{g}_{i j}(x) \quad \bar{\psi}_{k}(x)$ and $\bar{F}_{i j}$ are components of $\bar{g}, \psi, \bar{F}$ and "," is a covariant derivative on $K_{n}, x=\left(x^{1}, x^{2}, \ldots, x^{n}\right)$ is a point of coordinate neighbourhood $U \subset M$. Equations (2) and (3) hold when $K_{n}$ and $\bar{K}_{n} \in C^{1}$, i.e. $g_{i j}(x)$ and $\bar{g}_{i j}(x) \in C^{1}$ in any coordinate neighbourhood $U$.

The following conditions are necessary for a holomorphically projective mapping:

$$
\begin{gather*}
\bar{R}_{i j k}^{h}=R_{i j k}^{h}+\psi_{i j} \delta_{k}^{h}-\psi_{i k} \delta_{j}^{h}+\psi_{i \alpha} F_{j}^{\alpha} F_{k}^{h}-\psi_{i \alpha} F_{k}^{\alpha} F_{j}^{h}+2 \psi_{j \alpha} F_{k}^{\alpha} F_{i}^{h},  \tag{4}\\
\bar{R}_{i j}=R_{i j}+(n-1) \psi_{i j} . \tag{5}
\end{gather*}
$$

Here $R_{i j k}^{h}$ is the Riemannian curvature tensor, $R_{i j}$ is the Ricci tensor, and

$$
\psi_{i j}=\psi_{i, j}-\psi_{i} \psi_{j}+\bar{\psi}_{i} \bar{\psi}_{j} .
$$

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On the other hand, necessary and sufficient condition for existence of non-trivial holomorphically projective mappings of the given pseudo-Kähler space onto pseudo-Kähler spaces is existence of a solution for the system of equations [7, 13, 16, 18, 19]

$$
\begin{gather*}
a_{i j, k}=\lambda_{i} g_{j k}+\lambda_{j} g_{i k}+\bar{\lambda}_{i} F_{j k}+\bar{\lambda}_{j} F_{i k},  \tag{6}\\
n \lambda_{i, j}=\mu g_{i j}+a_{\alpha i} R_{j}^{\alpha}-a_{\alpha \beta} R_{\cdot{ }_{i j} \cdot}^{\beta},  \tag{7}\\
(n-1) \mu_{, k}=2(n+1) \lambda_{\alpha} R_{k}^{\alpha}+a_{\alpha \beta}\left(2 R_{\cdot k, \cdot}^{\alpha}-R_{\cdot, \cdot k}^{\alpha \beta}\right) \tag{8}
\end{gather*}
$$

with respect to a regular symmetric tensor $a_{i j}$, a co-vector $\lambda_{i} \neq 0$ and a function $\mu$. Here
 and $g^{i j}$ are elements of the matrix inverse to $g_{i j}$.

According to the known solutions of the above system of differential equations the metrics of the resulting image spaces under holomorphically projective mappings can be determined from the equations [13, 19]:

$$
\begin{gather*}
a_{i j}=e^{2 \psi} \bar{g}^{\alpha \beta} g_{\alpha i} g_{\beta j} ;  \tag{9}\\
\lambda_{i}=-e^{2 \psi} \psi_{\alpha} \bar{g}^{\alpha \beta} g_{\beta i} . \tag{10}
\end{gather*}
$$

The important invariants under holomorphically projective mappings are the generalized Thomas' parameters

$$
\begin{equation*}
\bar{T}_{i j}^{h}=T_{i j}^{h} ; \quad T_{i j}^{h}=\Gamma_{i j}^{h}-\frac{1}{n+2}\left(\delta_{i}^{h} \Gamma_{j \alpha}^{\alpha}+\delta_{j}^{h} \Gamma_{i \alpha}^{\alpha}-F_{i}^{h} F_{j}^{\beta} \Gamma_{\beta \alpha}^{\alpha}-F_{j}^{h} F_{i}^{\beta} \Gamma_{\beta \alpha}^{\alpha}\right) \tag{11}
\end{equation*}
$$

and the tensor of holomorphically projective curvature

$$
\begin{equation*}
\bar{P}_{i j k}^{h}=P_{i j k}^{h} ; \quad P_{i j k}^{h}=R_{i j k}^{h}-\frac{1}{n+2}\left(\delta_{k}^{h} R_{j i}-\delta_{j}^{h} R_{k i}-F_{k}^{h} F_{j}^{\alpha} R_{\alpha i}+F_{j}^{h} F_{k}^{\alpha} R_{\alpha i}+2 F_{i}^{h} F_{j}^{\alpha} R_{\alpha k}\right) \tag{12}
\end{equation*}
$$

## 3 Basic equations for Einstein tensor-preserving geodesic mappings

We call a holomorphically projective mapping Einstein tensor-preserving if it satisfies:

$$
\begin{equation*}
\bar{E}_{i j}=E_{i j}, \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
E_{i j}=R_{i j}-\frac{R}{n} g_{i j} \tag{14}
\end{equation*}
$$

is the Einstein tensor and $R=R_{\alpha \beta} g^{\alpha \beta}$ is the scalar curvature.
If this is the case, the deformation tensor for the Ricci tensor takes the form:

$$
\begin{equation*}
T_{i j}=\bar{R}_{i j}-R_{i j}=\frac{\bar{R}}{n} \bar{g}_{i j}-\frac{R}{n} g_{i j} . \tag{15}
\end{equation*}
$$

On the other hand, accounting (5) we obtain

$$
\begin{equation*}
T_{i j}=\bar{R}_{i j}-R_{i j}=(n-1) \psi_{i j} . \tag{16}
\end{equation*}
$$

Comparing we get:

$$
\begin{equation*}
\psi_{i j}=\frac{\bar{R}}{n(n-1)} \bar{g}_{i j}-\frac{R}{n(n-1)} g_{i j} \tag{17}
\end{equation*}
$$

Substituting the last expression into (4) and using the notation

$$
\begin{equation*}
Y_{i j k}^{h}=R_{i j k}^{h}-\frac{R}{n(n-1)}\left(\delta_{k}^{h} g_{i j}-\delta_{j}^{h} g_{i k}+F_{k}^{h} F_{i}^{\alpha} g_{\alpha j}-F_{j}^{h} F_{i}^{\alpha} g_{\alpha k}+2 F_{i}^{h} F_{j}^{\alpha} g_{\alpha k}\right) \tag{18}
\end{equation*}
$$

(and similarly with bar) we find

$$
\begin{equation*}
\bar{Y}_{i j k}^{h}=Y_{i j k}^{h} . \tag{19}
\end{equation*}
$$

Here $Y_{i j k}^{h}$ are components of the tensor of $h$-concircular curvature, where $Y$ is an analog of the tensor of concircular curvature [13, 17, 18, 19, 24].

Hence we have proved:
Theorem 3.1 The tensor of h-concircular curvature is invariant under Einstein tensor-preserving holomorphically projective mappings.

Let us apply covariant differentiation to the formula (10):

$$
\begin{equation*}
\lambda_{i, j}=-e^{2 \psi} \psi_{\alpha, j} \bar{g}^{\alpha \beta} g_{\beta i}+e^{2 \psi} \psi_{\alpha} \psi_{\beta} \bar{g}^{\alpha \beta} g_{j i}+e^{2 \psi} \psi_{j} \psi_{\alpha} \bar{g}^{\alpha \beta} g_{\beta i} . \tag{20}
\end{equation*}
$$

By (9) and (17), we get

$$
\begin{equation*}
\lambda_{i, j}=\mu g_{i j}+\frac{R}{n(n-1)} a_{i j}, \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu=e^{2 \psi}\left(\psi_{\alpha} \psi_{\beta} \bar{g}^{\alpha \beta}-\frac{\bar{R}}{n(n-1)}\right) . \tag{22}
\end{equation*}
$$

Obviously using (9), (10), from (20) and (21) we get (17), and consequently also (13), hence we have proved:

Theorem 3.2 A pseudo-Riemannian space admits an Einstein tensor-preserving holomorphically projective mapping if and only if the conditions (6), (21) and (22) are satisfied.

We say that a Kähler space $K_{n}$ belongs to the class $K_{n}[B]$ if it admits a geodesic mapping and the corresponding vector satisfies $[12,14,18]$

$$
\begin{equation*}
\lambda_{i, j}=\mu g_{i j}+B a_{i j} \tag{23}
\end{equation*}
$$

for some function $B$.
So we have actually proved that a Kähler space $K_{n}$ admitting Einstein tensor-preserving holomorphically projective mappings belongs to the class $K_{n}[B]$ where $B=-\frac{R}{n(n-1)}$.

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# ON HOLOMORPHICALLY PROJECTIVE MAPPINGS ONTO ALMOST HERMITIAN SPACES 

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#### Abstract

In this paper we consider holomorphically projective mappings $f$ from (pseudo-) Riemannian spaces $V_{n}$ onto non-Kähler almost Hermitian spaces $\bar{H}_{n}$. If in the point $x_{o}$ it satisfies the following initial condition $\bar{g}\left(f\left(x_{o}\right)\right)=k \cdot g\left(x_{o}\right)$, where $g$ and $\bar{g}$ are the metrics of $V_{n}$ and $\bar{H}_{n}$, then $f$ is homothetic. Key words and phrases. holomorphically projective mapping, equiaffine space, affineconnected space, (pseudo-) Riemannian space, almost Hermitian space. Mathematics Subject Classification. Primary 53B35, 53C24, 53C55.


## 1 Introduction

Diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the current main directions in differential geometry. A large number of papers are devoted to geodesic, quasigeodesic, almost geodesic, holomorphically projective and other mappings (see [1]-[6], [12]-[27]). On the other hand, one line of thought is now the most important one, namely, the investigation of special affine-connected, Riemannian, Kählerian and Hermitian spaces.

In this paper, we present some new results obtained for holomorphically projective mappings from equiaffine spaces $A_{n}$ onto almost Hermitian spaces $\bar{H}_{n}$, which is not Kählerian.

By a Hermitian space $H_{n}$ we mean a (pseudo-) Riemannian manifold together with an affinor structure $F_{i}^{h}$ satisfying the conditions

$$
\begin{equation*}
F_{\alpha}^{h} F_{i}^{\alpha}=-\delta_{i}^{h}, \quad g_{\alpha(i} F_{j)}^{\alpha}=0 \tag{1}
\end{equation*}
$$

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hold. Here $g_{i j}$ is the metric tensor on $H_{n}, \delta_{i}^{h}$ is the Kronecker symbol and $(i, j)$ denotes a symmetrization without normalization.

A natural classification containg 16 types of Hermitian spaces has been done by A. Gray and L.M. Hervella [7]. Kählerian spaces $K_{n}$ are special cases of Hermitian spaces, which have a covariantly constant structure $F_{i}^{h}$.

In many papers holomorphically projective mappings and transformations of Hermitian spaces $H_{n} \rightarrow \bar{H}_{n}$ are studied (for example see [?], [12], [14], [17], [22], [24], [27]). These are special cases of $F_{1}$-planar mappings. In [15], [17], $F_{1}$-planar mappings from the space $A_{n}$ with affine connection onto a Riemannian space $\bar{V}_{n}$ are defined and studied. These are characterized w.r.t. a common coordinate system $x$ by the following equations
a) $\bar{\Gamma}_{i j}^{h}(x)=\Gamma_{i j}^{h}(x)+\delta_{(i}^{h} \psi_{j)}+F_{(i}^{h} \varphi_{j)}$,
b) $\bar{g}_{\alpha(i} F_{j)}^{\alpha}=0$,
where $\Gamma_{i j}^{h}$ and $\bar{\Gamma}_{i j}^{h}$ are the objects of affine connection on $A_{n}$ and $\bar{V}_{n}$, respectively, $\bar{g}_{i j}$ is the metric tensor of $\bar{V}_{n}, \psi_{i}(x), \varphi_{i}(x)$ are covectors, and $F_{i}^{h}(x)\left(\operatorname{Rank}\left\|F_{i}^{h}-\rho \delta_{i}^{h}\right\|>1\right)$ is the affinor structure on $A_{n}$ and $\bar{V}_{n}$.

Equations (2) are equivalent to the equations
a) $\bar{g}_{i j, k}=2 \psi_{k} \bar{g}_{i j}+\psi_{(i} \bar{g}_{j) k}+\varphi_{(i} \bar{F}_{j) k}$,
b) $\bar{g}_{\alpha(i} F_{j)}^{\alpha}=0$,
where $\bar{F}_{i j} \stackrel{\text { def }}{=} \bar{g}_{i \alpha} F_{j}^{\alpha}$. Here and in what follows, comma denotes covariant derivative on $A_{n}$.
In [15] it is proved that a general solution of the system (3) for a given space $A_{n}$ and a given structure $F_{i}^{h}$ depends on finitely many parameters.

An $F_{1}$-planar mapping is called $F_{2}$-planar if the covector $\psi_{i}$ is a gradient, i.e. $\psi_{i}=\mathrm{d} \psi / \mathrm{d} x^{i}$, and $F_{3}$-planar if $\psi_{i}=\varphi_{\alpha} F_{i}^{\alpha}$. If $A_{n}$ is an equiaffine space, then an $F_{3}$-planar mapping is $F_{2}$-planar. The following Theorem holds [15]:

Theorem 1.1 An equiaffine space $A_{n}$ admits $F_{3}$-planar mappings onto $\bar{V}_{n}$ if and only if $a$ regular symmetric tensor $a^{i j}$ and a vector $\xi^{i}$ satisfie the following equations:

$$
\begin{align*}
& \text { a) } a^{i j}{ }_{, k}=\xi^{\alpha} F_{\alpha}^{(i} \delta_{k}^{j)}+\xi^{(i} F_{k}^{j)}, \\
& \text { b) } a^{\alpha(i} F_{\alpha}^{j)}=0 . \tag{4}
\end{align*}
$$

Solutions of (3) and (4) are connected by relations

$$
\begin{align*}
& a^{i j}=\mathrm{e}^{-2 \psi} \bar{g}^{i j}, \\
& \xi^{i}=-\mathrm{e}^{-2 \psi} \bar{g}^{i \alpha} \varphi_{\alpha} \tag{5}
\end{align*}
$$

where $\left\|\bar{g}^{i j}\right\|=\left\|\bar{g}_{i j}\right\|^{-1}$.

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## 2 Holomorphically projective mappings

An $F_{3}$-planar mapping from a space $A_{n}$ with an affine connection onto a Hermitian space $\bar{H}_{n}$, for which formulas (2) are satisfied and $F_{i}^{h}$ is the almost complex structure of $\bar{H}_{n}$, is called a holomorphically projective mapping.

For this mapping it holds:

$$
\begin{equation*}
F_{i, j}^{h}=F_{i \mid j}^{h}, \tag{6}
\end{equation*}
$$

where "," and "।" are the covariant derivatives in $A_{n}$ and $\bar{H}_{n}$ respectively.
In the following we will study holomorphically projective mappings from an equiaffine space $A_{n}$ onto a non Kählerian space $\bar{H}_{n}$, in which $F_{i \mid j}^{h} \neq 0$. In this case (6) implies

$$
\begin{equation*}
F_{i, j}^{h} \neq 0 . \tag{7}
\end{equation*}
$$

Study the second differential prolongation of the conditions (4b) was proved following Theorem [19]:
Theorem 2.1 Let $A_{n}$ be an equiaffine space with an affine connection, $F$ be a covariantly non constant almost complex structure (i.e., an affinor $F_{i}^{h}$ such that $F_{\alpha}^{h} F_{i}^{\alpha}=-\delta_{i}^{h}$ and $F_{i, j}^{h} \neq 0$ ). Then $A_{n}$ admits a holomorphically projective mapping onto a non-Kählerian Hermitian space $\bar{H}_{n}$, if and only if the following system of linear differential equations of Cauchy type is solvable with respect to the unknown functions $a^{i j}$ :

$$
\begin{equation*}
a^{i j}{ }_{, k}=\xi^{\alpha} F_{\alpha}^{(i} \delta_{k}^{j)}+\xi^{(i} F_{k}^{j)} \tag{8}
\end{equation*}
$$

where $\xi^{i}=a^{\alpha \beta} T_{\alpha \beta}^{i}$. Further, the matrix ( $a^{i j}$ ) satisfies $\operatorname{det}\left\|a^{i j}\right\| \neq 0$ and the algebraic conditions

$$
\begin{align*}
& a^{i j}=a^{j i} \\
& a^{i j}=a^{\alpha \beta} F_{\alpha}^{i} F_{\beta}^{j} . \tag{9}
\end{align*}
$$

Here $T_{\alpha \beta}^{i}$ is tensor which are explicitly expressed using the objects defined in $A_{n}$, i.e. affine connection $A_{n}$ and affinor $F_{i}^{h}$.

This Theorem is a generalization of the results in [?], [12]-[17], [24]-[26].
The system (8) does not have more than one solution for the initial Cauchy conditions $a^{i j}\left(x_{o}\right)=a_{o}^{i j}$ under the conditions (9). Therefore the general solution of (8) does not depend on more than $N_{o}=n^{2} / 4$ parameters. The question of existence of a solution of (8) leads to the study of integrability conditions, which are linear equations w.r. to the unknowns variables $a^{i j}$ with objects from the space $A_{n}$.

## 3 Holomorphically projective mappings with initial conditions

It is known, that Riemannian manifolds have equiaffine connection, hence the Theorem 2.1 holds also in the case if $A_{n}$ is (pseudo-) Riemannian manifolds with the metric tensor $g$.

We proved following:

Theorem 3.1 Let $V_{n}$ be a (pseudo-) Riemannian space, $F$ be a covariantly non constant almost complex structure, and let $f$ be a holomorphically projective mapping from the (pseudo-) Riemannian space $V_{n}$ onto a non-Kählerian Hermitian space $\bar{H}_{n}$ with the condition $\bar{g}\left(\bar{x}_{o}\right)=k \cdot g\left(x_{o}\right)$, where $g$ and $\bar{g}$ are metrics of $V_{n}$ and $\bar{H}_{n}, \bar{x}_{o}=f\left(x_{o}\right), k \in \mathbb{R}$.

Then $f$ is a homothetic mapping, i.e.

$$
\begin{equation*}
\bar{g}=k \cdot g, k=\text { const. } \tag{10}
\end{equation*}
$$

Proof. Let us suppose that the assumptions of Theorem 3.1 hold on neighbourhood $U$. Equations (3) and (4) hold and we get from them the system of Cauchy type (8).

This set is the closed system of partial differential equations of the Cauchy type with respect to unknown functions $a^{i j}(x)$.

For the initial condition for $x_{o} \in U$

$$
\begin{equation*}
\bar{g}_{i j}\left(x_{o}\right)=k \cdot g_{i j}\left(x_{o}\right), \tag{11}
\end{equation*}
$$

which is firstly equivalent to

$$
\begin{equation*}
\bar{g}^{i j}\left(x_{o}\right)=k^{*} \cdot g^{i j}\left(x_{o}\right) \tag{12}
\end{equation*}
$$

where $g^{i j}$ are components of the inverse matrix of metric tensor $g$, and, moreower, accepted the formula (9), this initial condition is equivalent to

$$
\begin{equation*}
a^{i j}\left(x_{o}\right)=k^{*} \cdot g^{i j}\left(x_{o}\right) \tag{13}
\end{equation*}
$$

there is no more that one unique solution of equation (8).
On the other hand,

$$
a^{i j}=k^{*} \cdot g^{i j}, \quad k^{*}=\text { const },
$$

is a trivial solution of equations (8) and the initial conditions of (13) is satisfied. This solution, accepted the formula (9) is equivalent to

$$
\bar{g}=k \cdot g
$$

where $k$ is a function on $V_{n}$, this means, $f$ is conformal mapping between $V_{n}$ and $\bar{H}_{n}$.
We substitute into the formula (3a) and after analysis we determine, that $k$ is a constant. It means that a mapping $f$ is homothetic.

This Theorem is a generalization of the results in [2], [3], [?].

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# ON HOLOMORPHICALLY PROJECTIVE MAPPINGS WITH CERTAIN INITIAL CONDITIONS 

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#### Abstract

In this paper we studied holomorphically projective mapping $f$ between (pseudo -) Kähler spaces $K_{n}$ and $\bar{K}_{n}$ with the folowing initial condition $\bar{g}\left(f\left(x_{o}\right)\right)=k \cdot g\left(x_{o}\right)$, where $g$ and $\bar{g}$ are the metrics of $K_{n}$ and $\bar{K}_{n}$. We proved if in the point $x_{o} \in V_{n}$ the tensor of the holomorphically projective curvature does not vanish, then $f$ is homothetic. This statement holds as well as for hyperbolical Kähler spaces.


Key words and phrases. holomorphically projective mapping, initial conditions, Kähler space, hyperbolical Kähler spaces.

Mathematics Subject Classification. Primary 53B35, 53C24, 53C55.

## 1 Introduction

Many monographs and papers are devoted to the theory of geodesic and holomorphically projective mappings, see [1]- [19]. Geodesic and holomorphically projective mappings were studied with a complementary condition for proportional metrics displayed spaces, in a certain subset of the points $[1,2,3]$, see $[8,9,11,12]$. It turns out that, even under this weaker conditions, the mapping is homothetic, i.e. $\bar{g}=$ const $\cdot g$.

We prove that under certain circumstances a condition for proportional metrics holds only for a single pair of points $x_{o} \mapsto f\left(x_{o}\right)$.

We suppose that the metrics studied on Kähler spaces $K_{n}$ have generally signature metrics, i.e. Kähler space in our sense, is classical Kähler or (pseudo-) Kähler. We talk about classical Kähler spaces or (pseudo-) Kähler spaces, as well as hyperbolical Kähler spaces.

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## 2 Main properties of holomorphically projective mappings

A Riemannian space $K_{n}$ is called a Kähler space if it is endowed, besides a metric tensor $g$, with an affinor structure $F$ satisfying the following relations $[9,12,13,14]$

$$
F^{2}=e \operatorname{Id}, \quad e= \pm 1, \quad g(X, F X)=0, \quad \nabla F=0
$$

Here $X$ are all tangent vectors of $T K_{n}$ and $\nabla$ is a connection of $K_{n}$. The structure $F$ is a complex structure. If $e=-1$ then $K_{n}$ is a classical Kähler space or a (pseudo-) Kähler spaces. If $e=1$ then $K_{n}$ is a hyperbolical Kähler space.

It is known, that the diffeomorphism $f$ between Kähler spaces $K_{n}$ and $\bar{K}_{n}$ is called a holomorphically projective mapping, if $f$ maps any analytical planar curve of $K_{n}$ onto a analytical planar curve of $\bar{K}_{n}$. Due to the diffeomorphism $f$, we can suppose that $\bar{M}=M$, where $M$ is "common" manifolds on which the metrics $g$ and $\bar{g}$ and the complex structure $F$ on $K_{n}$ and $\bar{K}_{n}$ are defined.

A holomorphically projective mapping $f$ from $K_{n}$ onto $\bar{K}_{n}$ preserves the structures and is characterized by the following condition

$$
\begin{equation*}
(\bar{\nabla}-\nabla)(X, Y)=\psi(X) Y+\psi(Y) X+e \psi(F X) F Y+e \psi(F Y) F X \tag{1}
\end{equation*}
$$

for any vector fields $X, Y$, where $\bar{\nabla}$ and $\nabla$ are affine connections of $K_{n}$ and $\bar{K}_{n} \psi$ is an 1-form.
If $\psi=0$, then a holomorphically projective mapping is called trivial or affine.
The mapping from $K_{n}$ onto $\bar{K}_{n}$ is holomorphically projective if the equations hold

$$
\begin{align*}
\nabla_{Z} \bar{g}(X, Y)= & 2 \psi(Z) \bar{g}(X, Y)+\psi(X) \bar{g}(Y, Z)+\psi(Y) \bar{g}(X, Z) \\
& +\bar{\psi}(X) \bar{F}(Y, Z)+\bar{\psi}(Y) \bar{F}(X, Z) \tag{2}
\end{align*}
$$

where $X, Y, Z$ are tangent vectors, $\bar{\psi}(X)=e \psi(F X), \bar{F}(X, Z)=\bar{g}(X, F Z)$. The equations (2) we rewrite in local coordinates:

$$
\begin{equation*}
\bar{g}_{i j, k}=2 \psi_{k} \bar{g}_{i j}+\psi_{i} \bar{g}_{j k}+\psi_{j} \bar{g}_{i k}+e \bar{\psi}_{i} \bar{F}_{j k}+e \bar{\psi}_{j} \bar{F}_{i k}, \tag{3}
\end{equation*}
$$

where $\bar{g}_{i j}(x), \psi_{k}(x), \bar{\psi}_{k}(x)$ and $F_{i j}$ are components of $\bar{g}, \psi, \bar{\psi}, F$ and "," is a covariant derivative on $K_{n}, x=\left(x^{1}, x^{2}, \ldots, x^{n}\right)$ is a point of coordinate neighbourhood $U \subset M$. Equations (2) and (3) hold when $K_{n}$ and $\bar{K}_{n} \in C^{1}$, i.e. $g_{i j}(x)$ and $\bar{g}_{i j}(x) \in C^{1}$ in any coordinate neighbourhood $U$.

The tensor of holomorphically projective curvature of the Kähler space $K_{n}$ has a following invariant form:

$$
\begin{align*}
& P(X, Y, Z)=R(X, Y) Z-\frac{1}{n+2}(\operatorname{Ric}(X, Y) \cdot Z-\operatorname{Ric}(X, Z) \cdot Y  \tag{4}\\
& +e \operatorname{Ric}(X, F Y) \cdot F X-e \operatorname{Ric}(X, F Z) \cdot F Y-2 e \operatorname{Ric}(F Y, Z) \cdot F X),
\end{align*}
$$

or locally

$$
\begin{equation*}
P_{i j k}^{h}=R_{i j k}^{h}-\frac{1}{n+2}\left(\delta_{k}^{h} R_{j i}-\delta_{j}^{h} R_{k i}+e F_{j}^{\alpha} R_{\alpha i}-e F_{k}^{\alpha} R_{\alpha i}-2 e F_{j}^{\alpha} R_{\alpha k}\right), \tag{5}
\end{equation*}
$$

where $R_{i j k}^{h}$ and $R_{i j}=R_{i j \alpha}^{\alpha}$ are components of Riemannian and Ricci tensors, respectively. The tensor of the holomorphically projective curvature is invariant under holomorphically projective mappings $K_{n} \rightarrow \bar{K}_{n}$, i.e. $P=\bar{P}$.

## 3 Fundamental equations of holomorphically projective mapping

Domashev and Mikeš [4] for (elliptical) Kähler spaces, and Kurbatova [6] for hyperbolical Kähler spaces, proved that the equation (3) is equivalent to (see $[9,12,13,14]$ )

$$
\begin{equation*}
a_{i j, k}=\lambda_{i} g_{j k}+\lambda_{j} g_{i k}-e \lambda_{\alpha} F_{i}^{\alpha} F_{j}^{\beta} g_{\beta k}-e \lambda_{\alpha} F_{j}^{\alpha} F_{i}^{\beta} g_{\beta k} \tag{6}
\end{equation*}
$$

whereas

$$
a_{i j}=e^{2 \psi} \bar{g}^{\alpha \beta} g_{\alpha i} g_{\beta j}, \quad \lambda_{i}=e^{2 \Psi} \bar{g}^{\alpha \beta} \psi_{\alpha} g_{\beta j}
$$

$\bar{g}^{i j}$ is the inverse matrix to $\bar{g}_{i j}$ and $\Psi$ is a function which generates a gradient like a vector $\psi(X)=X \Psi$.

In paper [7, 6], see [9, 12, 13, 14], from equation (6) it follows that a Kähler space $K_{n}$ admits holomorphically projective mappings onto $\bar{K}_{n}$ if only if the following system of differential equations with covariant derivatives of Cauchy type has a solution with respect to the symmetric tensor $a_{i j}(x), \operatorname{det}\left(a_{i j} \neq 0\right)$, the covector $\lambda_{i}(x)$ and the function $\mu(x)$ :

$$
\begin{align*}
\text { a) } a_{i j, k} & =\lambda_{i} g_{j k}+\lambda_{j} g_{i k}-e \lambda_{\alpha} F_{i}^{\alpha} F_{j}^{\beta} g_{\beta k}-e \lambda_{\alpha} F_{j}^{\alpha} F_{i}^{\beta} g_{\beta k} \\
\text { b) } n \lambda_{i, j} & =\mu g_{i j}+a_{i \alpha} R_{j}^{\alpha}-a_{\alpha \beta} R_{. i j .}^{\alpha \beta} .  \tag{7}\\
\text { c) } \quad \mu_{, i} & =2 \lambda_{\alpha} R_{i}^{\alpha} .
\end{align*}
$$

where $R_{. i j}^{h k}=R_{i j \alpha}^{h} g^{\alpha h}, R_{i}^{h}=R_{i \alpha} g^{\alpha h}, R_{i j k}^{h}$ and $R_{i j}$ are components of the Riemannian and Ricci tensors.

The system (7) has no more than one solution for initial conditions in the point $x_{o}$ :

$$
\begin{equation*}
a_{i j}\left(x_{o}\right)=a_{i j}^{o}, \quad \lambda_{i}\left(x_{o}\right)=\lambda_{i}^{o}, \quad \mu\left(x_{o}\right)=\mu^{o} . \tag{8}
\end{equation*}
$$

Evidently for

$$
\begin{equation*}
a_{i j}\left(x_{o}\right)=k^{*} \cdot g_{i j}\left(x_{o}\right), \quad \lambda_{i}\left(x_{o}\right)=0, \quad \mu\left(x_{o}\right)=0 \tag{9}
\end{equation*}
$$

the initial condition corresponds to a trivial solution $a=k^{*} \cdot g, \quad k^{*}=$ const on all $K_{n}$. It means, that $K_{n}$ and $\bar{K}_{n}$ are homothetic, i.e. $\bar{g}=k \cdot g, k=$ const.

In the work by al Lamy, Škodová, Mikeš [15] there was proved in the neighbourhood on which the tensor $P$ is non vanishing it is possible to express the vector field $\lambda_{i}$ as a linear function of components $a_{i j}$ with coefficients which are defined by the objects of the space $K_{n}$.

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In principle this result follows as well as from the study of a degree of holomorphically projective mapping $[6,8,14]$.

It means that in the neighbourhood $U \subset K_{n}$ where $P(x) \neq 0$ equations of Cauchy type with respect only to unknown functions $a_{i j}(x)$ hold

$$
\begin{equation*}
a_{i j, k}=\lambda_{i} g_{j k}+\lambda_{j} g_{i k}+e \lambda_{\alpha} F_{i}^{\alpha} F_{j}^{\beta} g_{\beta k}+e \lambda_{\alpha} F_{j}^{\alpha} F_{i}^{\beta} g_{\beta k} \tag{10}
\end{equation*}
$$

where $\lambda_{i}=\Omega_{i}^{\alpha \beta}(x) a_{\alpha \beta}, \Omega_{i}^{\alpha \beta}(x)$ are objects of $K_{n}$. It is evident that the solution

$$
\begin{equation*}
\bar{g}_{i j}=k \cdot g_{i j}, \quad k=\mathrm{const} \tag{11}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
a_{i j}=k^{*} \cdot g_{i j}, k^{*}=\mathrm{const} \tag{12}
\end{equation*}
$$

## 4 Holomorphically projective mapping with the condition $\bar{g}\left(x_{0}\right)=k \cdot g\left(x_{0}\right)$

We proved following:
Lemma 4.1 Let $K_{n}=(M, g, F)$ and $\bar{K}_{n}=(M, \bar{g}, F)$, are Kähler spaces of the class $C^{3}, n \geq 3$, and let in coordinate neighbourhood $U \subset M$, the tensor of holomorphically projective curvature $P(x) \not \equiv 0, \forall x \in U$.

If in the point $x_{o} \in U$ it satisfies $a=k \cdot g$, and spaces $K_{n}$ and $\bar{K}_{n}$ have the same analytical planar curves then metrics $g$ and $\bar{g}$ on $U$ are homothetics, i.e. $\bar{g}(x)=k \cdot g(x)$, for all $x \in U$.

Proof. Let us suppose that the assumptions of Lemma 4.1 hold on neighbourhood $U$. Equations (3) and (6) hold and we get from them the system of Cauchy type (7).

This set is the closed system of partial differential equations of the Cauchy type with respect to unknown functions $a_{i j}(x)$.

For the initial condition for $x_{o} \in U$

$$
\begin{equation*}
\bar{g}_{i j}\left(x_{o}\right)=k \cdot g_{i j}\left(x_{o}\right), \tag{13}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
a_{i j}(x)=k^{*} \cdot g_{i j}(x), \tag{14}
\end{equation*}
$$

there is no more that one unique solution.
On the other hand, $a=k^{*} \cdot g, k^{*}=$ const, is a trivial solution of equations (7) and the initial conditons of (9) is satisfied. The solution of $a=k^{*} \cdot g$, is equivalent to $\bar{g}=k \cdot g$.
The Lemma implies:
Theorem 4.2 Let $f$ be a holomorphically projective mapping between Kähler spaces $K_{n}$ and $\bar{K}_{n}$ with the condition $\bar{g}\left(\bar{x}_{o}\right)=k \cdot g\left(x_{o}\right)$, where $g$ and $\bar{g}$ are metrics of $K_{n}$ and $\bar{K}_{n}, \bar{x}_{o}=f\left(x_{o}\right), k \in \mathbb{R}$. If the tensor of holomorphically projective curvature does not vanish in the point $x_{o} \in K_{n}$ then $f$ is a homothetic mapping, i.e.

$$
\begin{equation*}
\bar{g}=k \cdot g, k=\text { const } . \tag{15}
\end{equation*}
$$

Proof. Let $f$ be a holomorphically projective mapping between (pseudo-)Kähler spaces $K_{n}$ and $\bar{K}_{n}$. We suppose $K_{n}, \bar{K}_{n} \in C^{3}$ and $\bar{g}\left(f\left(x_{0}\right)=k \cdot g\left(x_{0}\right)\right)$ Because $P\left(x_{o}\right) \neq 0$, then there exists neighbourhood $U$ at the point $x_{o}$, so that $P(x) \not \equiv 0$, for all points $x \in U$. It follows from 4.1 that there is only one solution of Levi-Civita equation in the form $\bar{g}(x)=k \cdot g(x), \forall x \in U, k=$ const, and also

$$
\begin{equation*}
a(x)=k^{*} \cdot g(x), \forall x \in U, k^{*}=\text { const. } \tag{16}
\end{equation*}
$$

It means, that in the neighbourhood $U$, the set of equations (7) has the solution:

$$
\begin{equation*}
a_{i j}(x)=k \cdot g_{i j}(x), \lambda_{i}(x)=0, \mu(x)=0, \text { for } x \in U \tag{17}
\end{equation*}
$$

These conditions guarantee that the system of equations in the point $x_{0}$ has the initial condition (9). If equations (7) fulfil the initial condition (9) in the point $x_{0} \in U$ then the Kähler spaces $K_{n}$ and $\bar{K}_{n}$ are homothetic. It follows from this that the initial conditions globally generate only trivial solutions $\bar{g}=k \cdot g, k=$ const. These equations characterize homothetic mappings.

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# ON RINGS OF ENDOMORPHISMS OF CERTAIN FREE MODULES 

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#### Abstract

This article deals with ring of endomorphisms of free finite dimensional modules over a local algebra $A$ (s.c. $A$-spaces). The correspondences between a set of submodules of given $A$-space and the set of rigt (left) anihilators of the ring of endomorfisms are studied. The criterion for sumbodules to be a direct summand of given $A$-space (s.c. $A$-subspace) is obtained in the article.


Key words and phrases. local ring, free module, $A$-space, $A$-subspace,ring of endomorphisms, ideal, anihilator, projection
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## 1 Introduction

Geometry of subspaces of vector spaces has fundamental importance for projective geometry. In this case it is well known that there exists isomorphism, resp. antiisomorphism, between an ordered set of subspaces of given vector spaces and the ordered set of left, resp. right, anihilators of the ring of endomorphisms of given vector spaces. The structure of subspaces of such vectore space is fully determined by the structure of ideals of the associated ring of endomormphisms (see e. g. [1]).

In this article we replace the vector space by a free module. It will be shown that the structure of submodules of a free modules is more "fruitful" because besides free submodules there may exists submodules with no bases. Therefore it is actual to find criteria for a submodule of $M$ to be a free submodule, i.e. an $A$-subspace ${ }^{1}$. We will consider modules over local algebras of special types, plural algberas and Weil algebras. This structures are widely applicable e.g. in statistics or differential geometry (for example [6], [10], [11], [13], [17]) and also in dynamics

[^5]or robotics ([2], [5], [15] or [16]).
Let a local ring $A$ be given. In what follows we will denote by $M$ an $A$-space in the following sense of B. R. McDonald [14]:

Definition 1.1 Let $A$ be a local ring. Let $M$ be a finitely generated $A$-module. Then $M$ is an $A$-space of finite dimension if there exist $\boldsymbol{e}_{1}, \ldots, \boldsymbol{e}_{n} \in M$ with
(a) $M=A \boldsymbol{e}_{1} \oplus \cdots \oplus A \boldsymbol{e}_{n}$,
(b) the map $A \rightarrow A \boldsymbol{e}_{i}$ defined by $1 \mapsto \boldsymbol{e}_{i}$ is an isomorphism for $i, i=1, \ldots, n$.

The set $\left\{\boldsymbol{e}_{1}, \ldots, \boldsymbol{e}_{n}\right\} \subseteq M$ will be called an $A$-basis of $M$. A submodule $K \subseteq \subseteq M$ will be called an $A$-subspace of $M$ if it is a direct summand of $M$.

Clearly, every $A$-space in the sense of this definition is a free finite dimensional module over a local ring $A$ and vice versa.

Thorough the following we shall consider the linear algebra $A$ having as a vector space over an arbitrary commutative field $T$ a basis

$$
\begin{equation*}
\left\{1, \eta, \eta^{2}, \ldots, \eta^{m-1}\right\} \text { with } \eta^{m}=0 \tag{1}
\end{equation*}
$$

It means that any element $\beta$ of $A$ may be uniquely written by the following form

$$
\beta=\sum_{i=0}^{m-1} b_{i} \eta^{i}
$$

It is easy to see that $A$ is a local ring with maximal ideal $\mathfrak{a}=(\eta)$ and $\mathfrak{a}, \ldots, \mathfrak{a}^{m-1}$ are just all nontrivial proper ideal in $A$. By $M$ the $A$-spaces just over such algebras will be denoted.

The algebra $A$, s.c. plural algebra of order $m$ in the sense of [4], is isomorphic to the factor ring of polynomials $T[x] /\left(x^{m}\right)$. Therefore, in the case $T=\mathbb{R}$, algebra $A$ may be considered as a Weil algebra of the order $m$ and width 1 .

Remark 1.2 Let $M$ be an A-space over $A$. It is well known corollary of Nakayama lemma that all bases of $M$ have the same number of elements (called dimension of $M$ ) and from every set of generators a basis of $M$ may be selected. It is also known that every $A$-subspace is a free submodule of M (see [14]).

Moreover in our case the $A$-space $M$ has the following properties (proved in [7]):

1. Any linearly independent system can be completed to a basis of $M$,
2. A submodule of $M$ is a free submodule if and only if it is a direct summand of $M$.
3. An intersection of two $A$-subspaces of $M$ is an $A$-subspace of $M$ if and only if the sum of them is a $A$-subspace of $M .^{2}$

It follows from this that in our case $A$-subspaces of $M$ are just all free submodules of $M^{3}$.
Let us define an endomorphism $\eta \in \operatorname{End}(M)$ induced by a very natural way:

$$
\begin{equation*}
\forall \boldsymbol{x} \in M: \eta(\boldsymbol{x})=\eta \boldsymbol{x} \tag{2}
\end{equation*}
$$

The properties of sets of generators of arbitrary $A$-submodule of $M$ are brought by the following proposition.

Theorem 1.3 Let $K$ be a nontrivial submodule of $A$-space $M$. Then there exists a system $\mathcal{B}_{0}, \ldots, \mathcal{B}_{r}$ of subsets on $M$ such that

1. $\mathcal{B}_{0} \cup \cdots \cup \mathcal{B}_{r-1} \cup \mathcal{B}_{r}$ is a basis of $M$,
2. $\eta^{m-r} \mathcal{B}_{0} \cup \eta^{m-r+1} \mathcal{B}_{1} \cup \cdots \cup \eta^{m-1} \mathcal{B}_{r-1}$ is a set of generators of $K$.

In this case $r, 1 \leq r \leq m$, is an integer with $K \subseteq \operatorname{Ker} \eta^{r} \wedge K \not \subset \operatorname{Ker} \eta^{r-1}$.

Proof. Let us denote $\vartheta=\eta \mid K, \vartheta \in \operatorname{End}(K)$.
Since $M$ is a free module, it follows from [9] that $\eta^{j} M=\operatorname{Ker} \eta^{m-j}, 0 \leq j \leq m$. It implies

$$
\begin{equation*}
K \cap \eta^{j} M=\operatorname{Ker} \vartheta^{m-j}, 0 \leq j \leq m \tag{3}
\end{equation*}
$$

Let us denote $r \in \mathbb{N}, 1 \leq r \leq m$, such that $K \subseteq \operatorname{Ker} \eta^{r} \wedge K \not \subset \operatorname{Ker} \eta^{r-1}$. Respecting the fact $K$ is a vector space over $A /(\eta)=T$ we have $\vartheta$ is a nilpotent endomorphism of $K$ and therefore we obtain elements $\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{s_{0}}, \boldsymbol{u}_{s_{0}+1}, \ldots, \boldsymbol{u}_{s_{1}}, \boldsymbol{u}_{s_{1}+1}, \ldots, \boldsymbol{u}_{s_{2}}, \ldots, \boldsymbol{u}_{s_{r-2}+1}, \ldots, \boldsymbol{u}_{s_{r-1}}$ of $K$ such that

$$
\eta^{r-k} \boldsymbol{u}_{1}, \ldots, \eta^{r-k} \boldsymbol{u}_{s_{0}}, \eta^{r-k-1} \boldsymbol{u}_{s_{0}+1}, \ldots, \eta^{r-k-1} \boldsymbol{u}_{s_{1}}, \ldots, \boldsymbol{u}_{s_{r-k-1}+1}, \ldots, \boldsymbol{u}_{s_{r-k}}
$$

form a $T$-basis $\operatorname{Ker} \vartheta^{k} \bmod \operatorname{Ker} \vartheta^{k-1}, 1<k<r-1$.
It follows from $\left\{\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{s_{0}}\right\} \subseteq \operatorname{Ker} \vartheta^{r} \subseteq \eta^{m-r} M$ (see (3)) that there exist $\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{s_{0}} \in M$ with

$$
\begin{equation*}
\boldsymbol{u}_{i}=\eta^{m-r} \boldsymbol{v}_{i}, 1 \leq i \leq s_{0} . \tag{4}
\end{equation*}
$$

By the same way, $\left\{\boldsymbol{u}_{s_{r-k-1}+1}, \ldots, \boldsymbol{u}_{s_{r-k}}\right\} \subseteq \operatorname{Ker} \vartheta^{k}$ implies the existence of elements $\boldsymbol{v}_{s_{r-k-1}+1}, \ldots, \boldsymbol{v}_{s_{r-k}} \in M$ with

$$
\begin{equation*}
\boldsymbol{u}_{i}=\eta^{m-k} \boldsymbol{v}_{i}, s_{r-k-1}+1 \leq i \leq s_{r-k}, \text { for } k=1, \ldots, r-1 \tag{5}
\end{equation*}
$$

Let us denote $\mathcal{B}_{0}=\left\{\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{s_{0}}\right\}$ and $\mathcal{B}_{r-k}=\left\{\boldsymbol{v}_{s_{r-k-1}+1}, \ldots, \boldsymbol{v}_{s_{r-k}}\right\}, k=r-1, \ldots, 1$

[^6]a) Let us show the linear independence of the set $\mathcal{B}_{0} \cup \ldots \cup \mathcal{B}_{r-1}$. Supposing
$$
\sum_{i=1}^{s_{r-1}} \xi_{i} \boldsymbol{v}_{i}=\boldsymbol{o}
$$
and denoting (according to the (1)) $\xi_{i}=\sum_{j=0}^{m-1} x_{i j} \eta^{j}, 1 \leq i \leq s_{r-1}$, where $x_{i j} \in T$, we obtain:
\[

$$
\begin{equation*}
\sum_{j=0}^{m-1} \eta^{j} \sum_{i=1}^{s_{r-1}} x_{i j} \boldsymbol{v}_{i}=\boldsymbol{o} \tag{6}
\end{equation*}
$$

\]

Multiplying this equality by $\eta^{m-1}$ and using (4), (5) we may write:

$$
\sum_{i=1}^{s_{0}} x_{i 0}\left(\eta^{r-1} \boldsymbol{u}_{i}\right)+\sum_{i=s_{0}+1}^{s_{1}} x_{i 0}\left(\eta^{r-2} \boldsymbol{u}_{i}\right)+\cdots+\sum_{i=s_{r-2}+1}^{s_{r-1}} x_{i 0} \boldsymbol{u}_{i}=\boldsymbol{o} .
$$

It is a linear combination of elements of a $T$-basis of submodule $K$ (with coefficients from $T$ ) which implies $x_{i 0}=0$, for $i=1, \ldots, s_{r-1}$. Therefore (6) may be written in the form

$$
\sum_{j=1}^{m-1} \eta^{j} \sum_{i=1}^{s_{r-1}} x_{i j} \boldsymbol{v}_{i}=\boldsymbol{o}
$$

Multiplying this equality by $\eta^{m-2}$ and using again (4), (5), we obtain $x_{i 1}=0, i=1, \ldots, s_{r-1}$. Then (6) may be written by

$$
\sum_{j=2}^{m-1} \eta^{j} \sum_{i=1}^{s_{r-1}} x_{i j} \boldsymbol{v}_{i}=\boldsymbol{o}
$$

Let us (6) multiply by $\eta^{m-3}, \ldots, \eta$, consecutively. This gives that all $x_{i j}$ are zero and $\xi_{1}=\xi_{2}=$ $\cdots=\xi_{s_{r-1}}=0$, consequently.

Since $\mathcal{B}_{0} \cup \cdots \cup \mathcal{B}_{r-1}$ is linearly independent, it may be completed (by a set $\mathcal{B}_{r}$ ) to a $A$-basis of $M$ (according to the Remark 1.2).
b) Let us show $\eta^{m-r} \mathcal{B}_{0} \cup \eta^{m-r+1} \mathcal{B}_{1} \cup \cdots \cup \eta^{m-1} \mathcal{B}_{r-1}$ generates (over $A$ ) the module $K$. Respecting (4) a (5) we may write:

$$
\begin{aligned}
& \boldsymbol{x}= \sum_{\substack{1 \leq i \leq s_{0} \\
0 \leq j \leq r-1}} \mathrm{x}_{i j}\left(\eta^{j} \boldsymbol{u}_{i}\right)+\sum_{\substack{s_{0}<i \leq s_{1} \\
0 \leq j \leq r-2}} x_{i j}\left(\eta^{j} \boldsymbol{u}_{i}\right)+\cdots+\sum_{\substack{s_{r-3}<i \leq s_{r}-2 \\
0 \leq j \leq 1}} x_{i j}\left(\eta^{j} \boldsymbol{u}_{i}\right)+\sum_{s_{r-2}<i \leq s_{r-1}} x_{i j} \boldsymbol{u}_{i}= \\
&=\sum_{\substack{1 \leq i \leq s_{0} \\
0 \leq j \leq r-1}} x_{i j}\left(\eta^{j+m-r} \boldsymbol{v}_{i}\right)+\sum_{\substack{s_{0}<i \leq s_{1} \\
0 \leq j \leq r-2}} x_{i j}\left(\eta^{j+m-r+1} \boldsymbol{v}_{i}\right)+\cdots+\sum_{\substack{s_{r-3}<i \leq s_{r} \\
0 \leq j \leq 1}} x_{i j}\left(\eta^{j+m-2} \boldsymbol{v}_{i}\right)+ \\
&+\sum_{s_{r-2}<i \leq s_{r-1}} x_{i j} \eta^{m-1} \boldsymbol{v}_{i}=
\end{aligned}
$$

$$
\begin{aligned}
=\sum_{\substack{1 \leq i \leq s_{0} \\
0 \leq k \leq m-1}}\left(x_{i k} \eta^{k}\right)\left(\eta^{m-r} \boldsymbol{v}_{i}\right)+ & \sum_{\substack{s_{0}<i \leq s_{1} \\
0 \leq k \leq m-1}}\left(x_{i k} \eta^{k}\right)\left(\eta^{m-r+1} \boldsymbol{v}_{i}\right)+\cdots+\sum_{\substack{s_{r-3}<i \leq s_{r}-2 \\
0 \leq k \leq m-1}}\left(x_{i k} \eta^{k}\right)\left(\eta^{m-2} \boldsymbol{v}_{i}\right)+ \\
& +\sum_{\substack{s_{r-2}<i \leq s_{r}-1 \\
0 \leq k \leq m-1}}\left(x_{i k} \eta^{k}\right)\left(\eta^{m-1} \boldsymbol{v}_{i}\right)= \\
=\sum_{1 \leq i \leq s_{0}} \xi_{i}\left(\eta^{m-r} \boldsymbol{v}_{i}\right)+ & \sum_{s_{0}<i \leq s_{1}} \xi_{i}\left(\eta^{m-r+1} \boldsymbol{v}_{i}\right)+\cdots+\sum_{s_{r-3}<i \leq s_{r-2}} \xi_{i}\left(\eta^{m-2} \boldsymbol{v}_{i}\right)+ \\
& +\sum_{s_{r-2}<i \leq s_{r-1}} \xi_{i}\left(\eta^{m-1} \boldsymbol{v}_{i}\right)
\end{aligned}
$$

If an element $\boldsymbol{x}$ of $M$ may be expressed as a linear combination (over $T$ ) of elements of a $T$-basis of submodule $K$ then it may be also expressed as a linear combination of elements of the set $\eta^{m-r} \mathcal{B}_{0} \cup \eta^{m-r+1} \mathcal{B}_{1} \cup \cdots \cup \eta^{m-1} \mathcal{B}_{r-1}$ with coefficients from $A$ and vice versa.

## 2 Ring of endomorphisms of $A$-space

Theorem 2.1 Let $K$ be a submodule of $A$-space $M$. Then there exist endomorphisms $f, g \in$ $\operatorname{End}(M)$ such that ${ }^{4}$ :

$$
\text { Ker } f=K, \operatorname{Im} g=K
$$

Proof. Evidently, this theorem is true for the trivial submodule. Now, let us consider a nontrivial submodule $K$.
Let $\mathcal{B}_{0}, \ldots, \mathcal{B}_{r}$, be a system of subsets according to the theorem 1.3. Denotig for every $j, 0 \leq$ $j \leq r$, by $M_{j}$ the $A$-subspace with the basis $\mathcal{B}_{j}$ we have:

$$
\begin{gather*}
M_{0} \oplus M_{1} \oplus \cdots \oplus M_{r}=M  \tag{7}\\
\eta^{m-r} M_{0}+\eta^{m-r+1} M_{1}+\cdots+\eta^{m-r+j} M_{j}+\cdots+\eta^{m-1} M_{r-1}=K \tag{8}
\end{gather*}
$$

Let us define an endomorphism $f$ of $M$ by:

$$
\begin{equation*}
f \mid M_{j}=\eta^{r-j}, 0 \leq j \leq r . \tag{9}
\end{equation*}
$$

Clearly, $K \subseteq \operatorname{Ker} f($ see (8)).
Now, considering $\boldsymbol{x} \in \operatorname{Ker} f, \boldsymbol{x}=\sum_{j=0}^{r} \boldsymbol{x}_{j}, \boldsymbol{x}_{j} \in M_{j}$ we have (see 9):

$$
\boldsymbol{o}=f(\boldsymbol{x})=\sum_{j=0}^{r} f\left(\boldsymbol{x}_{j}\right)=\sum_{j=0}^{r}\left(\eta^{r-j} \boldsymbol{x}_{j}\right) .
$$

[^7]Since $\eta^{r-j} \boldsymbol{x}_{j} \in M_{j}$, using (7) we have: $\eta^{r-j} \boldsymbol{x}_{j}=\boldsymbol{o}, 0 \leq j \leq r$. Respecting that all $M_{j}$ are $A$-spaces we get (see (3)): $\operatorname{Ker}\left(\eta \mid M_{j}\right)^{k}=\eta^{m-k} M_{j}, 0 \leq k \leq m, 0 \leq j \leq r$. It means that $\boldsymbol{x}_{j} \in \eta^{m-r+j} M_{j}$ and (with respect to the (8)) $\boldsymbol{x}_{j} \in K, 0 \leq j \leq r$. This implies $\boldsymbol{x} \in K$.
We have $\operatorname{Ker} f=K$.
Let us define an endomorphism $g$ of $M$ by:

$$
\begin{equation*}
g \mid M_{j}=\eta^{m-r+j}, 0 \leq j \leq r . \tag{10}
\end{equation*}
$$

Evidently, (7) and (8) give $\operatorname{Im} g \subseteq K$.
If $\boldsymbol{x} \in K$ then (with respect to (8) and (10)) we may write $\left(\boldsymbol{y}_{j} \in \mathbf{M}_{j}\right)$ :

$$
\boldsymbol{x}=\sum_{j=0}^{r} \eta^{m-r+j} \boldsymbol{y}_{j}=\sum_{j=0}^{r} g\left(\boldsymbol{y}_{j}\right)=g(\boldsymbol{y}), \boldsymbol{y} \in M
$$

therefore $\boldsymbol{x} \in \operatorname{Im} g$. We see $\operatorname{Im} g=K$.

Now, let us generalize Galois triangle theorem, which is known for vectore space (see [1]). We will show that the ring of endomorphisms of $A$-space $M$ and the structure of its ideals reflect "fully" the structure of $A$-spaces, its submodules and subspaces.

## Denotation 2.1

1. By $P$ we will denote the ring of endomorphisms the of $A$-space $M$; the composition of endomorphisms $f, g \in P$ will be define by $(f g)(\boldsymbol{x})=g(f(\boldsymbol{x}))$.
2. For every $J \subseteq P$ we will by $\mathbf{L}(J)$ denote a left anihilator of a subset $J$, i.e.

$$
\mathbf{L}(J)=\{f \in P ; \forall g \in J: f g=o\} .
$$

By $\mathbf{R}(J)$ we will denote the right anihilator of $J$, i.e.

$$
\mathbf{R}(J)=\{f \in P ; \forall g \in J: g f=o\} .
$$

3. By $\mathcal{L}(P)$ and $\mathcal{R}(P)$ we will denote the set of all left and right anihilators of ring $P$, respectivelly.
4. By $\mathcal{L}_{0}(P)$ and $\mathcal{R}_{0}(P)$ we will denote the set of all left and right principle ideals of $P$ generated by an idempotent element of $P$, respectivelly.
5. By $\mathcal{U}(M)$ we will denote the set of all $A$-submodules of $A$-space $M$.
6. By $\mathcal{U}_{0}(M)$ we will denote the set of all $A$-subspaces of $A$-space $M$.
7. For any submodule $S \in \mathcal{U}(M)$ we will denote

$$
\begin{gathered}
\mathbf{N}(S)=\{f \in P ; \forall \boldsymbol{x} \in S: f(\boldsymbol{x})=\boldsymbol{o}\}, \mathbf{Q}(S)=\{f \in P ; \forall \boldsymbol{x} \in M: f(\boldsymbol{x}) \in S\} . \\
\text { (Equivalently, } \mathbf{N}(S)=\{f \in P ; S \subseteq \operatorname{Ker} f\}, \mathbf{Q}(S)=\{f \in P ; \operatorname{Im} f \subseteq S\} . \text { ) }
\end{gathered}
$$

8. For any subset $J \subseteq P$ we will denote

$$
\begin{gathered}
\mathbf{K}(J)=\{\boldsymbol{x} \in M ; \forall f \in J: f(\boldsymbol{x})=\boldsymbol{o}\}, \\
\mathbf{M}(J)=\{\boldsymbol{x} \in M ; \exists f \in J, \exists \boldsymbol{y} \in M: \boldsymbol{x}=f(\boldsymbol{y})\} \\
\text { (Equivalently, } \mathbf{K}(J)=\bigcap_{f \in J} \operatorname{Ker} f, \mathbf{M}(J)=\bigcup_{f \in J} \operatorname{Im} f . \text { ) }
\end{gathered}
$$

Remark 2.2 It is clear to see that for all $J \subseteq P, S \subseteq \mathcal{U}(M)$ the sets $\mathbf{L}(J)$ and $\mathbf{Q}(S)$ are left ideals of $P$ as well as $\mathbf{R}(J)$ and $\mathbf{N}(S)$ are right ideals of the ring $P$.

The following properties follow from the definition above for all $U, S \in \mathcal{U}(P), J, H \subseteq P$ :

$$
\begin{gathered}
J \subseteq H \Rightarrow \mathbf{K}(J) \supseteq \mathbf{K}(H), \mathbf{M}(J) \subseteq \mathbf{M}(H), \mathbf{R}(J) \supseteq \mathbf{R}(H), \mathbf{L}(J) \supseteq \mathbf{L}(H) \\
U \subseteq S \Rightarrow \mathbf{N}(U) \supseteq \mathbf{N}(S), \mathbf{Q}(U) \subseteq \mathbf{Q}(S)
\end{gathered}
$$

The proof of followig lemma is based on the theorem 2.1.
Lemma 2.3 For any submodule $S$ of $M$ it holds:

$$
\mathbf{K}(\mathbf{N}(S))=S, \mathbf{M}(\mathbf{Q}(S))=S
$$

The following lemma may be proved analogously as in the case of vectore spaces (see [1]).
Lemma 2.4 For any subset $J \subseteq P$ it holds:

$$
\mathbf{N}(\mathbf{M}(J))=\mathbf{R}(J), \mathbf{Q}(\mathbf{K}(J))=\mathbf{L}(J)
$$

Two following lemmas follow from lemmas 2.3 and 2.4:
Lemma 2.5 For any submodule $S$ of $M$ it holds:

$$
\mathbf{N}(S)=\mathbf{R}(\mathbf{Q}(S)), \mathbf{Q}(S)=\mathbf{L}(\mathbf{N}(S))
$$

Lemma 2.6 For any right anihilator $H \in \mathcal{R}(\mathbf{P})$ and any left anihilator $J \in \mathcal{L}(P)$ it holds:

$$
\mathbf{N}(\mathbf{K}(H))=H, \mathbf{Q}(\mathbf{M}(J))=J
$$

Remark 2.7 It follows from lemma 2.5: $\mathbf{N}(S)$ belongs to $\mathcal{R}(P)$ and $\mathbf{Q}(S)$ belongs to $\mathcal{L}(P)$; for any $S \in \mathcal{U}(M)$.

From the lemmas above the first main theorem of this section follows. It may be called Galois theorem I.

## Theorem 2.8

1. Operators $\mathbf{N}$ and $\mathbf{K}$ are mutually inverse antiisomorphisms of ordered sets $(\mathcal{U}(M), \subseteq) a$ $(\mathcal{R}(P), \subseteq)$.
2. Operators $\mathbf{Q}$ and $\mathbf{M}$ are mutually inverse isomorphisms of ordered sets $(\mathcal{U}(M), \subseteq) a$ $(\mathcal{L}(P), \subseteq)$.
3. Operators $\mathbf{L} a \mathbf{R}$ are mutually inverse antiisomorphisms of ordered sets $(\mathcal{R}(P), \subseteq) a$ $(\mathcal{L}(P), \subseteq)$.
4. The following diagram is commutative.


Now let us study the proper subset $\mathcal{U}_{0}(M) \subset \mathcal{U}(M)$ of all $A$-subspaces of module $M$. Since in our case $A$-subspaces are just all direct summands of $M$, it will be useful to use the properties of projections of $M$.

The notion a projection of $A$-space $M$ we will use in the usual sense - as an idempotent endomorphism of $M$. Using the known properties of projections (see e.g. [3]) and remark 1.2 we obtain that in our case kernel and image of any idempotent endomorphism form $A$-subspaces of $M$.

Lemma 2.9 For any $A$-subspace $S$ of $M$ it holds:

1. $\mathbf{N}(S) \in \mathcal{R}_{0}(P)$,
2. $\mathbf{Q}(S) \in \mathcal{L}_{0}(P)$.

Proof. Ad (a): Let $S \in \mathcal{U}_{0}(M)$. The definition of the operator $\mathbf{N}$ implies that $\mathbf{N}(S)$ is a right ideal of $P$. Since $S \in \mathcal{U}_{0}(M)$, it is a direct summand of $M$. Therefore there exists a $T \subseteq \subseteq M$ with $\mathbf{M}=S \oplus T$. It gives that we may construct a projection $f M$ onto $T$ parallely $S=\operatorname{Ker} f$. It is easy to see, for any endomorphisms $f, g \in P$ where $f$ is a projection, it holds:

$$
\text { Ker } f \subseteq \operatorname{Ker} g \Leftrightarrow f g=g
$$

Considering an arbitrary endomorphism $g \in \mathbf{N}(S)$ we have $g=f g$ and $\mathbf{N}(S) \subseteq f P$, consequently. The inverse inclusion is clear, therefore $\mathbf{N}(S) \in \mathcal{R}_{0}(P)$.

The part (b) may be proved analogously.

Lemma 2.10 If $J \in \mathcal{R}_{0}(P)$, then $\mathbf{K}(J) \in \mathcal{U}_{0}(M)$.

Proof. Let $f$ be an idempotent generator of an ideal $J, J=f P$. Using the definition of operator $\mathbf{K}$ we have: $\boldsymbol{x} \in \mathbf{K}(J)$ iff $(f p)(\boldsymbol{x})=\boldsymbol{o}$, for certain $p \in P$. This is equivalent to $f(\boldsymbol{x})=\boldsymbol{o}$, which means $\operatorname{Ker} f=\mathbf{K}(J)$. As $f$ is a projection, its kernel belongs to $\mathcal{U}_{0}(M)$.

Lemma 2.11 If $J \in \mathcal{L}_{0}(P)$, then $\mathbf{M}(J) \in \mathcal{U}_{0}(M)$.

Proof. Denoting $f$ an idempotent generator of an ideal $J$ and putting $S=\operatorname{Im} f$, we see $S \in \mathcal{U}_{0}(M)$. For arbitrary $\boldsymbol{x} \in \mathbf{M}(J)$ we may write: $\boldsymbol{x} \in \mathbf{M}(J)$ iff $\boldsymbol{x}=p f(\boldsymbol{y})=f(p(\boldsymbol{y}))$ for certain $p \in P$. It gives $\boldsymbol{x} \in \operatorname{Im} f$. We have $\mathbf{M}(J)=S$.

Now we may formulated a theorem on 1-1 correspondences between ordered sets $\mathcal{L}_{0}(P)$, $\mathcal{R}_{0}(P)$ and $\mathcal{U}_{0}(M)$. This theorem (Galois theorem II) with the theorem 2.8 forms a Galois triangle theory for $A$-spaces.

The structure of ideals of ring of endomorphisms of an $A$-space is "wealthier" than in the case of vectore spaces (e.g. [1]) or totally reducible modules (see [12]).

## Theorem 2.12

1. Operators $\mathbf{N} \mid \mathcal{U}_{0}(M)$ and $\mathbf{K} \mid \mathcal{R}_{0}(P)$ are mutually inverse antiisomorphisms of ordered sets $\left(\mathcal{U}_{0}(M), \subseteq\right)$ and $\left(\mathcal{R}_{0}(P), \subseteq\right)$.
2. Operators $\mathbf{Q} \mid \mathcal{U}_{0}(M)$ and $\mathbf{M} \mid \mathcal{L}_{0}(P)$ are mutually inverse isomorphisms of ordered sets $\left(\mathcal{U}_{0}(M), \subseteq\right)$ and $\left(\mathcal{L}_{0}(P), \subseteq\right)$.
3. Operators $\mathbf{L} \mid \mathcal{R}_{0}(P)$ and $\mathbf{R} \mid \mathcal{L}_{0}(P)$ are mutually inverse antiisomorphisms of ordered sets $\left(\mathcal{R}_{0}(P), \subseteq\right)$ and $\left(\mathcal{L}_{0}(P), \subseteq\right)$.
4. The following diagram is commutative. By indices 0 the corresponding restrictions of the operators are denoted.


Finally, the criterion for $A$-submodule to be an $A$-subspace follows from the both Galois theorems 2.8 and 2.12.

Theorem 2.13 Let a submodule $K$ of an $A$-space $M$ be given. Then the following conditions are equivalent.

1. $K$ is an $A$-subspace of $M$,
2. the set of endomorphisms the images of which are contained in $K$ forms a left principle ideal of $P$ generated by an idempotent element of $P$,
3. the set of endomorphisms the kernels of which contain $K$ forms a left principle ideal of $P$ generated by an idempotent element of $P$.

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# ON AN APPLICATION OF THE $A$-SPACES OVER PLURAL ALGEBRAS IN PHYSICS <br> JUKL Marek, (CZ) 


#### Abstract

This article deals with $A$-spaces over local rings of a special types, s.c. plural algebras. The $A$-spaces representation of physical phenomenons are studied. Especially, a concrete representation of electromagnetic fields and Maxwell equations is offer in this article. It is shown that the $A$-space formalism may be simpler that usual representation by Euclidean spaces.


Key words and phrases. local ring, free module, $A$-space, plural algebra, dual numbers, maxwell equations, electromagnetic field
Mathematics Subject Classification. Primary 13C10, 51C10; Secondary 78A25.

## $1 A$-spaces over plural algebras

The 3-dimensional Euklidean space $E=\mathbb{R}^{3}$ is a space which is in physics the most often used to the description of physical phenomenons. Vector physical quantities are usually represented by vector fields in this Euclidean space.

It will be show further, that any Euclidean space may be isomorphically embedded into a certain free module over local rings ( $A$-space) of the same dimension. Therefore it is useful to show a representation of certain physical quantities by fields of elements of such $A$-space. As an example of this formalism the representation of (physical quantities of) the electromagnetic field will be presented.

Let us construct a special linear algebra $A$ (s. c. plural algebra, see e.g. [2]). Plural algebra of order $m$ is any linear algebra $A$ having as a vector space over a commutative field $T$ a basis

$$
\left\{1, \eta, \eta^{2}, \ldots, \eta^{m-1}\right\} \text { with } \eta^{m}=0^{1}
$$

[^8]
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It means that any element $\beta$ of $A$ may be uniquely written by the following form

$$
\beta=\sum_{i=0}^{m-1} b_{i} \eta^{i} .
$$

It is easy to see that $A$ is a local ring with maximal ideal $\mathfrak{a}=(\eta)$ and $\mathfrak{a}, \ldots, \mathfrak{a}^{m-1}$ are just all nontrivial proper ideal in $A$.

The plural algebra $A$ of order $m$ is isomorphic to the factor ring of polynomials $T[x] /\left(x^{m}\right)$. Therefore, in the case $T=\mathbb{R}$, algebra $A$ may be considered as a Weil algebra of the order $m$ and width 1 (see e.g. [10]).

Let us remark that in the case $T=\mathbb{R}$ and $m=2$ we obtain well known dual numbers introduced by Clifford [3] more than a hundred years ago.

Let us denote $M=A^{n}$. In [7] and in [8] this $A$-modules $M$ and their submodules are studied. $M$ is an $n$-dimensional $A$-space in the sense of B. R. McDonald [12] such that $A$-subspaces of $M$ are just all direct summand of $M$. Let us register a fact

$$
\begin{equation*}
M=T^{n} \oplus \eta T^{n} \oplus \cdots \oplus \eta^{m-1} T^{n} \tag{1}
\end{equation*}
$$

Moreover, it is shown in [7] that properties of such $A$-spaces are very "near" to properties of vectore spaces over $A / \mathfrak{a}$. This fact predetermines this modules for representation of physical phenomenons.

Let $E$ be an $n$-dimensional Euclidean space and let $A$ be a plural algebra of an arbitrary order $m$ over $T=\mathbb{R}$. Let us construct the $A$-space $M$ of the same dimension. Then the mapping $\iota: E \rightarrow M$ defined by

$$
\iota:\left(x_{1}, x_{2}, \ldots x_{n}\right) \mapsto\left(x_{1}, x_{2}, \ldots x_{n}\right)
$$

is an isomorphic embedding $E$ into $M^{2}$.
The relation (1) presents a possibility how to express arbitrary $m$ vectors of an Euclidean space $E$ by one element of the $A$-space $M$. This is described by a mapping $\alpha: E^{m} \rightarrow M$ defined by

$$
\begin{equation*}
\alpha:\left(\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{m}\right) \mapsto \boldsymbol{x}_{0}+\eta \boldsymbol{x}_{1}+\ldots \eta^{m-1} \boldsymbol{x}_{m} . \tag{2}
\end{equation*}
$$

This is a way how to simplify a mathematical representations of some physical phenomenons which are desribed by a set of real vectors. Formally, it may seem analogous to the description of a $m$-touple of "real" 3 -dimensional vectors by the $3 m$-touple of their coordinates, i. e. by one vector of $E^{\prime}=\mathbb{R}^{3 m}$. It is important to say that a dimension of the $A$-space $M$ is the same as dimension of $E$, which inter alia means that every transformation of the $A$-space $M$ is still given by a matrix of the type $3 \times 3$ (represented by $9 m$ real numbers), but any transformations of the real vector space $E^{\prime}$ will be given by a matrix of the type $3 m \times 3 m$ (represented by $9 m^{2}$ real numbers).

[^9]This fact is a motivation for concrete using of dual numbers mentioned above for example for representation rigid body dynamics or for representation of movement of robots etc. It is well known that the first dual numbers formalisms in mechanics were introduced by Kotelnikov ${ }^{3}$ and Study [17] in the early 20th century. But this topic is still actual - we may mentioned today papers - e.g. [1], [5], [14] or [16].

Let us remark that in these mentioned papers the structure of the set of "motors", i.e. elements in the form $\boldsymbol{x}=\boldsymbol{x}_{0}+\eta \boldsymbol{x}_{1}$, where $\boldsymbol{x}_{0}, \boldsymbol{x}_{1} \in E$, is not studied systematically, and therefore it is not identified as an $A$-space (free finite dimensional module over local ring).

Let us, very briefly, introduce an example of dual numbers representation. Brodsky and Shoham [16] introduced for motion of rigid body composed of translational motion (linear momenum $\boldsymbol{p}$ ) and of rotational motion (angular momentum $\boldsymbol{b}$ ) only one quantity $\hat{\boldsymbol{b}}$, dual momentum of the body, defined by

$$
\hat{\boldsymbol{b}}=\boldsymbol{p}+\eta \boldsymbol{b}
$$

Defining dual force momentum $\hat{\boldsymbol{N}}$ (with respect to a point $B$ ) by

$$
\hat{\mathbf{N}}_{B}=\boldsymbol{F}+\eta \boldsymbol{N}_{B},
$$

where $\boldsymbol{F}$ is a total force acting on a body and $\boldsymbol{N}_{B}$ is a total momentum of force (with respect to a point $B$ ) acting on a body, then we have the dynamic equation in the form

$$
\hat{\boldsymbol{N}}_{B}=\frac{\mathrm{d} \hat{\boldsymbol{b}}_{\mathrm{B}}}{\mathrm{dt}} .
$$

It means that the motion of the rigid body is possible to describe by only one physical quantity which fulfils only one dynamic equation (instead two dynamic equations in real coordinates).

Plural numbers formalisms may be also widely applicated in differential geometry (for example [6], [10], [11], [13], [19] or [20]). The first application of dual numbers in geometry was done by Segre [15].

In the following section of this paper we will offer a proposal of plural numbers formalism for a representation of electromagnetic field.

## 2 Maxwell equations in $A$-spaces representation

In this section we will by $A$ denote the plural algebra over real numbers of order 2 (set of dual numbers) and by $M$ we will denote 3 -dimensional $A$-space, i.e. $M=A^{3}$.
The electomagnetic field is represented by four vector fields - by the vector field of electric field intensity $\boldsymbol{E}$, by the vector field of magnetic induction $\boldsymbol{B}$, by the vector field of electric induction $\boldsymbol{D}$ and by the vector field of magnetic field intensity $\boldsymbol{H}$. It is known that this four fields are not independent, they are fixed by the following two relations:

$$
\begin{equation*}
\boldsymbol{D}=\varepsilon \boldsymbol{E}, \boldsymbol{B}=\mu \boldsymbol{H} \tag{3}
\end{equation*}
$$

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where $\varepsilon$ is a permitivity and $\mu$ is a permeability of the medium. Therefore the electromagnetic field may be fully described by a couple of vector fields, for example by $\boldsymbol{E}$ and $\boldsymbol{H}$.

The quantities representing electromagnetic fields satisfy to the well known Maxwell equations which form the foundation of classical electrodynamics. The differential form of them is (see e.g. [4]):

$$
\begin{align*}
& \operatorname{rot} \boldsymbol{H}=\boldsymbol{j}+\frac{\partial \boldsymbol{D}}{\partial \mathrm{t}}  \tag{4}\\
& \operatorname{rot} \boldsymbol{E}=-\frac{\partial \boldsymbol{B}}{\partial \mathrm{t}}  \tag{5}\\
& \operatorname{div} \boldsymbol{D}=\rho  \tag{6}\\
& \operatorname{div} \boldsymbol{B}=0 \tag{7}
\end{align*}
$$

where $\boldsymbol{j}$ is an electric current density and $\rho$ is a charge density.
Beeing inspired by in previous section mentioned representation of dynamics we offer a proposal of representation of electromagnetic field by a field of one element $\hat{\boldsymbol{V}}$ instead two real vector fields $\boldsymbol{E}, \boldsymbol{H}$ of the 3 -dimensional $A$-space $M$. It may be called plural electromagnetic field intensity. We will defined it by

$$
\begin{equation*}
\hat{\boldsymbol{V}}=\boldsymbol{E}+\eta \boldsymbol{H} \tag{8}
\end{equation*}
$$

Physical sense of plural quantity $\hat{\boldsymbol{V}}$ is given by the circulation of it along a closed curve $L$. If $\mathcal{E}$ is electromotive voltage and $\mathcal{M}$ is magnetomotive potential difference along a choosen curve, then we may write

$$
\oint_{L} \hat{\boldsymbol{V}} \mathrm{~d} \boldsymbol{l}=\oint_{L} \boldsymbol{E} \mathrm{~d} \boldsymbol{l}+\eta \oint_{L} \boldsymbol{H} \mathrm{~d} \boldsymbol{l}=\mathcal{E}+\eta \mathcal{M}
$$

which justifies the definition of plural quantity $\hat{\mathcal{E}}$ by

$$
\hat{\mathcal{E}}=\mathcal{E}+\eta \mathcal{M}
$$

For circulation of electromagnetic field intensity we obtain

$$
\begin{equation*}
\oint_{L} \hat{\boldsymbol{V}} \mathrm{~d} \boldsymbol{l}=\hat{\mathcal{E}} \tag{9}
\end{equation*}
$$

Now, let us introduce beside $\hat{\boldsymbol{V}}$ the second plural quantity $\hat{\boldsymbol{W}}$, called plural induction, by

$$
\begin{equation*}
\hat{\boldsymbol{W}}=\eta \boldsymbol{D}-\boldsymbol{B} \tag{10}
\end{equation*}
$$

Let us derive the sense of the product $\hat{\boldsymbol{W}} \times \hat{\boldsymbol{V}}$. With respect to the (8), (10) and (3) we have:

$$
\hat{\boldsymbol{W}} \times \hat{\boldsymbol{V}}=(-\boldsymbol{B} \times \boldsymbol{E})+\eta(\boldsymbol{D} \times \boldsymbol{E}-\boldsymbol{B} \times \boldsymbol{H})=\boldsymbol{E} \times \boldsymbol{B}
$$

which means

$$
\begin{equation*}
\hat{\boldsymbol{W}} \times \hat{\boldsymbol{V}}=\mu \boldsymbol{P} \tag{11}
\end{equation*}
$$

where $\boldsymbol{P}$ is a Poynting vector ${ }^{4}$. The product $\hat{\boldsymbol{W}} \times \hat{\boldsymbol{V}}$ is equal to the vector which determines the direction of electromagnetic field propagation. The norm of it is equal to the density of electromagnetic power flow multiplyied by the permeability of a medium. This fact represents fysical justification of the definition of $\hat{\boldsymbol{W}}$ by (10).

Naturally, both plural quantities $\hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}$ are fixed by a relation which may be derived from constitutive relations (3):

$$
\begin{equation*}
\hat{\boldsymbol{W}}=\hat{\varepsilon} \hat{\boldsymbol{V}} . \tag{12}
\end{equation*}
$$

where $\hat{\varepsilon}$ is an operator of plural permitivity defined on $M$ by ${ }^{5}$

$$
\hat{\varepsilon}=\left(\eta \varepsilon-\mu \frac{\mathrm{d}}{\mathrm{~d} \eta}\right)
$$

Now we may formulated the Maxwell equations (4) - (7) by elements $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{W}}$ of $A$-space $M$. Respecting (3) we get the following relations for a curl operation and divergence of plural quantity $\hat{\boldsymbol{V}}$ :

$$
\begin{aligned}
\operatorname{rot} \hat{\boldsymbol{V}} & =-\frac{\partial \boldsymbol{B}}{\partial \mathrm{t}}+\eta \frac{\partial \boldsymbol{D}}{\partial \mathrm{t}}+\eta \boldsymbol{j} \\
\operatorname{div} \hat{\boldsymbol{V}} & =\operatorname{div} \boldsymbol{E}+\eta \operatorname{div} \boldsymbol{H}
\end{aligned}
$$

The four Maxwell equations (4)-(7) in real coordinates may be with respet to (10) equivalently written as a following couple of equations:

$$
\begin{gather*}
\operatorname{rot} \hat{\boldsymbol{V}}=\frac{\partial \hat{\boldsymbol{W}}}{\partial \mathrm{t}}+\eta \boldsymbol{j}  \tag{13}\\
\operatorname{div} \hat{\boldsymbol{V}}=\frac{\rho}{\varepsilon} \tag{14}
\end{gather*}
$$

In the free space we may Maxwell equations write by

$$
\begin{gathered}
\operatorname{rot} \hat{\boldsymbol{V}}=\frac{\partial \hat{\boldsymbol{W}}}{\partial \mathrm{t}} \\
\operatorname{div} \hat{\boldsymbol{V}}=0
\end{gathered}
$$

We have shown that electromagnetic field may be instead of two (or four) real vector field fully described by one (or two) field od elements of the $A$-space $M$. The plural quantities have not only formal sense but they also have a concrete physical interpretation. Four Maxwell equations may be equivalently expressed by a couple of equation (13), (14) of plural quantities.

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# APPLICATION OF DUAL QUATERNIONS ALGORITHM FOR GEODETIC DATUM TRANSFORMATION 

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#### Abstract

The paper briefly introduces the theory of quaternions and dual quaternions. Consequently, dual quaternions are used to represent rotation and translation parameters and the formulae is derived for computation of rotation, translation and scale parameters in the Bursa-Wolf geodetic datum transformation model from two set of co-located 3D coordinates. The following part is devoted to some practical applications of dual quaternions with a numerical case study to demonstrate the application of the derived formulae.


Key words and phrases. Quaternions, dual quaternions, geodetic datum transformation.
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## 1 Introduction

Different types of coordinate systems and cartographic projections have been widely used in geodesy, cartography, mapping or geographic information systems. For example, we need to process coordination data from a different state territory, as well as we need to transform coordinates from a digitizer to the world geodetic coordinate system or, conversely, we have to compute with coordinates obtained by GPS (Global Positioning System).

As we have mentioned, different states or groups of states use local coordinate systems by selecting the reference area and the type of a projection, which conform best to their area. In recent years a trend has been to introduce a global coordinate system, enabling an easier cooperation between neighboring countries to exchange their information. These include for example UTM (Universal Transverse Mercator) and WGS $84^{1}$ (World Geodetic System).

[^12]
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In the datum transformation case, we need generally to compute seven transformation parameters, namely three rotations, three translation and one scaling. In several papers, a scale parameter is associated with the transformation between coordinates and not with the transformation between coordinate systems. We can find this alternative approach without using scale parameters in [12]. This technique can be also provided by the dual quaternions which we will presented here.

William Kingdon Clifford (1845-1879) invented dual quaternions in the nineteenth century [3] to represent rigid transformations. There is a close connection to a classical result of spatial kinematics known as Chasles theorem, see [7] for more details. Chasles theorem states that any rigid transformation can be described by a screw, i.e., a rotation about an axis followed by a translation in the direction of this axis. So, dual quaternions are convenient to describe a composition of rotations and translations.

In this paper we will show how to represent datum transformation by using dual quaternions. The main contribution is simplification of the original solution of datum transformation.

## 2 Preliminaries

Dual quaternions can be considered as standard quaternions whose elements are the dual numbers. Their structure is convenient for instance for describing rigid transformations, which are compositions of rotations and translations. Let us therefore start our discussion with recalling some fundamental facts, see e.g. [4].

### 2.1 Quaternions and rotations

A quaternion $\mathcal{Q}$ can be defined as follows

$$
\begin{equation*}
\mathcal{Q}=1 q_{0}+\mathbf{i} q_{1}+\mathbf{j} q_{2}+\mathbf{k} q_{3}, \tag{1}
\end{equation*}
$$

where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis elements called quaternion units satisfying the relations $\mathbf{i}^{2}=\mathbf{k}^{2}=\mathbf{j}^{2}=$ $\mathbf{i} \mathbf{j} \mathbf{k}=-1, \mathbf{i j}=\mathbf{k}, \mathbf{j} \mathbf{i}=-\mathbf{k}$ and $q_{0}, q_{1}, q_{2}, q_{3}$ are real numbers. Otherwise, it can be written as comprising scalar and vector parts $q_{0}$ and $\mathbf{q}=q_{1} \mathbf{i}+q_{2} \mathbf{j}+q_{3} \mathbf{k}$. Hence we write $\mathcal{Q}=\left(q_{0}, \mathbf{q}\right)$. A Pure quaternion is a quaternion with zero scalar part, i.e., $\mathcal{Q}=(0, \mathbf{q})$. The corresponding conjugate quaternion is denoted as

$$
\begin{equation*}
\mathcal{Q}^{*}=q_{0}-\mathbf{i} q_{1}-\mathbf{j} q_{2}-\mathbf{k} q_{3} . \tag{2}
\end{equation*}
$$

The norm of a quaternion is defined as

$$
\begin{equation*}
\|\mathcal{Q}\|=\sqrt{q_{0}^{2}+q_{1}^{2}+q_{2}^{2}+q_{3}^{2}}=\sqrt{\mathcal{Q} \mathcal{Q}^{*}} \tag{3}
\end{equation*}
$$

The product of two quaternions $\mathcal{Q}$ and $\mathcal{P}$ is defined by

$$
\begin{equation*}
\mathcal{Q P}=\left[q_{0} p_{0}-\mathbf{q} \cdot \mathbf{p}, \mathbf{q} \times \mathbf{p}+q_{0} \mathbf{p}+p_{0} \mathbf{q}\right], \tag{4}
\end{equation*}
$$

where the symbols • and $\times$ are dot product and cross product in $\mathbb{R}^{3}$. A quaternion is called the unit quaternion if $\|\mathcal{Q}\|=1$. If $\mathcal{Q}$ is the unit quaternion then there exists a unit vector $\mathbf{n}$, an angle $\theta \in\langle-\pi, \pi\rangle$ such that

$$
\begin{equation*}
\mathcal{Q}=(\cos \theta, \mathbf{n} \sin \theta) \tag{5}
\end{equation*}
$$

The special orthogonal group is defined as

$$
\begin{equation*}
\mathbf{S O}(3)=\left\{\mathbf{A} \in \mathbf{G L}(3, \mathbb{R}) \mid \mathbf{A}^{T} \mathbf{A}=\mathbf{I} \wedge \operatorname{det} \mathbf{A}=1\right\} . \tag{6}
\end{equation*}
$$

The matrix $\mathbf{A}$ represents a rotation in $\mathbb{R}^{3}$ about the origin if and only if $\mathbf{A} \in \mathbf{S O}(3)$, see [8] and the references given there. In the following statement we can easily see how unit quaternions can represent rotations.

A unit quaternion $\mathcal{Q}=(\cos \theta, \mathbf{n} \sin \theta)$ represents the rotation of a vector $\mathbf{p}$ by the angle $2 \theta$ along the axis given by $\mathbf{n}$. The vector $\mathbf{p}$ is represented by pure quaternion $\mathcal{P}=(0, \mathbf{p})$. Rotated vector, represented as a pure quaternion, is

$$
\begin{equation*}
\widehat{\mathcal{P}}=\mathcal{Q} \mathcal{P} \mathcal{Q}^{*} \tag{7}
\end{equation*}
$$

We can find an elegant proof of this in [4]. First, it is shown how a vector $\mathbf{p}$ is rotated by $\theta$ along $\mathbf{n}$, using sine, cosine and the scalar and the vector products. Then is shown that the same result is obtained through a rotation described by quaternions. Each element in $\mathbf{S O}(3)$ can be expressed using quaternions as (7), see [5] for instance.

### 2.2 Matrix representation of quaternions

A quaternion $\mathcal{Q}$ can be expressed by a $4 \times 4$ matrix, see [11]

$$
\mathcal{Q}=q_{0}+\mathbf{i} q_{1}+\mathbf{j} q_{2}+\mathbf{k} q_{3}=\left[\begin{array}{rrrr}
q_{0} & q_{1} & q_{2} & q_{3}  \tag{8}\\
-q_{1} & q_{0} & -q_{3} & q_{2} \\
-q_{2} & q_{3} & q_{0} & -q_{1} \\
-q_{3} & -q_{2} & q_{1} & q_{0}
\end{array}\right]
$$

The quaternion product $\mathcal{S}=\mathcal{Q} \widehat{\mathcal{Q}}$ is described in the matrix form as

$$
\mathcal{S}=\left[\begin{array}{l}
s_{0}  \tag{9}\\
s_{1} \\
s_{2} \\
s_{3}
\end{array}\right]=\left[\begin{array}{rrrr}
q_{0} & -q_{1} & -q_{2} & -q_{3} \\
q_{1} & q_{0} & -q_{3} & q_{2} \\
q_{2} & q_{3} & q_{0} & -q_{1} \\
q_{3} & -q_{2} & q_{1} & q_{0}
\end{array}\right]\left[\begin{array}{l}
\hat{q}_{0} \\
\hat{q}_{1} \\
\hat{q}_{2} \\
\hat{q}_{3}
\end{array}\right]=\left[\begin{array}{rrrr}
\hat{q}_{0} & -\hat{q}_{1} & -\hat{q}_{2} & -\hat{q}_{3} \\
\hat{q}_{1} & \hat{q}_{0} & \hat{q}_{3} & -\hat{q}_{2} \\
\hat{q}_{2} & -\hat{q}_{3} & \hat{q}_{0} & \hat{q}_{1} \\
\hat{q}_{3} & \hat{q}_{2} & -\hat{q}_{1} & \hat{q}_{0}
\end{array}\right]\left[\begin{array}{l}
q_{0} \\
q_{1} \\
q_{2} \\
q_{3}
\end{array}\right] .
$$

Now we have a basic idea of matrix representation of the quaternions. Let us try to apply this formulae for the quaternion representation of rotations. Equation (7) can be expressed in matrix form as

$$
\widehat{\mathcal{P}}=\mathcal{Q P} \mathcal{Q}^{*}=\left[\begin{array}{rrrr}
q_{0} & -q_{1} & -q_{2} & -q_{3}  \tag{10}\\
q_{1} & q_{0} & -q_{3} & q_{2} \\
q_{2} & q_{3} & q_{0} & -q_{1} \\
q_{3} & -q_{2} & q_{1} & q_{0}
\end{array}\right]\left[\begin{array}{rrrr}
0 & -p_{1} & -p_{2} & -p_{3} \\
p_{1} & 0 & -p_{3} & p_{2} \\
p_{2} & p_{3} & 0 & -p_{1} \\
p_{3} & -p_{2} & p_{1} & 0
\end{array}\right]\left[\begin{array}{r}
q_{0} \\
-q_{1} \\
-q_{2} \\
-q_{3}
\end{array}\right] .
$$

## 3 Dual numbers and dual quaternions

This section provides a brief introduction to the theory of dual numbers and dual quaternions. We focus only on the basics of this algebra. More details can be found in [2], [3] or [10].

### 3.1 Dual numbers and dual quaternions

Dual numbers were invented by Clifford in 1873, see [2] for more details. They are similar to complex numbers because any dual number $z_{d}$ can be written as

$$
\begin{equation*}
z_{d}=a+\varepsilon a_{\varepsilon} \tag{11}
\end{equation*}
$$

where $a$ is the non-dual part, $a_{\varepsilon}$ the dual part and $\varepsilon$ is a basis element called dual unit. The defining condition for the dual unit is $\varepsilon^{2}=0$. The dual conjugate is analogous to the complex conjugate, i.e.,

$$
\begin{equation*}
\overline{z_{d}}=a-\varepsilon a_{\varepsilon} . \tag{12}
\end{equation*}
$$

The multiplication of two dual numbers is given as

$$
\begin{equation*}
z_{d} \hat{z}_{d}=a \hat{a}+\varepsilon\left(a \hat{a_{\varepsilon}}+a_{\varepsilon} \hat{a}\right) \tag{13}
\end{equation*}
$$

Finally, note that pure dual numbers, i.e., dual numbers with $a=0$, do not have an inverse. This is a fundamental difference from complex numbers because every non-zero complex number has the inverse.

A dual quaternion $\mathcal{Q}_{d}$ can be written as the sum of two standard quaternions

$$
\begin{equation*}
\mathcal{Q}_{d}=\mathcal{Q}+\varepsilon \mathcal{Q}_{\varepsilon} \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{Q}=q_{0}+q_{1} \mathbf{i}+q_{2} \mathbf{j}+q_{3} \mathbf{k} \quad \text { and } \quad \mathcal{Q}_{\varepsilon}=q_{0 \varepsilon}+q_{1 \varepsilon} \mathbf{i}+q_{2 \varepsilon} \mathbf{j}+q_{3 \varepsilon} \mathbf{k} \tag{15}
\end{equation*}
$$

are real quaternions and $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are usual quaternion units. The dual unit $\varepsilon$ commutes with quaternion units, for example $\mathbf{i} \varepsilon=\varepsilon \mathbf{i}$. A dual quaternion can also be considered as an 8 -tuple of real numbers, or as

$$
\begin{align*}
\mathcal{Q}_{d} & =q_{0 d}+q_{1 d} \mathbf{i}+q_{2 d} \mathbf{j}+q_{3 d} \mathbf{k} \\
& =\left(q_{0}+\varepsilon q_{0 \varepsilon}\right)+\left(q_{1}+\varepsilon q_{1 \varepsilon}\right) \mathbf{i}+\left(q_{2}+\varepsilon q_{2 \varepsilon}\right) \mathbf{j}+\left(q_{3}+\varepsilon q_{3 \varepsilon}\right) \mathbf{k} \tag{16}
\end{align*}
$$

where $q_{0 d}$ is the scalar part (a dual number), $\left(q_{1 d}, q_{2 d}, q_{3 d}\right)$ is the vector part (a dual vector), see [10]. The product of two dual quaternions $\mathcal{Q}_{d}$ and $\widehat{\mathcal{Q}}_{d}$ is defined as

$$
\mathcal{Q}_{d} \widehat{\mathcal{Q}}_{d}=\widehat{\mathcal{Q}}+\varepsilon\left(\mathcal{Q} \widehat{\mathcal{Q}}_{\varepsilon}+\mathcal{Q}_{\varepsilon} \widehat{\mathcal{Q}}\right)
$$

Multiplication of dual quaternions is associative, distributive, but not commutative. The conjugation of a dual quaternion is defined using the classical quaternion conjugation

$$
\begin{equation*}
\mathcal{Q}_{d}{ }^{*}=Q^{*}+\varepsilon \mathcal{Q}_{\varepsilon}^{*} . \tag{17}
\end{equation*}
$$

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However, the dual number conjugation (12) can be applied to dual quaternion conjugation and we get dual conjugate dual quaternion

$$
\begin{equation*}
\overline{\mathcal{Q}_{d}^{*}}=\mathcal{Q}^{*}-\varepsilon \mathcal{Q}_{\varepsilon}^{*} . \tag{18}
\end{equation*}
$$

The norm of a dual quaternion is a dual scalar and is defined as

$$
\begin{equation*}
\left\|\mathcal{Q}_{d}\right\|=\sqrt{\left(q_{0}+\varepsilon q_{0 \varepsilon}\right)^{2}+\left(q_{1}+\varepsilon q_{1 \varepsilon}\right)^{2}+\left(q_{2}+\varepsilon q_{2 \varepsilon}\right)^{2}+\left(q_{3}+\varepsilon q_{3 \varepsilon}\right)^{2}}=\sqrt{\mathcal{Q}_{d}^{*} \mathcal{Q}_{d}} . \tag{19}
\end{equation*}
$$

Dual quaternion is called unit dual quaternion if $\left\|\mathcal{Q}_{d}\right\|=1$. Note that a dual quaternion $\mathcal{Q}_{d}$ is unit if and only if

$$
\begin{equation*}
\|\mathcal{Q}\|=1 \quad \wedge \quad \mathcal{Q} \cdot \mathcal{Q}_{\varepsilon}=0 \tag{20}
\end{equation*}
$$

If we have a vector $\mathbf{p}=\left(p_{1}, p_{2}, p_{3}\right)$, we define the associated unit dual quaternion as

$$
\begin{equation*}
\mathcal{P}_{d}=1+\varepsilon\left(p_{1} \mathbf{i}+p_{2} \mathbf{j}+p_{3} \mathbf{k}\right), \tag{21}
\end{equation*}
$$

which satisfy the previous statement.
A new method to represent rigid transformations is based on using dual quaternions. Dual quaternions capture in their inner structure the basic information about this transformationsnamely the axis of rotation and the rotation angle about the axis and the translation along it. Composition of this transformations corresponds to multiplication of dual quaternions.

Suppose that $\mathbf{p}=\left(p_{1}, p_{2}, p_{3}\right)$ is a position vector of a point $P, \mathbf{t}=\left(t_{1}, t_{2}, t_{3}\right)$ is a translation vector and $\mathcal{Q}=(\cos \theta, \mathbf{n} \sin \theta)$ is a unit quaternion. Then we can express the image of the point $P$ after this translation and this rotation as

$$
\begin{equation*}
\hat{\mathcal{P}}_{d}=\mathcal{Q}_{d} \mathcal{P}_{d} \overline{\mathcal{Q}_{d}^{*}} \tag{22}
\end{equation*}
$$

where $\mathcal{P}_{d}, \mathcal{Q}_{d}$ are the unit dual quaternions and $\mathcal{T}$ is the pure quaternion fulfilling

$$
\begin{equation*}
\mathcal{Q}_{d}=\mathcal{Q}+\varepsilon \mathcal{Q}_{\varepsilon}=\mathcal{Q}+\varepsilon \frac{\mathcal{T} \mathcal{Q}}{2}, \quad \mathcal{T}=t_{1} \mathbf{i}+t_{2} \mathbf{j}+t_{3} \mathbf{k} \quad \text { and } \quad \mathcal{P}_{d}=1+\varepsilon\left(p_{1} \mathbf{i}+p_{2} \mathbf{j}+p_{3} \mathbf{k}\right) . \tag{23}
\end{equation*}
$$

To sum up, unit dual quaternions naturally represent rotation when the dual part $\mathcal{Q}_{\varepsilon}=0$, see (7).

### 3.2 Matrix representation of dual quaternions

We can represent a dual quaternion $\mathcal{Q}_{d}=\mathcal{Q}+\varepsilon \mathcal{Q}_{\varepsilon}$ by a $8 \times 8$ matrix, see [11] for more information,

$$
\mathcal{Q}_{d}=\left[\begin{array}{ll}
\mathcal{Q} & \mathcal{Q}_{\varepsilon}  \tag{24}\\
0 & \mathcal{Q}
\end{array}\right],
$$

where $\mathcal{Q}$ and $\mathcal{Q}_{\varepsilon}$ are the matrix forms of the type (8). The dual conjugate dual quaternion in $8 \times 8$ matrix representation is defined as

$$
\overline{\mathcal{Q}_{d}^{*}}=\mathcal{Q}^{*}-\varepsilon \mathcal{Q}_{\varepsilon}^{*}=\left[\begin{array}{cc}
\mathcal{Q}^{*} & -\mathcal{Q}_{\varepsilon}^{*}  \tag{25}\\
0 & \mathcal{Q}^{*}
\end{array}\right]
$$

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This representation is often used in computer processing, see [1]. In this matrix form are included well-known relations for quaternion units or dual unit. Now we re-write (22) to $8 \times 8$ matrix form. The unit dual quaternions $\mathcal{Q}_{d}$ and $\overline{\mathcal{Q}_{d}^{*}}$ are expressed as (24) and (25). If we multiply the unit dual quaternions $\widehat{\mathcal{P}}_{d}=\mathcal{Q}_{d} \mathcal{P}_{d} \overline{\mathcal{Q}_{d}^{*}}$ in $8 \times 8$ matrix form, we get $8 \times 8$ matrix form again, i.e.,

$$
\widehat{\mathcal{P}}_{d}=\left[\begin{array}{rrrrrrrr}
1 & 0 & 0 & 0 & 0 & \hat{p}_{1} & \hat{p}_{2} & \hat{p}_{3}  \tag{26}\\
0 & 1 & 0 & 0 & -\hat{p}_{1} & 0 & -\hat{p}_{3} & \hat{p}_{2} \\
0 & 0 & 1 & 0 & -\hat{p}_{2} & \hat{p}_{3} & 0 & -\hat{p}_{1} \\
0 & 0 & 0 & 1 & -\hat{p}_{3} & -\hat{p}_{2} & \hat{p}_{1} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right],
$$

where

$$
\begin{align*}
\hat{p}_{1}= & q_{0}^{2}+q_{1}^{2}-q_{2}^{2}-q_{3}^{2}+2\left[q_{1}\left(p_{2} q_{2}+p_{3} q_{3}\right)-q_{1} q_{0 \varepsilon}+q_{0}\left(p_{3} q_{2}-p_{2} q_{3}\right)+q_{0} q_{1 \varepsilon}-\right.  \tag{27}\\
& \left.q_{3} q_{2 \varepsilon}+q_{2} q_{3 \varepsilon}\right], \\
\hat{p}_{2}= & p_{2}\left(q_{0}^{2}-q_{1}^{2}+q_{2}^{2}-q_{3}^{2}\right)+2\left[q_{1} q_{2}+p_{3}\left(q_{2} q_{3}-q_{0} q_{1}\right)-q_{2} q_{0 \varepsilon}+q_{3}\left(q_{0}+q_{1 \varepsilon}\right)+\right.  \tag{28}\\
& \left.q_{0} q_{2 \varepsilon}-q_{1} q_{3 \varepsilon}\right], \\
\hat{p}_{3}= & p_{3}\left(q_{0}^{2}-q_{1}^{2}-q_{2}^{2}+q_{3}^{2}\right)+2\left[p_{2}\left(q_{0} q_{1}+q_{2} q_{3}\right)-q_{3} q_{1 \varepsilon}-q_{2}\left(q_{0}+q_{2 \varepsilon}\right)+\right.  \tag{29}\\
& \left.q_{1}\left(q_{3}+q_{2 \varepsilon}\right)+q_{0} q_{3 \varepsilon}\right] .
\end{align*}
$$

This unit dual quaternion can be written as $\widehat{\mathcal{P}}_{d}=1+\varepsilon\left(\hat{p}_{1} \boldsymbol{i}+\hat{p}_{2} \boldsymbol{j}+\hat{p}_{3} \boldsymbol{k}\right)$, i.e., the point $P$ with the position vector $\mathbf{p}=\left(p_{1}, p_{2}, p_{3}\right)$ is rotated and then translated to the point $\hat{P}$ with the position vector $\hat{\mathbf{p}}=\left(\hat{p}_{1}, \hat{p}_{2}, \hat{p}_{3}\right)$.

## 4 Dual quaternion datum transformation model

This section brings the main contribution of the paper. We use dual quaternions for description of datum transformation, where matrix representations of dual quaternions help us to simplify manipulations of equations.

Datum transformation models contain seven or six parameters which depends on the fact whether we want to know the scale parameter. Our goal is to estimated all required parameters from co-located coordinates on two different datums. Bursa-Wolf similarity transformation model can be written as

$$
\begin{equation*}
\mathbf{s}_{i}=\mathbf{t}+k \mathbf{R} \mathbf{p}_{i} \tag{30}
\end{equation*}
$$

where $\mathbf{s}_{i}=\left(s_{1 i}, s_{2 i}, s_{3 i}\right)^{T} \in \mathbb{R}^{3}, \mathbf{p}_{i}=\left(p_{1 i}, p_{2 i}, p_{3 i}\right)^{T} \in \mathbb{R}^{3}, i=1 \ldots n$ are two sets of the co-located coordinates in the two different systems, $\mathbf{t}=\left(t_{1}, t_{2}, t_{3}\right)^{T} \in \mathbb{R}^{3}$ denotes the three translation parameters, $k$ denotes the scale parameter and $\mathbf{R} \in \mathbf{S O}(3)$ is the rotation matrix containing three rotation parameters. In order to determine the mentioned parameters, the number of the co-located coordinates $\mathbf{s}_{i}, \mathbf{p}_{i}$ must be greater than or equal to three. We will proceed similar way to optimization problem as in [9].

First define centrobaric coordinates $\triangle \mathbf{s}_{i}=\left(\triangle s_{1 i}, \triangle s_{2 i}, \triangle s_{3 i}\right)^{T}, \triangle \mathbf{p}_{i}=\left(\triangle p_{1 i}, \triangle p_{2 i}, \triangle p_{3 i}\right)^{T}$, $i=1 \ldots n$ for the sets of the co-located coordinates as

$$
\begin{equation*}
\triangle \mathbf{s}_{i}=\mathbf{s}_{i}-\frac{1}{n} \sum_{i=1}^{n} \mathbf{s}_{i}, \quad \text { and } \quad \triangle \mathbf{p}_{i}=\mathbf{p}_{i}-\frac{1}{n} \sum_{i=1}^{n} \mathbf{p}_{i} \tag{31}
\end{equation*}
$$

If we substitute (31) into (30), we obtain

$$
\begin{align*}
\triangle \mathbf{s}_{i} & =\mathbf{t}+k \mathbf{R}\left(\triangle \mathbf{p}_{i}+\frac{1}{n} \sum_{i=1}^{n} \mathbf{p}_{i}\right)-\frac{1}{n} \sum_{i=1}^{n} \mathbf{s}_{i}  \tag{32}\\
& =\triangle \mathbf{t}+k \mathbf{R} \triangle \mathbf{p}_{i}, \quad \text { where } \quad \triangle \mathbf{t}=\mathbf{t}+k \mathbf{R} \frac{1}{n} \sum_{i=1}^{n} \mathbf{p}_{i}-\frac{1}{n} \sum_{i=1}^{n} \mathbf{s}_{i} \tag{33}
\end{align*}
$$

Equation (32) is over-determined therefore we denote the residual vector $\mathbf{v}_{i} \in \mathbb{R}^{3}, i=1 \ldots n$ as

$$
\begin{equation*}
\mathbf{v}_{i}=\triangle \mathbf{s}_{i}-\triangle \mathbf{t}-k \mathbf{R} \triangle \mathbf{p}_{i} \tag{34}
\end{equation*}
$$

Now we get the following optimization problem to solve required parameters

$$
\begin{equation*}
\min _{k, \Delta \mathbf{t}, \mathbf{R}} \sum_{i=1}^{n} \mathbf{v}_{i}^{T} \mathbf{v}_{i}=\min _{k, \Delta \mathbf{t}, \mathbf{R}} \sum_{i=1}^{n}\left(\triangle \mathbf{s}_{i}-\triangle \mathbf{t}-k \mathbf{R} \triangle \mathbf{p}_{i}\right)^{T}\left(\triangle \mathbf{s}_{i}-\triangle \mathbf{t}-k \mathbf{R} \triangle \mathbf{p}_{i}\right) \tag{35}
\end{equation*}
$$

It is possible to express residual vector $\mathbf{v}_{i}$ in the form of dual quaternions. First we modify (34). The scale parameter $k \in \mathbb{R}$ is a constant therefore

$$
\begin{align*}
\mathbf{v}_{i} & =\triangle \mathbf{s}_{i}-\triangle \mathbf{t}-k \mathbf{R} \triangle \mathbf{p}_{i} \\
& =\triangle \mathbf{s}_{i}-\triangle \mathbf{t}-\mathbf{R} k \triangle \mathbf{p}_{i} \\
& =-\left(\triangle \mathbf{l}_{i}+\mathbf{R} \triangle \mathbf{r}_{i}\right), \quad \text { where } \quad \triangle \mathbf{l}_{i}=\triangle \mathbf{t}-\triangle \mathbf{s}_{i} \quad \text { and } \quad \triangle \mathbf{r}_{i}=k \triangle \mathbf{p}_{i} \tag{36}
\end{align*}
$$

Equation (36) expresses a rotation of the vector $\triangle \mathbf{r}_{i}$ and then a translation given by the translation vector $\Delta \mathrm{l}_{i}$. We can express this equation according to (22) using dual quaternions in the form

$$
\begin{equation*}
\mathcal{V}_{d_{i}}=-\mathcal{Q}_{d_{i}} \mathcal{R}_{d_{i}} \overline{\mathcal{Q}_{d_{i}}^{*}}, \tag{37}
\end{equation*}
$$

where $\mathcal{R}_{d_{i}}$ is a unit dual quaternion

$$
\begin{equation*}
\mathcal{R}_{d_{i}}=1+\varepsilon\left(\Delta r_{1 i} \mathbf{i}+\triangle r_{2 i} \mathbf{j}+\triangle r_{3 i} \mathbf{k}\right)=1+k \varepsilon\left(\triangle p_{1 i} \mathbf{i}+\triangle p_{2 i} \mathbf{j}+\triangle p_{3 i} \mathbf{k}\right) \tag{38}
\end{equation*}
$$

and $\mathcal{Q}_{d_{i}}$ is a unit dual quaternion

$$
\begin{align*}
\mathcal{Q}_{d_{i}}=\mathcal{Q}+\varepsilon \mathcal{Q}_{\varepsilon} & =\mathcal{Q}+\varepsilon \frac{\mathcal{L}_{i} \mathcal{Q}}{2}, \quad \text { where } \mathcal{Q}=q_{0}+q_{1} \mathbf{i}+q_{2} \mathbf{j}+q_{3} \mathbf{k}  \tag{39}\\
& \text { and } \quad \mathcal{L}_{i}=\left(\triangle t_{1}-\triangle s_{1 i}\right) \mathbf{i}+\left(\triangle t_{2}-\triangle s_{2 i}\right) \mathbf{j}+\left(\triangle t_{3}-\triangle s_{3 i}\right) \mathbf{k} \tag{40}
\end{align*}
$$

The quaternion $\mathcal{Q}$ is a unit quaternion and $\mathcal{L}$ is a pure quaternion. Since $\mathcal{Q}_{d_{i}}$ is a unit dual quaternion, we must apply the conditions (20), i.e.,

$$
\begin{equation*}
\|\mathcal{Q}\|=1 \quad \wedge \quad \mathcal{Q} \cdot \mathcal{Q}_{\varepsilon}=0 \tag{41}
\end{equation*}
$$

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From equation (37) we get the unit dual quaternion of the form $\mathcal{V}_{d_{i}}=1+\varepsilon\left(v_{1 i} \mathbf{i}+v_{2 i} \mathbf{j}+v_{3 i} \mathbf{k}\right)$, see Section 3.2, corresponding to the vector $\mathbf{v}_{i}=\left(v_{1 i}, v_{21}, v_{3 i}\right)^{T}$, where terms $v_{1 i}, v_{21}, v_{3 i}$ contains seven parameters to be solved, i.e., $q_{0}, \cdots, q_{3}, \Delta t_{1}, \cdots, \Delta t_{3}, k$. Further, the transformation parameters can be determined by solving this optimization problem

$$
\begin{align*}
\min _{q_{0}, \cdots, q_{3}, \Delta t_{1}, \cdots, \Delta t_{3}, k} \sum_{i=1}^{n} \mathbf{v}_{i}^{T} \mathbf{v}_{i} & =\left(v_{1 i}, v_{21}, v_{3 i}\right)^{T}\left(v_{1 i}, v_{21}, v_{3 i}\right),  \tag{42}\\
\|\mathcal{Q}\| & =1 \wedge \mathcal{Q} \cdot \mathcal{Q}_{\varepsilon}=0 . \tag{43}
\end{align*}
$$

We can use nonlinear method to solve this minimization problem, i.e., Lagrange multipliers. This method can also accommodate multiple constraints. Therefore, the rotation matrix $\mathbf{R}$ is

$$
\begin{equation*}
\mathbf{R}=\left[q_{0}^{2}-\left(q_{1}, q_{2}, q_{3}\right)^{T}\left(q_{1}, q_{2}, q_{3}\right)\right] \mathbf{I}+2\left[\left(q_{1}, q_{2}, q_{3}\right)\left(q_{1}, q_{2}, q_{3}\right)^{T}+q_{0} \mathbf{C}\left(q_{1}, q_{2}, q_{3}\right)\right], \tag{44}
\end{equation*}
$$

where $\mathbf{I}$ denotes the $3 \times 3$ identity matrix and $\mathbf{C}\left(q_{1}, q_{2}, q_{3}\right)=\left[\begin{array}{ccc}0 & -q_{3} & q_{2} \\ q_{3} & 0 & -q_{1} \\ -q_{2} & q_{1} & 0\end{array}\right]$. The rotation angles can be computed by using

$$
\begin{equation*}
\theta_{x}=\arctan \frac{r_{23}}{r_{33}}, \quad \theta_{y}=\arcsin \left(-r_{13}\right), \quad \theta_{z}=\arctan \frac{r_{12}}{r_{11}} \tag{45}
\end{equation*}
$$

where $r_{i j}$ is the element of the rotation matrix $\mathbf{R}$ in the $i$-th row and $j$-th column and $\theta_{x}, \theta_{y}, \theta_{z}$ are the rotation angles around corresponding axes. Finally, translation vector $\mathbf{t}$ can be determined by using (33) as

$$
\begin{equation*}
\mathbf{t}=\Delta \mathbf{t}-k \mathbf{R} \frac{1}{n} \sum_{i=1}^{n} \mathbf{p}_{i}+\frac{1}{n} \sum_{i=1}^{n} \mathbf{s}_{i} . \tag{46}
\end{equation*}
$$

The dual quaternion transformation algorithm can be summarized in the following steps:

## Algorithm 1 Dual quaternion transformation algorithm

INPUT: Cartesian coordinates of $n$ stations given in a local and a global reference system.

1. Compute centrobaric coordinates $\triangle \mathbf{s}_{i}=\left(\triangle s_{1 i}, \triangle s_{2 i}, \triangle s_{3 i}\right)^{T}, \triangle \mathbf{p}_{i}=\left(\triangle p_{1 i}, \triangle p_{2 i}, \triangle p_{3 i}\right)^{T}$ using (31).
2. Express the unit dual quaternion $\mathcal{V}_{d_{i}}$ with the parameters $q_{0}, \cdots, q_{3}, \Delta t_{1}, \cdots, \Delta t_{3}, k$ using (37) and determine corresponding vector $\mathbf{v}_{i}=\left(v_{1 i}, v_{2 i}, v_{3 i}\right)^{T}$.
3. Compute required parameters $q_{0}, \cdots, q_{3}, \Delta t_{1}, \cdots, \Delta t_{3}, k$ by (42) determined by conditions (43).
4. Compute rotation matrix $\mathbf{R}$ using (44) and then rotation angles $\theta_{x}, \theta_{y}, \theta_{z}$ using (45).
5. Compute translation vector $\mathbf{t}$ using (46).

OUTPUT:Three rotation parameters $\theta_{x}, \theta_{y}, \theta_{z}$, three translation parameters $t_{1}, t_{2}, t_{3}$ and the scale parameter $k$.

## 5 Computed example

Numerical example is presented to demonstrate the functionality of the designed method. We consider the Cartesian coordinates of seven stations given in a local and a global reference system (WGS 84) as shown in Table 1, adopted from [6].

Table 1: Coordinates for local system (system $A$ ), coordinates for WGS-84 (system $B$ ).

|  | System $A$ |  |  | System $B$ |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Station name | $X(m)$ | $Y(m)$ | $Z(m)$ | $X(m)$ | $Y(m)$ | $Z(m)$ |
| Solitude | 4157870.237 | 664818.678 | 4775416.524 | 4157222.543 | 664789.307 | 4774952.099 |
| Buoch Zeil | 4149691.049 | 688865.785 | 4779096.588 | 4149043.336 | 688836.443 | 4778632.188 |
| Hohenneuffen | 4173451.354 | 690369.375 | 4758594.075 | 4172803.511 | 690340.078 | 4758129.701 |
| Kuehlenberg | 4177796.064 | 643026.700 | 4761228.899 | 4177148.376 | 642997.635 | 4760764.800 |
| Ex Mergelaec | 4137659.549 | 671837.337 | 4791592.531 | 4137012.190 | 671808.029 | 4791128.215 |
| Ex Hof Asperg | 4146940.228 | 666982.151 | 4784324.099 | 4146292.729 | 666952.887 | 4783859.856 |
| Ex Kaisersbach | 4139407.506 | 702700.227 | 4786016.645 | 4138759.902 | 702670.738 | 4785552.196 |

Now we compute the transformation parameters $\theta_{x}, \theta_{y}, \theta_{z}, t_{1}, t_{2}, t_{3}, k$ from the local geodetic system to WGS-84. We use the CAS system Mathematica to find the transformation parameters, where it is convenient express dual quaternions in the $8 \times 8$ matrix form. The optimization problem was solved using Lagrange multipliers.

Table 2: Quaternion and translation parameters.

| Quaternion $\mathcal{Q}$ |  | Translation $\triangle \mathbf{t}$ |  |
| :--- | ---: | ---: | ---: |
| $q_{0}$ | -0.9999999987474 | $\triangle t_{1}$ | $-6.649 \times 10^{-10}$ |
| $q_{1}$ | -0.0000024204319 | $\triangle t_{2}$ | $-3 \times 10^{-13}$ |
| $q_{2}$ | 0.0000021663738 | $\triangle t_{3}$ | $2.658 \times 10^{-10}$ |
| $q_{3}$ | 0.0000024073178 |  |  |

The quaternion $\mathcal{Q}$ and the translation $\triangle \mathbf{t}$ are shown in Table 2. The rotation matrix $\mathbf{R}$ computed from (44) is expressed in (47). The final list of results given from (45), (46) and Lagrange multipliers are listed in Table 3.

$$
\mathbf{R}=\left[\begin{array}{ccc}
0.9999999974902 & 4.8146252 \times 10^{-6} & -4.3327593 \times 10^{-6}  \tag{47}\\
-4.8146461 \times 10^{-6} & 0.9999999974879 & -4.8408533 \times 10^{-6} \\
4.3327360 \times 10^{-6} & 4.8408742 \times 10^{-6} & 0.9999999974901
\end{array}\right]
$$

Table 3: Final results of dual quaternion transformation algorithm.

| Rotation angles |  | Translation $\mathbf{t}$ |  | Scale $k$ |  |
| :---: | ---: | ---: | :---: | :---: | :---: |
| $\theta_{x}$ | $-0.99850^{\prime \prime}$ | $t_{1}$ | $641.8908 m$ |  |  |
| $\theta_{y}$ | $0.89370^{\prime \prime}$ | $t_{2}$ | $68.6570 m$ | $k$ |  |
| $\theta_{z}$ | $0.99309^{\prime \prime}$ | $t_{3}$ | $416.4101 m$ |  |  |

In addition to this transformation, we compute the error matrix $\mathbf{E}$, i.e., the difference between coordinates of the system $A$ and the new coordinates of the system $A^{\prime}$. The coordinates
of the system $A^{\prime}$ are determined using the computed transformation parameters $\mathbf{R}, \mathbf{t}, k$ and the substitution (30). The transformation parameters and transformed coordinates are equal to the parameters described by [6] and [9].

Table 4: Residual matrix of transformation parameters.

| Residual matrix $\mathbf{E}(m)$ |  |  |  |
| :--- | ---: | ---: | ---: |
| Station name | $X(m)$ | $Y(m)$ | $Z(m)$ |
| Solitude | 0.0940 | 0.1351 | 0.1402 |
| Buoch Zeil | 0.0588 | -0.0497 | 0.0137 |
| Hohenneuffen | -0.0399 | -0.0879 | -0.0081 |
| Kuehlenberg | 0.0203 | -0.0221 | -0.0875 |
| Ex Mergelaec | 0.0919 | 0.0139 | -0.0055 |
| Ex Hof Asperg | -0.0118 | 0.0065 | -0.0546 |
| Ex Kaisersbach | -0.0294 | 0.0041 | 0.0017 |

## 6 Conclusion

There is a strong motivation for dealing with the problem of finding parameters of transformation of two coordinate systems, the local and the world one. Datum transformation is widely used in geodesy. This paper describes one of the methods for determination of the datum transformation parameters. We use a nonlinear transformation model. In this model we can easily use a description by the dual quaternions. The dual quaternions allow us to describe any rigid transformation, i.e., a composition of rotations and translations. Main advantages of this approach are the simplification of the original solution of the datum transformation. The maximal error of the method can be estimated by the error matrix and is similar to other methods based on e.g. quaternions. This paper presents one numerical example to demonstrate the introduced formula describing the datum transformation. Another advantage of the novel approach lies in the fact that there is not need a linearization of the nonlinear transformation model.

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# GENERAL CONFORMAL ALMOST SYMPLECTIC N-LINEAR CONNECTIONS ON SECOND ORDER COTANGENT BUNDLE 

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#### Abstract

In the present paper starting from the notions of conformal almost symplectic N -linear connection and conformal almost symplectic structure we define the notions of: conformal almost symplectic N -linear connection and general conformal almost symplectic N -linear connection. We determine the set of all general conformal almost symplectic N linear connections in the case when the nonlinear connection is arbitrary and we find important particular cases.


Key words and phrases. Second order cotangent bundle, nonlinear connection, Nlinear connection, almost symplectic structure, conformal almost symplectic structure, conformal almost symplectic N -linear connection, general conformal almost symplectic N -linear connection.
Mathematics Subject Classification. 53B40,58B20,53C05,53C20,53C60

## 1 Introduction

The differential geometry of the second order cotangent bundle $\left(T^{* 2} M, \pi^{*^{2}}, M\right)$ was introduced and studied by R. Miron in [3], R. Miron, D. Hrimiuc, H. Shimada, V.S. Sabău in [5], Gh. Atanasiu and M. Târnoveanu in [1], etc.

In the present section we keep the general setting from R. Miron, D. Hrimiuc, H. Shimada, V.S. Sabău, [5] and subsequently we recall only some needed notions. For more details see [5] .

Let $M$ be a real $n$-dimensional manifold and let $\left(T^{*^{2}} M, \pi^{*^{2}}, M\right)$ be the dual of the 2 -tangent bundle, or 2 -cotangent bundle. A point $u \in T^{*^{2}} M$ can be written in the form $u=(x, y, p)$, having the local coordinates $\left(x^{i}, y^{i}, p_{i}\right),(i=1,2, \ldots, n)$.

A change of local coordinates on the $3 n$ dimensional manifold $T^{*^{2}} M$ is

$$
\left\{\begin{array}{l}
\bar{x}^{i}=\bar{x}^{i}\left(x^{1}, \ldots, x^{n}\right), \operatorname{det}\left(\frac{\partial \bar{x}^{i}}{\partial x^{j}}\right) \neq 0,  \tag{1.1}\\
\bar{y}^{i}=\frac{\partial \bar{x}^{i}}{\partial x^{j}} \cdot y^{j} \\
\bar{p}_{i}=\frac{\partial x^{j}}{\partial \bar{x}^{i}} \cdot p_{j},(i, j=1,2, \ldots, n)
\end{array}\right.
$$

We denote by $T^{*^{2}} M=T^{*^{2}} M-\{0\}$ where $0: M \longrightarrow T^{*^{2}} M$ is the null section of the projection $\pi^{*^{2}}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*^{2}} M,\left(T T^{*^{2}} M, \tau^{*^{2}}, T^{*^{2}} M\right)$, where $\tau^{*^{2}}$ is the canonical projection and the vertical distribution $V: u \in T^{*^{2}} M \longrightarrow V(u) \subset T_{u} T^{*^{2}} M$, locally generated by the vector fields $\left\{\left.\frac{\partial}{\partial y^{2}}\right|_{u},\left.\frac{\partial}{\partial p_{i}}\right|_{u}\right\}$, $\forall u \in T^{*^{2}} M$.

We denote with $N$ a nonlinear connection on the manifold $T^{* 2} M$, with the local coefficients $\left(N^{j}{ }_{i}(x, y, p), N_{i j}(x, y, p)\right),(i, j=1,2, \ldots, n)$.

Hence, the tangent space of $T^{* 2} M$ in the point $u \in T^{* 2} M$ is given by the direct sum of vector spaces:

$$
\begin{equation*}
T_{u} T^{* 2} M=N(u) \oplus W_{1}(u) \oplus W_{2}(u), \forall u \in T^{* 2} M \tag{1.2}
\end{equation*}
$$

A local adapted basis to the direct decomposition (1.2) is given by:

$$
\begin{equation*}
\left\{\frac{\delta}{\delta x^{i}}, \frac{\partial}{\partial y^{i}}, \frac{\partial}{\partial p_{i}}\right\},(i=1,2, \ldots, n) \tag{1.3}
\end{equation*}
$$

where:

$$
\begin{equation*}
\frac{\delta}{\delta x^{i}}=\frac{\partial}{\partial x^{i}}-N^{j}{ }_{i} \frac{\partial}{\partial y^{j}}+N_{i j} \frac{\partial}{\partial p_{j}} . \tag{1.4}
\end{equation*}
$$

With respect to the coordinates transformations (1.1), we have the rules:

$$
\begin{equation*}
\frac{\delta}{\delta x^{i}}=\frac{\partial \bar{x}^{j}}{\partial x^{i}} \frac{\delta}{\delta \bar{x}^{j}} ; \quad \frac{\partial}{\partial y^{i}}=\frac{\partial \bar{x}^{j}}{\partial x^{i}} \cdot \frac{\partial}{\partial \bar{y}^{j}} ; \quad \frac{\partial}{\partial p_{i}}=\frac{\partial x^{i}}{\partial \bar{x}^{j}} \cdot \frac{\partial}{\partial \bar{p}_{j}} . \tag{1.4}
\end{equation*}
$$

The dual basis of the adapted basis (1.3) is given by:

$$
\begin{equation*}
\left\{\delta x^{i}, \delta y^{i}, \delta p_{i}\right\} \tag{1.5}
\end{equation*}
$$

where:

$$
\begin{equation*}
\delta x^{i}=d x^{i}, \delta y^{i}=d y^{i}+N^{i}{ }_{j} d x^{j}, \delta p_{i}=d p_{i}-N_{j i} d x^{j} . \tag{1.5}
\end{equation*}
$$

With respect to (1.1), the covector fields (1.5) are transformed by the rules:

$$
\begin{equation*}
\delta \bar{x}^{i}=\frac{\partial \bar{x}^{i}}{\partial x^{j}} \delta x^{j}, \delta \bar{y}^{i}=\frac{\partial \bar{x}^{i}}{\partial x^{j}} \delta y^{j}, \delta \bar{p}_{i}=\frac{\partial x^{j}}{\partial \bar{x}^{i}} \delta p_{j} . \tag{1.5}
\end{equation*}
$$

Let $D$ be an $N$-linear connection on $T^{* 2} M$, with the local coefficients in the adapted basis: $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$.

An N-linear connection: $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$. determines the $h-, w_{1}-, w_{2}-$ covariant derivatives in the tensor algebra of d-tensor fields.

We consider on $T^{*^{2}} M$ an almost symplectic structure A given only by a nonsingular and skewsymmetric d-tensor field $a_{i j}$, of the type $(0,2)$ :

$$
\begin{equation*}
A(x, y, p)=\frac{1}{2} a_{i j}(x, y, p) d x^{i} \wedge d x^{j}+\frac{1}{2} a_{i j}(x, y, p) d y^{i} \wedge d y^{j}+\frac{1}{2} a^{i j}(x, y, p) \delta p_{i} \wedge \delta p_{j} \tag{1.6}
\end{equation*}
$$

The contravariant tensor field $a^{i j}$ is obtained from the equations:

$$
a_{i j} a^{j k}=\delta_{i}^{k} .
$$

Definition 1.1 An $N$-linear connection $D$ is called almost symplectic if:

$$
\begin{equation*}
a_{i j \mid k}=0,\left.\quad a_{i j}\right|_{k}=0,\left.\quad a_{i j}\right|^{k}=0 \tag{1.7}
\end{equation*}
$$

where $\mathrm{I}_{k}, \mathrm{I}_{k}$ and $\left.\right|^{k}$ denote the $h-, w_{1}-$ and $w_{2}-$ covariant derivatives with respect to D .
We associate to the lift A the operators of Obata's type given by:

$$
\begin{equation*}
\Phi_{h k}^{i j}=\frac{1}{2}\left(\delta_{h}^{i} \delta_{k}^{j}-a_{h k} a^{i j}\right), \Phi_{h k}^{* i j}=\frac{1}{2}\left(\delta_{h}^{i} \delta_{k}^{j}+a_{h k} a^{i j}\right) . \tag{1.8}
\end{equation*}
$$

Obata's operators have the same properties as the ones associated with a Finsler space [4].
Let $\mathcal{A}_{2}\left(T^{*^{2}} M\right)$ be the set of all skewsymmetric d-tensor fields, of the type $(0,2)$ on $T^{*^{2}} M$. As is easily shown, the relation on $\mathcal{A}_{2}\left(T^{*^{2}} M\right)$ defined by (1.9):

$$
\begin{equation*}
\left(a_{i j} \sim b_{i j}\right) \Leftrightarrow\left((\exists) \lambda(x, y, p) \in \mathcal{F}\left(T^{*^{2}} M\right), a_{i j}(x, y, p)=e^{2 \lambda(x, y, p)} b_{i j}(x, y, p),\right) \tag{1.9}
\end{equation*}
$$

is an equivalence relation on $\mathcal{A}_{2}\left(T^{*^{2}} M\right)$.
Definition 1.2 The equivalent class $\hat{A}$ of $\mathcal{A}_{2}\left(T^{*^{2}} M\right) / \sim$ to which $A$ belongs, is called conformal almost symplectic structure on $T^{*^{2}} M$.

Thus:

$$
\begin{equation*}
\hat{A}=\left\{A^{\prime} \mid a_{i j}^{\prime}(x, y, p)=e^{2 \lambda(x, y, p)} a_{i j}(x, y, p), \lambda(x, y, p) \in \mathcal{F}\left(T^{* *^{2}} M\right)\right\} . \tag{1.10}
\end{equation*}
$$

## 2 General conformal almost symplectic $\mathbf{N}$-linear connections on $T^{*^{2}} M$

Definition 2.1 An N-linear connection, $D$, with local coefficients: $D \Gamma(N)=$
$=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$, is called general conformal almost symplectic $N$-linear connection with respect to $\hat{A}$ if:

$$
\begin{equation*}
a_{i j \mid k}=K_{i j k},\left.a_{i j}\right|_{k}=Q_{i j k},\left.a_{i j}\right|^{k}=\dot{Q}_{i j}^{k} \tag{2.1}
\end{equation*}
$$

where $\mathbf{I}_{k},\left.\right|_{k}$ and $\left.\right|^{k}$ denote the $h-, w_{1}-$ and $w_{2}-$ covariant derivatives with respect to $D$ and $K_{i j k}, Q_{i j k}, \dot{Q}_{i j}{ }^{k}$ are arbitrary tensor fields of the types $(0,3),(0,3)$ and $(2,1)$ respectively, with the properties:

$$
\begin{equation*}
K_{i j k}=-K_{j i k}, Q_{i j k}=-Q_{j i k}, \dot{Q}_{i j}^{k}=\dot{-} Q_{j i}^{k} \tag{2.2}
\end{equation*}
$$

Particularly we have:
Definition 2.2 An N-linear connection, $D$, with local coefficients: $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}\right.$, $C_{i}{ }^{j k}$ ), for which there exists the 1-form $\omega, \omega=\omega_{i} d x^{i}+\dot{\omega}_{i} \delta y^{i}+\ddot{\omega}^{i} \delta p_{i}$, such that:

$$
\begin{equation*}
a_{i j \mid k}=2 \omega_{k} a_{i j},\left.\quad a_{i j}\right|_{k}=2 \dot{\omega}_{k} a_{i j},\left.\quad a_{i j}\right|^{k}=2 \ddot{\omega}^{k} a_{i j} \tag{2.3}
\end{equation*}
$$

where $\boldsymbol{I}_{k},\left.\right|_{k}$ and $\left.\right|^{k}$ denote the $h-, w_{1}-$ and $w_{2}-$ covariant derivatives with respect to $D$ is called conformal almost symplectic $N$-linear connection, with respect to the conformal almost symplectic structure $\hat{A}$, corresponding to the 1 -form $\omega$ and is denoted by: $D \Gamma(N, \omega)$.

For any representative $A^{\prime} \in \hat{A}$,
$A^{\prime}=\frac{1}{2} a_{i j}^{\prime}(x, y, p) d x^{i} \wedge d x^{j}+\frac{1}{2} a_{i j}^{\prime}(x, y, p) d y^{i} \wedge d y^{j}+\frac{1}{2} a^{i j}(x, y, p) \delta p_{i} \wedge \delta p_{j}$, we have:
Theorem 2.1 For $a_{i j}^{\prime}=e^{2 \lambda} a_{i j}$, a conformal almost symplectic N-linear connection, with respect to the conformal almost symplectic structure $\hat{A}$, corresponding to the 1-form $\omega, D \Gamma(N, \omega)$ satisfies:

$$
\begin{equation*}
a_{i j \mid k}^{\prime}=2 \omega_{k}^{\prime} a_{i j}^{\prime},\left.\quad a_{i j}^{\prime}\right|_{k}=2 \dot{\omega}_{k}^{\prime} a_{i j}^{\prime},\left.\quad a_{i j}^{\prime}\right|^{k}=2 \ddot{\omega}^{\prime k} a_{i j}^{\prime} \tag{2.4}
\end{equation*}
$$

where $\omega^{\prime}=\omega+d \lambda$.

Since in Theorem 2.1. $\omega^{\prime}=0$ is equivalent to $\omega=d(-\lambda)$ we have:

Theorem 2.2 A conformal almost symplectic N-linear connection, with respect to the conformal almost symplectic structure $\hat{A}$, corresponding to the 1-form $\omega$, denoted by: $D \Gamma(N, \omega)$, is almost symplectic with respect to $A^{\prime} \in \hat{A}$, i.e. $a_{i j \mid k}^{\prime}=\left.a_{i j}^{\prime}\right|_{k}=\left.a_{i j}^{\prime}\right|^{k}=0$ if and only if $\omega$ is exact.

We shall determine the set of all general conformal almost symplectic N-linear connections, with respect to $\hat{A}$.

Let $\stackrel{0}{D} \Gamma(\stackrel{0}{N})=\left(\stackrel{0}{H^{i}}{ }_{j k}, \stackrel{0}{C}^{i}{ }_{j k}, C_{i}^{0}{ }^{j k}\right)$ be the local coefficients of a fixed $\stackrel{0}{N}$ - linear connection $\stackrel{0}{D}$, where $\left(\stackrel{0}{N}^{j}{ }_{i}(x, y, p), \stackrel{0}{N}_{i j}(x, y, p)\right),(i, j=1,2, \ldots, n)$ are the local coefficients of the nonlinear connection $\stackrel{0}{N}$.

Then any N-linear connection, D, with the local coefficients $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$, where $\left(N^{j}{ }_{i}(x, y, p), N_{i j}(x, y, p)\right),(i, j=1,2, \ldots, n)$ are the local coefficients of the nonlinear
connection $N$, can be expressed in the form ([7]):

$$
\left\{\begin{array}{l}
N_{j}^{i}={ }^{0} N^{i}{ }_{j}-A_{j}^{i},  \tag{2.5}\\
N_{i j}=N_{i j}-A_{i j}, \\
H^{i}{ }_{j k}=H^{i}{ }_{j k}+A_{k}{ }_{k} C^{i}{ }_{j l}-A_{k l} C_{j}{ }^{i l}-B^{i}{ }_{j k}, \\
C^{i}{ }_{j k}=C^{i}{ }_{j k}-D^{i}{ }_{j k}, \\
C_{i}^{j k}=C_{i}{ }^{j k}-D_{i}^{j k},(i, j, k=1,2, \ldots, n),
\end{array}\right.
$$

with

$$
\begin{equation*}
A_{i \mid j}^{k}=0, A_{i k \mid j}^{0}=0(i, j, k=1,2, \ldots, n), \tag{2.6}
\end{equation*}
$$

where ${ }_{\mathbf{I}}^{\mathbf{I}}$ denotes the h-covariant derivative with respect to $\stackrel{0}{D}$ and $\left(A_{j}^{i}, A_{i j}, B_{j k}^{i}, D_{j k}^{i}, D_{i}^{j k}\right)$ are the components of the difference tensor fields of D from ${ }^{D}$ ([2] for Finsler connections).

Using the relations (2.1), (2.5), (1.4) and the Theorem 1 given by R.Miron in [4] for the case of Finsler connections we obtain:
Theorem 2.3 Let $\stackrel{0}{D}$ be a given $\stackrel{0}{N}$-linear connection, with local coefficients ${ }^{D} \Gamma(\stackrel{0}{N})=$ $=\left(\begin{array}{c}{ }^{i}{ }^{i}{ }_{j k}, \stackrel{0}{C}^{i}{ }_{j k}, C_{i}^{j} \\ \end{array}\right)$. The set of all general conformal almost symplectic $N$-linear connections, with respect to $\hat{A}$, with local coefficients $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$ is given by:
with:

$$
\begin{equation*}
X_{i \mid j}^{k}=0, X_{i k \mid j}^{0}=0,(i, j, k=1,2, \ldots, n), \tag{2.8}
\end{equation*}
$$

where $\left.{ }^{0}\right|_{k}, \stackrel{0}{\mid}_{k}$ and ${ }^{0}$ denote the $h-, w_{1}-$ and $w_{2}-$ covariant derivatives with respect to $\stackrel{0}{D}$, $X^{i}, X_{i j}, X^{i}{ }_{j k}, Y^{i}{ }_{j k}, Z_{i}^{j k}$ are arbitrary d-tensor fields, $K_{i j k}, Q_{i j k}, \dot{Q}_{m i}{ }^{k}$ are arbitrary d-tensor fields of the types $(0,3),(0,3)$ and $(2,1)$ respectively, with the properties (2.2) and $\Phi$ is the operator of Obata's type gven by (1.8).

## Particular cases:

1. If we take $K_{i j k}=2 \omega_{k} a_{i j}, Q_{i j k}=2 \dot{\omega}_{k} a_{i j}, \dot{Q}_{i j}{ }^{k}=2 \ddot{\omega}^{k} a_{i j}$ in Theorem 2.3 we obtain:

Theorem 2.4 Let $\stackrel{0}{D}$ be a given $\stackrel{0}{N}$-linear connection, with local coefficients $\stackrel{0}{D} \Gamma(\stackrel{0}{N})=$ $=\left(\stackrel{0}{H}^{i}{ }_{j k}, \stackrel{0}{C^{i}}{ }_{j k}, C_{i}^{j}{ }^{j k}\right)$. The set of all conformal almost symplectic $N$-linear connections with respect to $\hat{A}$, corresponding to the 1-form $\omega$, with local coefficients $D \Gamma(N, \omega)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}^{j k}\right)$ is given by:

$$
\left\{\begin{array}{l}
N_{j}^{i}=\stackrel{0}{N^{i}}{ }_{0}-X^{i}{ }_{j}, N_{i j}=\stackrel{0}{N}_{i j}-X_{i j},  \tag{2.9}\\
H^{i}{ }_{j k}=H^{i}{ }_{j k}+X^{l}{ }_{k} C^{i}{ }_{j l}-X_{k l} C_{j}{ }^{i l}+ \\
+\frac{1}{2} g^{i m}\left(g{ }_{m j \mid k}{ }^{0}-2 \omega_{k} g_{m j}+\left.g_{m j}\right|_{l} X^{l}{ }_{k}-\right. \\
\quad 0 \\
\left.-\left.g_{m j}\right|^{l} X_{k l}\right)+\Phi_{s j}^{i r} X^{s}{ }_{r k}, \\
0^{i}{ }_{j k}=C^{i}{ }_{j k}+\frac{1}{2} g^{i m}\left(\left.g_{m j}\right|_{k}-2 \dot{\omega}_{k} g_{m j}\right)+ \\
+\Phi_{s j}^{i r} Y_{r k}^{s}, \\
C_{i}^{j k}=C_{i}{ }^{j k}+\frac{1}{2} g^{m j}\left(\left.g_{m i}\right|^{k}-2 \ddot{\omega}^{k} g_{m i}\right)+ \\
+\Phi_{s i}^{r j} Z_{r}^{s k},(i, j, k=1,2, \ldots, n),
\end{array}\right.
$$

with:

$$
\begin{equation*}
X_{i \mid j}^{k}=0, X_{i k \mid j}^{0}=0,(i, j, k=1,2, \ldots, n) \tag{2.10}
\end{equation*}
$$

where $\stackrel{0}{\mid}_{k}, \stackrel{0}{\mid}_{k}$ and $\left.\right|^{k}$ denote the $h-, w_{1}-$ and $w_{2}-$ covariant derivatives with respect to $\stackrel{0}{D}$, $X^{i}, X_{i j}, X^{i}{ }_{j k}, Y^{i}{ }_{j k}, Z_{i}^{j k}$ are arbitrary d-tensor fields, $\omega=\omega_{i} d x^{i}+\dot{\omega}_{i} \delta y^{i}+\ddot{\omega}^{i} \delta p_{i}$ is an arbitrary 1 -form and $\Phi$ is the operator of Obata's type given by (1.8).
2. If $X^{i}{ }_{j}=X_{i j}=X^{i}{ }_{j k}=Y^{i}{ }_{j k}=Z_{i}^{j k}=0$, in Theorem 2.3 we have:

Theorem 2.5 Let $\stackrel{0}{D}$ be a given $\stackrel{0}{N}$-linear connection, with local coefficients
$\stackrel{0}{D} \Gamma(\stackrel{0}{N})=\left(\stackrel{0}{H}^{i}{ }_{j k}, \stackrel{0}{C}^{i}{ }_{j k}, C_{i}^{0}{ }^{j k}\right)$. Then the following $N$-linear conection $K$, with local coefficients $K \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$ given by (2.11) is general conformal almost symplectic with respect
to $\hat{A}$ :

$$
\left\{\begin{array}{l}
H^{i}{ }_{j k}=\stackrel{0}{H}^{i}{ }_{j k}+\frac{1}{2} g^{i m}\left(g_{m j \mid k}^{0}-K_{m j k}\right),  \tag{2.11}\\
C^{i}{ }_{j k}=\stackrel{0}{C}^{i}{ }_{j k}+\frac{1}{2} g^{i m}\left(\left.g_{m j}\right|_{k}-Q_{m j k}\right), \\
C_{i}^{j k}=C_{i}{ }^{j k}+\frac{1}{2} g^{j m}\left(\left.g_{m i}\right|^{k}-\dot{Q}_{m i}{ }^{k}\right),
\end{array}\right.
$$

 $K_{i j k}, Q_{i j k}, \dot{Q}_{m i}{ }^{k}$ are arbitrary d-tensor fields of the types $(0,3),(0,3)$ and $(2,1)$ respectively, with the properties (2.2).
3. If we take a general conformal almost symplectic N-linear connection with respect to $\hat{A}$ as $\stackrel{0}{D}$, in Theorem 2.3. we have:

Theorem 2.6 Let ${ }_{D}^{D}$ be on $T^{*^{2}} M$ a fixed general conformal almost symplectic $N$-linear connection with respect to $\hat{A}$, with the local coefficients $\stackrel{0}{D} \Gamma(\stackrel{0}{N})=\left(\stackrel{0}{H}^{i}{ }_{j k}, \stackrel{0}{C}^{i}{ }_{j k}, C_{i}{ }^{j}{ }^{j k}\right)$. The set of all general conformal almost symplectic $N$-linear connections, with respect to $\hat{A}$, with local coefficients $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$ is given by:

$$
\left\{\begin{array}{l}
N_{j}^{i}=\stackrel{0}{N^{i}}{ }_{j}-X^{i}{ }_{j},  \tag{2.12}\\
N_{i j}=\stackrel{0}{N_{i j}}-X_{i j}, \\
H_{j k}^{i}=\stackrel{0}{H^{i}}{ }_{j k}+\left({ }_{C^{i}}{ }_{j l}+\frac{1}{2} g^{i m} Q_{m j l}\right) X_{k}^{l}-\left(C_{j}{ }^{i l}+\frac{1}{2} g^{i m} \dot{Q}_{m j}{ }^{l}\right) X_{k l}+\Phi_{s j}^{i r} X_{r k}^{s}, \\
C_{j k}^{i}=\stackrel{0}{C^{i}}{ }_{j k}+\Phi_{s j}^{i r} Y^{s}{ }_{r k}, \\
C_{i}^{j k}=C_{i}{ }^{j k}+\Phi_{s i}^{j r} Z_{r}^{s k}, \quad(i, j, k=1,2, \ldots, n),
\end{array}\right.
$$

with

$$
\begin{equation*}
X_{i \mid j}^{k}=0, X_{i k \mid j}^{0}=0,(i, j, k=1,2, \ldots, n), \tag{2.13}
\end{equation*}
$$

 $K_{i j k}, Q_{i j k}, \dot{Q}_{m i}{ }^{k}$ are arbitrary d-tensor fields of the types $(0,3),(0,3)$ and $(2,1)$ respectively, with the properties (2.2), $X_{j}^{i}, X_{i j}, X_{j k}^{i}, Y_{j k}^{i}, Z_{i}^{j k}$ are arbitrary d-tensor fields and $\Phi$ is the operator of Obata's type given by (1.8).
4. If we take $K_{i j k}=Q_{i j k}=\dot{Q}_{i j}{ }^{k}=0$ and $X^{i}{ }_{j}=X_{i j}=0$ in Theorem 2.3 we obtain the set of all almost symplectic $\stackrel{0}{N}$-linear connections in the case when the nonlinear connection is fixed:

Theorem 2.7 Let ${ }_{D}^{0}$ be a given $\stackrel{0}{N}$-linear connection, where $\stackrel{0}{N}$ is a fixed nonlinear connection, with local coefficients $\stackrel{0}{D} \Gamma(\stackrel{0}{N})=\left(\stackrel{0}{H}^{i}{ }_{j k},{ }^{0}{ }^{i}{ }_{j k}, C_{i}^{j k}\right)$. The set of all $\stackrel{0}{N}$-linear connections, with respect to $\hat{A}$ with local coefficients $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$ is given by:

$$
\left\{\begin{array}{l}
N_{j}^{i}={ }^{0} N^{i}{ }_{j},  \tag{2.14}\\
N_{i j}={ }^{0}{ }_{i j}, \\
H^{i}{ }_{j k}={ }^{0}{ }^{i}{ }_{j k}+\Phi_{s j}^{i r} X_{r k}^{s}, \\
C^{i}{ }_{j k}=C^{i}{ }_{j k}+\Phi_{s j}^{i r} Y^{s}{ }_{r k}, \\
C_{i}^{j k}=C_{i}{ }^{j k}+\Phi_{s i}^{j r} Z_{r}^{s k}, \quad(i, j, k=1,2, \ldots, n)
\end{array}\right.
$$

where $X^{i}{ }_{j k}, Y^{i}{ }_{j k}, Z_{i}^{j k}$ are arbitrary d-tensor fields and $\Phi$ is the operator of Obata's type given by (1.8).
5. If we take $K_{i j k}=Q_{i j k}=\dot{Q}_{i j}{ }^{k}=0$ in Theorem 2.3 we obtain the set of all almost symplectic N -linear connections in the case when the nonlinear connection is arbitrary:
Theorem 2.8 Let $\stackrel{0}{D}$ be a given $\stackrel{0}{N}$-linear connection, with local coefficients $\stackrel{0}{D} \Gamma(\stackrel{0}{N})=\left(\stackrel{0}{H}^{i}{ }_{j k}, \stackrel{0}{C}^{i}{ }_{j k}, C_{i}^{0}{ }^{j k}\right)$. The set of all almost symplectic $N$-linear connections, with local coefficients $D \Gamma(N)=\left(H^{i}{ }_{j k}, C^{i}{ }_{j k}, C_{i}{ }^{j k}\right)$ is given by:

$$
\left\{\begin{array}{l}
N_{j}^{i}=\stackrel{0}{N^{i}}{ }_{j}-X^{i}{ }_{j},  \tag{2.15}\\
N_{i j}=\stackrel{N}{N}_{i j}-X_{i j}, \\
H^{i}{ }_{j k}={ }_{H}{ }^{i}{ }_{j k}+X_{k}^{l}{ }_{k} C^{i}{ }_{j l}-X_{k l} C_{j}{ }^{i l}+\frac{1}{2} g^{i m}\left(g_{m j \mid k}{ }^{0}+\left.g_{m j}{ }^{0}\right|_{l} X_{k}^{l}-\left.g_{m j}\right|^{l} X_{k l}\right)+\Phi_{s j}^{i r} X_{r k}^{s}, \\
C^{i}{ }_{j k}=\stackrel{0}{C}^{i}{ }_{j k}+\left.\frac{1}{2} g^{i m} g_{m j}{ }_{k}\right|_{k}+\Phi_{s j}^{i r} Y_{r k}^{s}, \\
C_{i}^{j k}=C_{i}{ }^{j k}+\left.\frac{1}{2} g^{m j} g_{m j}\right|^{k}+\Phi_{s i}^{j r} Z_{r}{ }^{s k}
\end{array}\right.
$$

with:

$$
\begin{equation*}
X_{i \mid j}^{k}=0, X_{i k \mid j}^{0}=0,(i, j, k=1,2, \ldots, n) \tag{2.16}
\end{equation*}
$$

where ${ }^{\mathrm{I}_{k}}, \stackrel{0}{\mid}_{k}$ and ${ }^{0}$ denote the $h-, w_{1}-$ and $w_{2}-$ covariant derivatives with respect to $\stackrel{0}{D}$, $X^{i}{ }_{j}, X_{i j}, X^{i}{ }_{j k}, Y^{i}{ }_{j k}, Z_{i}^{j k}$ are arbitrary d-tensor fields and $\Phi$ is the operator of Obata's type given by (1.8).

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# ON THE INTEGRABILITY OF A STRUCTURE IN THE GEOMETRY OF THE 2-OSCULATOR BUNDLE 

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#### Abstract

In the paper herein we consider the group $\stackrel{m}{\mathcal{T}}_{N}{ }_{N}$ of transformations of semisymmetric metrical N-linear connections corresponding to the same nonlinear connection N on the 2-osculator bundle and we give its invariants, which are d-tensor fields. We study the 2 -forms on $T\left(O s c^{2} M\right)$ and we define the integrability of a 2-form. We give a result about the integrability of a d-metric structure. All this integrability relies only on the geometry of the 2 -osculator bundle $\left(O s c^{2} M, \pi, M\right)$.


Key words and phrases. 2-osculator bundle, d-metric structure, integrability, invariants, semi-symmetric metrical N -linear connection, transformations group.

Mathematics Subject Classification. Primary 53C05; Secondary 53C15, 53C21, 53C60, 53B05, 53B35, 53B40.

## 1 Introduction

The literature on the higher order Lagrange spaces geometry highlights the teoretical and practical importance of these spaces, (see eg.[9]-[12]).

A lot of mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus utilize multivariate Lagrangians of higher order acceleration, $L\left(x, \frac{d x}{d t}(t), \ldots, \frac{1}{k!} \frac{d^{k} x}{d t^{k}}(t)\right)$, (see. E.Cartan, [1] for $\mathrm{k}=2$, etc.).

From here one can see the reason of construction of the geometry of the total space of the hihger order accelerations (or the osculator bundle of hihger order) in local coordinates.

This construction is made by R.Miron and Gh. Atanasiu in the papers [11], [12], [13].
In the present section we recall the basic notions which are needed. For more detailes see [11],[12].

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Let $M$ be a real $C^{\infty}$-manifold with n dimensions and $\left(O s c^{2} M, \pi, M\right)$ its 2-osculator bundle, or the bundle of accelerations. The local coordinates on the 3 n -dimensional manifold $O s c^{2} M$ are denoted by $\left(x^{i}, y^{(1) i}, y^{(2) i}\right)=\left(x, y^{(1)}, y^{(2)}\right)=u,(i=1,2, \ldots n)$.

Let $\left\{\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial y^{(1) i}}, \frac{\partial}{\partial y^{(2) i}}\right\}$ be the natural basis of the tangent space, $T\left(O s c^{2} M\right)$, at the point $u \in O s c^{2} M$.

We set: $E=O s c^{2} M, \tilde{E}=O s c^{2} M \backslash\{0\}=\left\{\left(x, y^{(1)}, y^{(2)}\right) \in O s c^{2} M, \operatorname{rank}\left\|y^{(1) i}\right\|=1\right\}$.
Let us consider the natural 2-tangent structure on $E, J: \chi(E) \rightarrow \chi(E)$ given by:

$$
\begin{equation*}
J\left(\frac{\partial}{\partial x^{i}}\right)=\frac{\partial}{\partial y^{(1) i}}, J\left(\frac{\partial}{\partial y^{(1) i}}\right)=\frac{\partial}{\partial y^{(2) i}}, J\left(\frac{\partial}{\partial y^{(2) i}}\right)=0 . \tag{1.1}
\end{equation*}
$$

We denote with N a nonlinear connection on $E$ with the local coefficients $\left(N_{(1)}{ }_{j}^{i}, N_{(2)}{ }_{j}^{i}\right)$, $(i, j=1,2, \ldots, n)$.

Hence, the tangent space of $E$ in the point $u \in E$ is given by the direct sum of the linear vector spaces:

$$
\begin{equation*}
T_{u} E=N_{0}(u) \oplus N_{1}(u) \oplus V_{2}(u), \forall u \in E \tag{1.2}
\end{equation*}
$$

An adapted basis to the direct decomposition (1.2) is given by:

$$
\begin{equation*}
\left\{\frac{\delta}{\delta x^{i}}, \frac{\delta}{\delta y^{(1) i}}, \frac{\delta}{\delta y^{(2) i}}\right\},(i=1,2, \ldots, n) \tag{1.3}
\end{equation*}
$$

where:

$$
\begin{align*}
& \frac{\delta}{\delta x^{i}}=\frac{\partial}{\partial x^{i}}-N_{(1)}{ }^{j} \frac{\partial}{i \frac{\partial}{\partial y^{(1) j}}-N_{(2)}{ }^{j} \frac{\partial}{\partial y^{(2) j}},} \\
& \frac{\delta}{\delta y^{(1) i}}=\frac{\partial}{\partial y^{(1) i}}-N_{(1)}^{j} \frac{\partial}{\partial y^{(2) j}}, \quad \frac{\partial}{\delta y^{(2) i}}=\frac{\partial}{\partial y^{(2) i}} . \tag{1.4}
\end{align*}
$$

Let us consider the dual basis (cobasis) of (1.3):

$$
\begin{equation*}
\left\{d x^{i}, \delta y^{(1) i}, \delta y^{(2) i}\right\},(i=1,2, \ldots, n) \tag{1.5}
\end{equation*}
$$

where

$$
\begin{align*}
& \delta x^{i}=d x^{i}, \quad \delta y^{(1) i}=d y^{(1) i}+N_{(1)}{ }_{j}^{i} d x^{j}, \\
& \delta y^{(2) i}=d y^{(2) i}+N_{(1)}{ }_{j}^{i} d y^{(1) j}+\left(N_{(2)}{ }_{j}^{i}+N_{(1)}^{i}{ }_{m} N_{(1)}{ }^{m}{ }_{j}\right) d x^{j} . \tag{1.6}
\end{align*}
$$

Definition 1.1 A linear connection $D$ on $E, D: \chi(E) \times \chi(E) \rightarrow \chi(E)$ is called an $N$-linear connection on $E$ if it preserves by parallelism the horizontal distribution $N$ on $E$ and the 2-tangent structure $J$ is absolute parallel with respect to $D$.

An N-linear connection $D$ on $E$ is characterized by its coefficients in the adapted basis (1.3) in the form:

$$
\begin{align*}
& D_{\frac{\delta}{\delta x^{\delta}} \bar{\delta} \frac{\delta}{(\alpha) j}}=L^{i}{ }_{j k} \frac{\delta}{\delta y^{(\alpha) i}}, \\
& D_{\frac{\delta}{\delta y^{(\beta) k} k}} \frac{\delta}{\delta y^{(\alpha) j}} \tag{1.7}
\end{align*}=C_{(\beta) j k} \frac{\delta}{\delta y^{(\alpha) i}}, \quad\left(\beta=1,2 ; \alpha=0,1,2 ; y^{(0) i}=x^{i}\right) . ~ l
$$

The system of three functions:

$$
\begin{equation*}
D \Gamma(N)=\left(L_{j k}^{i}, C_{(1)}{ }_{j k}^{i}, C_{(2)}{ }_{j k}^{i}\right), \tag{1.8}
\end{equation*}
$$

are called the coefficients of the N -linear connection D .
The d-tensors of torsion of an N-linear connection D in the adapted basis (1.3) have the following expressions:

Definition 1.2 A covariant d-tensor field $g_{i j}$ on $\tilde{E}$, symmetric, $\operatorname{rank}\left\|g_{i j}\right\|=n$, and with constant signature is called a d-metric structure on $\tilde{E}$.

The Obata's operators asociated to the d-metric structure $g_{i j}$ are:

$$
\begin{equation*}
\Omega_{s j}^{i r}=\frac{1}{2}\left(\delta_{j}^{i} \delta_{j}^{r}-g_{s j} g^{i r}\right), \Omega_{s j}^{* i r}=\frac{1}{2}\left(\delta_{j}^{i} \delta_{j}^{r}+g_{s j} g^{i r}\right) . \tag{1.11}
\end{equation*}
$$

Obata's operators have the same properties as the ones associated with a Finsler space [14].

Definition 1.3 An N-linear connection $D$ on $E$, with local coefficients:
$D \Gamma(N)=\left(L^{i}{ }_{j k}, C_{(1)}{ }_{j k}^{i}, C_{(2)}^{i}{ }_{j k}\right)$ having the property:

$$
\begin{equation*}
g_{i j \mid k}=0,\left.\quad g_{i j}\right|_{k} ^{(\alpha)}=0, \quad(\alpha=1,2) \tag{1.12}
\end{equation*}
$$

where $g_{i j}$ is a d-metric structure on $\tilde{E}$ and I, ${ }^{(\alpha)}$ denote the $h$-and $v_{\alpha}$-covariant derivatives, $(\alpha=1,2)$ with respect to $D$, is said to be a metrical $N$-linear connection.

Definition 1.4 An $N$-linear connection, $D$, on $E$ is called semi-symmetric if the torsion tensor fields $T_{(0)}{ }^{i}{ }_{j k}, S_{(\alpha)}{ }_{j k}, \quad(\alpha=1,2)$ have the form:

$$
\left\{\begin{array}{l}
T_{(0)}{ }_{j k}^{i}=\frac{1}{n-1}\left(T_{(0) j} \delta_{k}^{i}-T_{(0) k} \delta_{j}^{i}\right)=\frac{1}{n-1} \mathcal{A}_{j k}\left\{T_{(0) j} \delta_{k}^{i}\right\},  \tag{1.13}\\
S_{(\alpha){ }_{j k}}^{i}=\frac{1}{n-1}\left(S_{(\alpha) j} \delta_{k}^{i}-S_{(\alpha) k} \delta_{j}^{i}\right)=\frac{1}{n-1} \mathcal{A}_{j k}\left\{S_{(\alpha) j} \delta_{k}^{i}\right\},(\alpha=1,2),
\end{array}\right.
$$

where $T_{(0) j}=T_{(0)}{ }^{i}{ }_{j i}, S_{(\alpha) j}=S_{(\alpha) j i}{ }^{i}$, are called $h-$ and $v_{\alpha}$-torsion vector fields $(\alpha=1,2)$.

## 2 The group of transformations of semi-symmetric metrical $N$-linear connections

Let us consider the transformations $t\left(\sigma_{j}, \tau_{(1) j}, \tau_{(2) j}\right): D \Gamma(N) \rightarrow D \bar{\Gamma}(N)$, of semi-symmetric metrical N -linear connections on E, corresponding to the same nonlinear connection N. Owing to [15] they are given by:

$$
\left\{\begin{array}{l}
\bar{L}_{j k}^{i}=L^{i}{ }_{j k}+\sigma_{j} \delta_{k}^{i}-g_{j k} g^{i s} \sigma_{s},  \tag{2.1}\\
\bar{C}_{(\alpha){ }_{j k}}=C_{(\alpha)}{ }^{i}{ }_{j k}+\tau_{(\alpha) j_{k}^{i} \delta_{k}^{i}-g_{j k} g^{i s} \tau_{(\alpha) s},},(\alpha=1,2),
\end{array}\right.
$$

where $\sigma_{j}=\frac{T_{(0) j}}{n-1}, \tau_{(\alpha) j}=\frac{S_{(\alpha) j}}{n-1}$ and $T_{(0) j}=T_{(0)}{ }^{i}{ }^{i}, \quad S_{(\alpha) j}=S_{(\alpha) j i}{ }^{i}, \quad(\alpha=1,2)$.
There is inferred:

Theorem 2.1 [16] The set $\stackrel{m}{\mathcal{T}}_{N}$ of all transformations $t\left(\sigma_{j}, \tau_{(1) j}, \tau_{(2) j}\right)$ : $D \Gamma(N) \rightarrow \bar{D} \Gamma(N)$ of semi-symmetric metrical N-linear connections, on $E$, corresponding to the same nonlinear connection $N$, given by (2.1), together with the mapping product:

$$
t\left(\bar{\sigma}_{j}, \bar{\tau}_{(1) j}, \bar{\tau}_{(2) j}\right) \circ t\left(\sigma_{j}, \tau_{(1) j}, \tau_{(2) j}\right)=t\left(\sigma_{j}+\bar{\sigma}_{j}, \tau_{(1) j}+\bar{\tau}_{(1) j}, \tau_{(2) j}+\bar{\tau}_{(2) j}\right)
$$

is an Abelian group. This group acts on the set of all semi-symmetric metrical $N$-linear connections corresponding to the same nonlinear connection $N$, transitively.

In order to determine invariants of the group $\stackrel{m s}{\mathcal{T}}_{N}$, we denote with:

$$
\begin{align*}
& t_{(\alpha)}{ }_{j k}^{i}=\mathcal{A}_{j k}\left\{\frac{\delta N_{(\alpha) j}^{i}}{\delta y^{(\alpha) k}}\right\}, \quad(\alpha=1,2), \tag{2.2}
\end{align*}
$$

where $\mathcal{S}_{i j k}\{\ldots\}$ denotes the cyclic summation: $\mathcal{S}_{i j k}\left\{A_{i j k}\right\}=A_{i j k}+A_{j k i}+A_{k i j}$ and with:

Remark 2.1 It is noted that: $t_{(\alpha)}{ }^{*}{ }_{i j k}, T_{(0)}{ }_{i j k}^{*}, R_{(0 \alpha)}{ }^{*}{ }_{i j k}, S_{(\alpha)}{ }^{*}{ }_{i j k}, R_{(12)}{ }^{*}{ }_{i j k},(\alpha=1,2)$ are alternate $\stackrel{1}{\mathcal{K}}_{(\alpha \alpha) i j k}, \stackrel{2}{\mathcal{S}}_{(22) i j k}, \stackrel{3}{\mathcal{S}}_{(\alpha \beta) i j k},(\alpha=1,2 ; \beta=1,2 ; \alpha \neq \beta)$, are alternate, with respect to: i,j and $\stackrel{2}{\mathcal{K}}_{(\alpha) i j k}, \mathcal{K}_{(\alpha \beta) i j k}, \stackrel{4}{\mathcal{S}}_{(21) i j k},(\alpha=1,2 ; \beta=1,2)$, are alternate with respect to: j,k.

Theorem 2.2 [16] The tensor fields: $t_{(\alpha){ }_{j k}}^{i}, R_{(0 \alpha) j k}{ }^{i}, P_{(12)}{ }_{j k}^{i}, P_{(21)}{ }_{j k}^{i}, t_{(\alpha)}{ }_{i j k}^{*}, T_{(0)}{ }_{i j k}, R_{(0 \alpha)}{ }^{*}{ }_{i j k}$, $S_{(\alpha)}{ }_{i j k}^{*}, R_{(12)}{ }^{*}{ }_{i j k}, C_{(\alpha)}{ }^{*}{ }_{i j k}, P_{(\alpha \beta)}{ }_{i j k}, Q_{(21)}{ }^{*}{ }_{i j k}, Q_{(22)}{ }^{*}{ }_{i j k}, P_{(\alpha \alpha)}{ }^{*}{ }_{i j k}, \stackrel{1}{\mathcal{K}}_{(\alpha \alpha) i j k}, \stackrel{2}{\mathcal{K}}_{(\alpha) i j k}, \mathcal{K}_{(\alpha \beta) i j k}, \hat{\mathcal{K}}_{(\alpha) i j k}$, $\stackrel{1}{\mathcal{S}}_{\text {ijk }}, \stackrel{2}{\mathcal{S}}_{(22) i j k}, \stackrel{3}{\mathcal{S}}_{(12) i j k}, \stackrel{3}{\mathcal{S}}_{(21) i j k}, \stackrel{4}{\mathcal{S}}_{(21) i j k},(\alpha=1,2 ; \beta=1,2 ; \alpha \neq \beta)$, are invariants of the group $\stackrel{m s}{\mathcal{T}_{N}}$.

## 3 2-forms on $T\left(O s c^{2} M\right)$

In the following we shall study the 2 -forms on $T\left(O s c^{2} M\right)$ and we shall define the integrability of a 2 -form.

Let $\Lambda^{k}\left(T\left(O s c^{2} M\right)\right)$ be the $\mathcal{F}$-module of all k -forms on the tangent bundle $T\left(O s c^{2} M\right)$, where $\mathcal{F}\left(T\left(O s c^{2} M\right)\right)$ is the ring of all differentiable functions on $T\left(O s c^{2} M\right)$. If N is a nonlinear connection given on $T\left(O s c^{2} M\right)$, then $\left\{d x^{i}, \delta y^{(1) i}, \delta y^{(2) i}\right\}$ makes a local basis of $\Lambda^{1}\left(T\left(O s c^{2} M\right)\right.$ ), which is dual to $\left\{\frac{\delta}{\delta x^{2}}, \frac{\delta}{\delta y^{(1) i}}, \frac{\delta}{\delta y^{(2) i}}\right\}$.

If $f \in \mathcal{F}\left(T\left(O s c^{2} M\right)\right)$, then the 1 -form df is written as:

$$
d f=\frac{\delta f}{\delta x^{i}} d x^{i}+\frac{\delta f}{\delta y^{(1) i}} \delta y^{(1) i}+\frac{\partial f}{\partial y^{(2) i}} \delta y^{(2) i}
$$

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and the exterior differentials of $\delta x^{i}, \delta y^{(1) i}, \delta y^{(2) i}$ are given by [11]:

If we express $\omega \in \Lambda^{1}\left(T\left(O s c^{2} M\right)\right.$ ), in the form:

$$
\begin{equation*}
d \omega=\tilde{\omega}_{i} d x^{i}+\dot{\omega}_{(1) i} \delta y^{(1) i}+\dot{\omega}_{(2) i} \delta y^{(2) i} \tag{3.3}
\end{equation*}
$$

the exterior differential $d \omega$ is given by:

$$
\begin{align*}
& d \omega=\frac{1}{2} \tilde{\omega}_{i j} d x^{j} \wedge d x^{i}+\stackrel{(10)}{\omega}_{i j} \delta y^{(1) j} \wedge d x^{i}+\stackrel{(20)}{\omega}_{i j} \delta y^{(2) j} \wedge d x^{i}+ \\
& +\frac{1}{2} \dot{\omega}_{(1) i j} \delta y^{(1) j} \wedge \delta y^{(1) i}+\stackrel{(21)}{\omega}_{i j} \delta y^{(2) j} \wedge \delta y^{(1) i}+\frac{1}{2} \dot{\omega}_{(2) i j} \delta y^{(2) j} \wedge \delta y^{(2) i} \tag{3.4}
\end{align*}
$$

where:

$$
\left\{\begin{array}{l}
\tilde{\omega}_{i j}=\frac{\delta \tilde{\omega}_{i}}{\delta x^{j}}-\frac{\delta \tilde{\omega}_{j}}{\delta x^{i}}+R_{(01)}{ }^{m}{ }_{i j} \dot{\omega}_{(1) m}+R_{(02)}{ }^{m}{ }_{i j} \dot{\omega}_{(2) m},  \tag{3.5}\\
\stackrel{(10)}{\omega}_{i j}=\frac{\delta \tilde{\omega}_{i}}{\delta y^{(1) j}}-\frac{\delta \dot{\omega}_{(1) j}}{\delta x^{i}}+B_{(11)}{ }^{m}{ }_{i j} \dot{\omega}_{(1) m}+B_{(12)}{ }^{m}{ }_{i j} \dot{\omega}_{(2) m}, \\
{ }^{(20)}{ }_{i j}=\frac{\partial \tilde{\omega}_{i}}{\partial y^{(2) j}}-\frac{\delta \dot{\omega}_{(2) j}}{\delta x^{i}}+B_{(21)}{ }^{m}{ }_{i j} \dot{\omega}_{(1) m}+B_{(22)}{ }^{m}{ }_{i j} \dot{\omega}_{(2) m}, \\
\dot{\omega}_{(1) i j}=\frac{\delta \dot{\omega}_{(1) i}}{\delta y^{(1) j}}-\frac{\delta \dot{\omega}_{(1) j}}{\delta y^{(1) i}}+R_{(12)} m_{i j} \dot{\omega}_{(2) m}, \dot{\omega}_{(2) i j}=\frac{\partial \dot{\omega}_{(2) i}}{\partial y^{(2) j}}-\frac{\partial \dot{\omega}_{(2) j}}{\partial y^{(2) i}}, \\
{ }^{(21)}{ }_{\omega i j}=\frac{\partial \dot{\omega}_{1(1) i}}{\partial y^{(2) j}}-\frac{\delta \dot{\omega}_{(2) i}}{\delta y^{(1) j}}+B_{(21)}{ }^{m}{ }_{i j} \dot{\omega}_{(2) m},
\end{array}\right.
$$

are tensor fields.
In fact, written as tensorial expressions we have:

Proposition 3.1 If an N-linear connection $D$, with local coefficients:
$D \Gamma(N)=\left(L_{j k}^{i}, C_{(1) j k}^{i}, C_{(2) j k}^{i}\right)$, is given on E, then the coefficients $\tilde{\omega}_{i j}, \stackrel{(10)}{\omega}_{i j}, \stackrel{(20)}{\omega}_{i j}, \dot{\omega}_{(1) i j}, \stackrel{(21)}{\omega}_{i j}$, $\dot{\omega}_{(2) i j}$ have the following expressions:

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Generally, $\omega \in \Lambda^{2}\left(T\left(O s c^{2} M\right)\right)$ is written in the form:

$$
\begin{align*}
& \omega=\frac{1}{2} \tilde{a}_{i j} d x^{i} \wedge d x^{j}+\tilde{b}_{i j} d x^{i} \wedge \delta y^{(1) j}+\tilde{c}_{i j} d x^{i} \wedge \delta y^{(2) j}+  \tag{3.6}\\
& +\frac{1}{2} \tilde{d}_{i j} \delta y^{(1) i} \wedge \delta y^{(1) j}+\tilde{e}_{i j} \delta y^{(1) i} \wedge \delta y^{(2) j}+\frac{1}{2} \tilde{f}_{i j} \delta y^{(2) i} \wedge \delta y^{(2) j}
\end{align*}
$$

where $\tilde{a}_{i j}=-\tilde{a}_{j i}, \tilde{f}_{i j}=-\tilde{f}_{j i}, \tilde{d}_{i j}=-\tilde{d}_{j i}$.
The exterior differential $d \omega$ is given by:

$$
\begin{align*}
& d \omega=\frac{1}{6} \stackrel{1}{\omega}_{i j k} d x^{i} \wedge d x^{j} \wedge d x^{k}+\frac{1}{2} \stackrel{2}{\omega}_{i j k} d x^{i} \wedge d x^{j} \wedge \delta y^{(1) k}+ \\
& +\frac{1}{2} \stackrel{3}{\omega}_{i j k} d x^{i} \wedge d x^{j} \wedge \delta y^{(2) k}+\frac{1}{2} \stackrel{4}{\omega}_{i j k} d x^{i} \wedge \delta y^{(1) j} \wedge \delta y^{(1) k}+ \\
& +{ }_{\omega}^{5} \omega_{i j k} d x^{i} \wedge \delta y^{(1) j} \wedge \delta y^{(2) k}+\frac{1}{2} \stackrel{6}{\omega}_{i j k} d x^{i} \wedge \delta y^{(2) j} \wedge \delta y^{(2) k}+  \tag{3.7}\\
& \quad+\frac{1}{6} \stackrel{7}{\omega}_{i j k} \delta y^{(1) i} \wedge \delta y^{(1) j} \wedge \delta y^{(1) k}+\frac{1}{2} \stackrel{8}{\omega}_{i j k} \delta y^{(1) i} \wedge \delta y^{(1) j} \wedge \delta y^{(2) k}+ \\
& \\
& \quad+\frac{1}{2} \stackrel{9}{\omega}_{i j k} \delta y^{(1) i} \wedge \delta y^{(2) j} \wedge \delta y^{(2) k}+\frac{1}{6} \omega_{i j k} \delta y^{(2) i} \wedge \delta y^{(2) j} \wedge \delta y^{(2) k}
\end{align*}
$$

where:

$$
\begin{align*}
& \stackrel{1}{\omega}_{i j k}=S_{i j k}\left\{\frac{\delta \tilde{a}_{i j}}{\delta x^{k}}+\tilde{b}_{i m} R_{(01)}{ }^{m}{ }_{j k}+\tilde{c}_{i m} R_{(02)}{ }^{m}{ }_{j k}\right\}, \\
& \stackrel{2}{\omega}_{i j k}=\frac{\delta \tilde{a}_{i j}}{\delta y^{(1) k}}+\tilde{d}_{k m} R_{(01)}{ }^{m}{ }_{i j}+\tilde{e}_{k m} R_{(02)}{ }^{m}{ }_{i j}+\mathcal{A}_{i j}\left\{\frac{\delta \tilde{b}_{j k}}{\delta x^{i}}+\tilde{b}_{i m} B_{(11)}{ }^{m}{ }_{j k}+\tilde{c}_{i m} B_{(12)}{ }^{m}{ }_{j k}\right\}, \\
& \stackrel{3}{\omega}_{i j k}=\frac{\partial \tilde{a}_{i j}}{\partial y^{(2) k}}-\tilde{e}_{m k} R_{(01)}{ }^{m}{ }_{i j}+\tilde{f}_{k m} R_{(02)}{ }^{m}{ }_{i j}+\mathcal{A}_{i j}\left\{\frac{\delta \tilde{c}_{j k}}{\delta x^{i}}+\tilde{b}_{i m} B_{(21)}{ }^{m}{ }_{j k}+\tilde{c}_{i m} B_{(22)}{ }^{m}{ }_{j k}\right\}, \\
& \stackrel{4}{\omega}_{i j k}=\frac{\delta \tilde{d}_{j k}}{\delta x^{i}}+\tilde{c}_{i m} R_{(12)}{ }^{m}{ }_{j k}+\mathcal{A}_{j k}\left\{\frac{\delta \tilde{b}_{i j}}{\delta y^{(1) k}}+\tilde{d}_{k m} B_{(11)}{ }^{m}{ }_{i j}+\tilde{e}_{k m} B_{(12)}{ }^{m}{ }_{i j}\right\}, \\
& \stackrel{5}{\omega}_{i j k}=\frac{\partial \tilde{b}_{i j}}{\partial y^{(2) k}}-\frac{\delta \tilde{c}_{i k}}{\delta y^{(1) j}} \tilde{\tilde{l}}+\frac{\delta \tilde{e}_{j k}}{\delta x^{i}}+\tilde{c}_{i m} B_{(21)}{ }^{m}{ }_{j k}-\tilde{d}_{j m} B_{(21)}{ }^{m}{ }_{i k}-\tilde{e}_{j m} B_{(22)}{ }^{m}{ }_{i k}-  \tag{3.8}\\
& -\tilde{e}_{m k} B_{(11)}^{m}{ }_{i j}+\tilde{f}_{k m} B_{(12)}{ }^{m}{ }_{i j}, \\
& \stackrel{6}{\omega}_{i j k}=\frac{\delta \tilde{f}_{j k}}{\delta x^{i}}+\mathcal{A}_{j k}\left\{\frac{\partial \tilde{c}_{i j}}{\partial y^{(2) k}}-\tilde{e}_{m k} B_{(21)}{ }^{m}{ }_{i j}+\tilde{f}_{k m} B_{(22)}{ }^{m}{ }_{i j}\right\}, \\
& \stackrel{7}{\omega}_{i j k}=S_{i j k}\left\{\frac{\delta \tilde{d}_{i j}}{\delta y^{(1) k}}+\tilde{e}_{i m} R_{(12)}{ }^{m}{ }_{j k}\right\}, \\
& {\stackrel{8}{\omega_{i j k}}}_{i}=\frac{\partial \tilde{d}_{i j}}{\partial y(2) k}+\tilde{f}_{k m} R_{(12)}{ }^{m}{ }_{i j}+\mathcal{A}_{i j}\left\{\frac{\delta \tilde{\delta}_{j k}}{\delta y^{(1) i}}+\tilde{e}_{i m} B_{(21)}{ }^{m}{ }_{j k}\right\}, \\
& \stackrel{9}{\omega}_{i j k}=\frac{\partial \tilde{f}_{j k}}{\delta y^{(1) i}}+\mathcal{A}_{j k}\left\{\frac{\partial \tilde{e}_{i j}}{\partial y^{(2) k}}+\tilde{f}_{k m} B_{(21)}{ }^{m}{ }_{i j}\right\}, \stackrel{10}{\omega}_{i j k}=S_{i j k}\left\{\frac{\partial \tilde{f}_{i j}}{\partial y^{(2) k}}\right\} \text {. }
\end{align*}
$$

Proposition 3.2 If an $N$-linear connection $D$, with the local coefficients:
$D \Gamma(N)=\left(L_{j k}^{i}, C_{(1) j k}^{i}, C_{(2) j k}^{i}\right)$, is given on $E$, then the coefficients $\stackrel{\alpha}{\omega}_{i j k},(\alpha=1,2, \ldots, 10)$ have the following expressions:

$$
\begin{aligned}
& \stackrel{1}{\omega}_{i j k}=S_{i j k}\left\{\tilde{a}_{i j \mid k}+\tilde{a}_{i m} T_{(0)}{ }^{m}{ }_{j k}+\tilde{b}_{i m} R_{(01)}{ }^{m}{ }_{j k}+\tilde{c}_{i m} R_{(02)}{ }^{m}{ }_{j k}\right\}, \\
& \stackrel{2}{\omega}_{i j k}=\tilde{a}_{i j}{ }_{\mid}^{(1)}{ }_{k}+\tilde{b}_{m k} T_{(0)}{ }_{j i}^{m}+\tilde{d}_{k m} R_{(01)}{ }^{m}{ }_{i j}+\tilde{e}_{k m} R_{(02)}{ }^{m}{ }_{i j}+\mathcal{A}_{i j}\left\{\tilde{b}_{j k \mid i}+\tilde{a}_{i m} C_{(1) j k}^{m}+\right. \\
& \left.+\tilde{b}_{i m} P_{(11)}{ }_{j k}^{m}+\tilde{c}_{i m} P_{(12)}{ }_{j k}^{m}\right\},
\end{aligned}
$$

$$
\begin{align*}
& \left.+\tilde{b}_{i m} P_{(21)}{ }_{j k}^{m}+\tilde{c}_{i m} P_{(22)}{ }_{j k}\right\} \text {, } \\
& \stackrel{4}{\omega}_{i j k}=\tilde{d}_{j k \mid \tilde{\sim}}+\tilde{b}_{i m} S_{(1)}{ }^{m}{ }_{j k}+\tilde{c}_{i m} R_{(12)}{ }^{m}{ }_{j k}+\mathcal{A}_{j k}\left\{\tilde{b}_{m j} \stackrel{(1)}{\mid}_{k}+\tilde{b}_{m j} C_{(1) i k}^{m}+\right. \\
& \left.+\tilde{d}_{m j} P_{(11)}{ }_{i k}^{m}+\tilde{e}_{j m} P_{(12)}{ }_{i k}^{m}\right\} \text {, } \\
& \stackrel{5}{\omega}_{i j k}=\tilde{b}_{i j}{ }^{(2)}{ }_{k}-\tilde{c}_{i k}{ }^{(1)}{ }_{j}+\tilde{e}_{j k \mid i}+\tilde{b}_{m j} C_{(2) i k}^{m}+\tilde{b}_{i m} C_{(2) j k}^{m}+\tilde{c}_{i m} Q_{(22)}{ }_{j k}^{m}- \\
& -\tilde{d}_{j m} P_{(21)}{ }_{i k}^{m}-\tilde{e}_{j m} P_{(22)}{ }_{i k}^{m}-\tilde{c}_{m k} C_{(1) i j}^{m}+\tilde{f}_{k m} P_{(12)}{ }_{i j}^{m}-\tilde{e}_{m k} P_{(11)}{ }_{i j}^{m} \text {, } \\
& \stackrel{6}{\omega}_{i j k}=\tilde{f}_{j k \mid i}+\tilde{c}_{i m} S_{(2)}{ }^{m}{ }_{j k}+\mathcal{A}_{i j}\left\{\tilde{c}_{i j} \stackrel{(2)}{\mid}_{k}+\tilde{c}_{m j} C_{(2) i k}^{m}+\tilde{e}_{m j} P_{(21)}{ }_{i k}^{m}+\tilde{f}_{m j} P_{(22)}{ }_{i k}^{m}\right\}, \\
& \stackrel{7}{\omega}_{i j k}=\mathcal{S}_{i j k}\left\{\tilde{d}_{i j}{ }^{(1)}{ }_{k}+\tilde{d}_{i m} S_{(1)}{ }^{m}{ }_{i k}+\tilde{e}_{i m} R_{(12)}{ }^{m}{ }_{j k}\right\} \text {, } \\
& \stackrel{8}{\omega}_{i j k}=\tilde{d}_{i j} \stackrel{(2)}{\mid}_{k}+\tilde{e}_{m k} S_{(1)}{ }^{m}{ }_{j i}+\tilde{f}_{k m} R_{(12)}{ }^{m}{ }_{i j}+\mathcal{A}_{i j}\left\{\tilde{e}_{j k}{ }^{(1)}{ }_{i}+\tilde{d}_{i m} C_{(2) j k}^{m}+\tilde{e}_{i m} Q_{(22)}{ }^{m}{ }_{j k}\right\}, \\
& \stackrel{9}{\omega}_{i j k}=\tilde{f}_{j k}{ }^{(1)}{ }_{i}+\tilde{e}_{i m} S_{(2)}{ }_{j k}^{m}+\mathcal{A}_{j k}\left\{\left.\tilde{e}_{i j}\right|_{k} ^{(2)}+\tilde{e}_{m j} C_{(2) i k}^{m}+\tilde{f}_{m j} Q_{(22)}{ }_{i k}^{m}\right\}, \\
& { }_{\omega}^{10}{ }_{i j k}=\mathcal{S}_{i j k}\left\{\left.\tilde{f}_{i j}\right|_{k} ^{(2)}+\tilde{f}_{i m} S_{(2)}{ }^{m}{ }_{j k}\right\} .
\end{align*}
$$

For $\omega \in \Lambda^{2}\left(T\left(O s c^{2} M\right)\right)$ written in the form (3.6) we put:

$$
A=\left(\begin{array}{ccc}
\tilde{a}_{i j} & \tilde{b}_{i j} & \tilde{c}_{i j} \\
-\tilde{b}_{j i} & \tilde{d}_{i j} & \tilde{e}_{i j} \\
-\tilde{c}_{j i} & -\tilde{e}_{j i} & \tilde{f}_{i j}
\end{array}\right)
$$

Definition 3.1 $A$ 2-form $\omega \in \Lambda^{2}\left(T\left(O s c^{2} M\right)\right.$ ), written in the form (3.6), for which the matrix $A$ is non-degenerate, is called integrable if: $d \omega=0$.

Theorem 3.1 $A$ 2-form $\omega \in \Lambda^{2}\left(T\left(O s c^{2} M\right)\right.$ ), for which the matrix $A$ is non-degenerate, is integrable if and only if the tensor fields $\stackrel{\alpha}{\omega}_{i j k},(\alpha=1,2, \ldots, 10)$ vanish, where $\stackrel{\alpha}{\omega}_{i j k}$, $(\alpha=1,2, \ldots, 10)$ are given in (3.8').

## 4 About the integrability of a d-metric structure on the 2 -osculator bundle

We shall give a result about the integrability of a d-metric structure on the 2-osculator bundle.

Assuming that a nonlinear connection N is given on E , then a d-metric structure $g_{i j}$ on the base manifold $O s c^{2} M$ is lifted to a 2 -form $\omega$ having the form (3.6), on $T\left(O s c^{2} M\right)$ in various ways. We consider the following $\omega$ of three single types: II,III,V and combined types $I I+I I I, I I+V, I I I+V, I I+I I I+V, \alpha I I+\beta I I I+\gamma V, \alpha, \beta, \gamma \in R(4.1)$,
where:

|  | $\tilde{a}_{i j}$ | $\tilde{b}_{i j}$ | $\tilde{c}_{i j}$ | $\tilde{d}_{i j}$ | $\tilde{e}_{i j}$ | $\tilde{f}_{i j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| II | 0 | $g_{i j}$ | 0 | 0 | 0 | 0 |
| III | 0 | 0 | $g_{i j}$ | 0 | 0 | 0 |
| V | 0 | 0 | 0 | 0 | $g_{i j}$ | 0 |
| $\mathrm{II}+\mathrm{III}$ | 0 | $g_{i j}$ | $g_{i j}$ | 0 | 0 | 0 |
| $\mathrm{II}+\mathrm{V}$ | 0 | $g_{i j}$ | 0 | 0 | $g_{i j}$ | 0 |
| $\mathrm{III}+\mathrm{V}$ | 0 | 0 | $g_{i j}$ | 0 | $g_{i j}$ | 0 |
| $\mathrm{II}+\mathrm{III}+\mathrm{V}$ | 0 | $g_{i j}$ | $g_{i j}$ | 0 | $g_{i j}$ | 0 |
| $\alpha \mathrm{II}+\beta \mathrm{III}+\gamma \mathrm{V}$ | 0 | $\alpha g_{i j}$ | $\beta g_{i j}$ | 0 | $\gamma g_{i j}$ | 0 |

Proposition 4.1 The coefficients $\stackrel{\alpha}{\omega}_{i j k},(\alpha=1,2, \ldots, 10)$, given in (4.1) of the exterior differentials of the 2-forms $\omega$ written at (3.7) are invariants of the group $\stackrel{m s}{\mathcal{T}}_{N}$.

Proof. Calculating directly from Proposition 3.2. and using the notations (2.3) and (2.4) we have for the type II $+\mathrm{III}+\mathrm{V}$, the coefficients given in the following table:

|  | $\mathrm{II}+\mathrm{III}+\mathrm{V}$ |
| :---: | :---: |
| $\stackrel{1}{\omega}_{i j k}$ | $R_{(01)}{ }^{*}{ }_{i j k}+R_{(02)}{ }^{*}{ }_{i j k}$ |
| $\stackrel{2}{\omega}_{i j k}$ | $\stackrel{1}{k_{(11) i j k}+\stackrel{\stackrel{1}{S}(12) i j k}{ }+g_{k m} R_{(02)}{ }^{m}{ }_{i j}}$ |
| $\stackrel{3}{\omega}_{i j k}$ | $\stackrel{2}{S_{(21) i j k}}+\stackrel{1}{k_{(22)} i j k}+g_{k m} R_{(01)}{ }^{m}{ }_{i j}$ |
| $\stackrel{4}{4}_{i j k}$ | $\stackrel{2}{k_{(11) i j k}+g_{i m} R_{(12)}{ }_{j k}+\stackrel{3}{k_{(12) i j k}}, ~}$ |
| $\begin{equation*} \stackrel{5}{\omega}_{i j k} \tag{4.2} \end{equation*}$ | $\stackrel{4}{k_{(2) i j k}+g_{i m} P_{(21)}{ }_{j k}^{m}+\stackrel{5}{k_{(1) i j k}}+\stackrel{1}{S_{i j k}}, ~}$ |
| $\stackrel{6}{\omega}_{i j k}$ | $\begin{aligned} & 2 \\ & k(2) i j k+\stackrel{3}{k}(21) i j k \\ & \hline \end{aligned}$ |
| $\hat{\omega}_{i j k}$ | $R_{(12)}{ }^{*}{ }_{i j k}$ |
| $\stackrel{8}{\omega}_{i j k}$ | $\mathcal{N}^{1}{ }_{(22) i j k}$ |
| $\stackrel{9}{\omega}_{i j k}$ | $\begin{aligned} & \stackrel{2}{k}(2) i j k \end{aligned}$ |
| $\stackrel{10}{\omega}_{\omega}^{\text {ijk }}$ | 0 |

etc. Corresponding to Definition 3.1. we have:

Definition 4.1 A d-metric structure $g_{i j}$ on a differentiable manifold $O s c^{2} M$ is called integrable of the types given in (4.1), with respect to the nonlinear connection $N$ if the corresponding lifted 2-forms on $T\left(O s c^{2} M\right)$ are integrable.

Remark 4.1 The 2-forms $\omega$ of the types considerated in (4.1) doesn't define a metrical structure on $O s c^{2} M$, because the coefficients $\tilde{a}_{i j}, \tilde{d}_{i j}$ and $\tilde{f}_{i j}$ are alternate and $\tilde{b}_{i j}, \tilde{c}_{i j}, \tilde{e}_{i j}$ cannot be the
coefficients of the metrical structure $g_{i j}\left(x, y^{(1)}, y^{(2)}\right)$, because the corresponding 2-forms $\omega$ are degenerate: for II +V we obtain $\omega=g_{i j} d x^{i} \wedge \delta y^{(1) j}+g_{i j} \delta y^{(1) i} \wedge \delta y^{(2) j}$. It follows that $A=\left(\begin{array}{ccc}0 & g_{i j} & 0 \\ -g_{j i} & 0 & g_{i j} \\ 0 & -g_{j i} & 0\end{array}\right)$ with $\operatorname{det} A=0$, etc.
Therefore it isn't possible to consider the problem of the integrability of a d-metric structure on $O s c^{2} M$.

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# HOLOMORPHICALLY PROJECTIVE MAPPINGS ONTO RIEMANNIAN TANGENT-PRODUCT SPACES 

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#### Abstract

We state the theorem which specifies fundamental equations of holomorphically projective mappings of a space with an affine connection and with an almost-product structure onto a Riemannian tangent-product spaces.


Key words and phrases. holomorphically projective mapping, tangent product space fundamental equations.
Mathematics Subject Classification. Primary 53B30, 53B35.

## 1 Introduction

Several results concerning holomorphically projective mappings and its generalizations were described in [1]-[19]. One of the topics studied here are holomorphically projective mappings of special Riemannian spaces with almost complex and almost product structures.

In the present work we study some properties of holomorphically projective mappings from spaces with affine connection and with almost-tangent structure onto Riemannian almosttangent spaces.

First we give the definitions of a Riemannian almost-tangent structure.

Definition 1.1 An almost-tangent structure on a differentiable manifold $M$ is an affinor $F$ $(\neq \mathrm{Id})$ which satisfies $F^{2}=0$.

Definition 1.2 A Riemannian almost-tangent structure on a manifold $M$ is an almost-tangent structure $F$ on $M$ such that

$$
\begin{equation*}
g(F X, Y)+g(X, F Y)=0, \tag{1}
\end{equation*}
$$

where $g$ is a metric tensor field on $M$ and $X, Y$ are any tangent vector fields of $T M$. A manifold $M$ with a Riemannian almost-tangent structure $F$ is called a Riemannian almost-tangent space and denoted by $V_{n}(g, F)$.

The classification of Riemannian almost-product spaces is described by A.M. Naveira in [13]. This is the analogue of the classification of almost Hermitian spaces by A. Gray and L. Hervella in [4]. The question of tangent and almost tangent structure is devoted by Vishnevsky [18].

Definition 1.3 Let $A_{n}$ be a manifold with an affine connection $\nabla$ and an almost-tangent structure $F$. A diffeomorphism $f$ from $A_{n}$ onto a Riemannian almost-tangent space $\bar{V}_{n}(\bar{g}, \bar{F})$ is called a holomorphically projective mapping if there exist a linear operator $\varphi$ such that the following conditions hold

$$
\begin{gather*}
\bar{\nabla}(X, Y)=\nabla(X, Y)+X \psi(Y)+Y \psi(X)+F X \varphi(Y)+F Y \varphi(X)  \tag{2}\\
\bar{F}(X)=F(X) \tag{3}
\end{gather*}
$$

where $\nabla$ and $\bar{\nabla}$ are affine connections of $A_{n}$ and $\bar{V}_{n}, X, Y$ are any tangent vector fields from $T A_{n}$ and their images in $T \bar{V}_{n}$, and $\psi(X)=\varphi(F X)$.

The condition (3) means that the holomorphically projective mapping preserves an almosttangent structure. Hence, in the following we suppose $\bar{F}=F$.

It is known that holomorphically projective mappings preserve $F$-planar curves. Here a $F$-planar curve is a curve $\gamma(t)$ such that any tangent vector $\frac{d \gamma}{d t}\left(t_{1}\right)$, when subjected to a parallel transport $\tau_{t_{1}, t_{2}}$, remains in the tangent plane spanned by the vectors $\frac{d \gamma}{d t}\left(t_{2}\right)$ and $F\left(\frac{d \gamma}{d t}\left(t_{2}\right)\right)$.

Hence, this type of holomorphically projective mappings could be defined by a more natural way, as mappings which preserve $F$-planar curves and satisfie the condition $\bar{\nabla}_{X} F(Y)=$ $\nabla_{X} F(Y)$, for all $X, Y$.

It follows from the equations (2) that holomorphically projective mappings are analogous to $F_{1}$ - and $F_{2}$-mappings studied in $[7,10,11]$.

## 2 Study of fundamental equations of holomorphically projective mappings of $A_{n}(\nabla, F)$ onto Riemannian almost-tangent spaces

Let $A_{n}$ be a manifold with an affine connection $\nabla$ and almost-tangent structure $F$. We shall study the problem of finding all Riemannian almost-tangent spaces $\bar{V}_{n}(\bar{g}, \bar{F})$ such that there exists a holomorphically projective mapping of $A_{n}$ onto $\bar{V}_{n}$.

Originally, similar problems were solved for geodesic mappings of Riemannian spaces, holomorphically projective mappings of Kählerian spaces, hyperbolic and parabolic Kählerian spaces $[9,10,16]$ and $F$-planar mappings onto Riemannian spaces [7].

In the following we use the local tensor notation which is traditionaly used in this field.
By a direct calculation we check that the equation (2) is equivalent to

$$
\begin{equation*}
\bar{g}_{i j, k}=2 \psi_{k} \bar{g}_{i j}+\psi_{(i} \bar{g}_{j) k}+\varphi_{(i} F_{j)}^{\alpha} \bar{g}_{\alpha k}, \tag{4}
\end{equation*}
$$

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where $\bar{g}_{i j}, F_{i}^{h}, \varphi, \psi_{i}\left(=\varphi_{\alpha} F_{i}^{\alpha}\right)$ are local components of the metric tensor $\bar{g}$, the almost-tangent structure $F$, and the operators $\varphi$ and $\psi$. We shall use the notation "," for the covariant derivative on $A_{n}$, and ( $i j$ ) denotes a symmetrization of indices.

We introduce the following notation:

$$
T_{\ldots \bar{i} \ldots} \equiv T_{\ldots \alpha \ldots} F_{i}^{\alpha}, \quad T^{\cdots \bar{i} \ldots} \equiv T^{\cdots \alpha \cdots} F_{\alpha}^{i}
$$

The solution $\bar{g}_{i j}$ of the equations (4) in the space $A_{n}(\nabla, F)$ which fulfills the conditions

$$
\begin{equation*}
\text { (a) } \quad \bar{g}_{i j}+\bar{g}_{i \bar{j}}=0, \quad \text { (b) } \quad\left|\bar{g}_{i j}\right| \neq 0 \tag{5}
\end{equation*}
$$

determines the metric of a Riemannian almost-tangent space $\bar{V}_{n}(\bar{g}, F)$, and for the inverse matrix $\bar{g}^{i j}\left(\left\|\bar{g}^{i j}\right\|=\left\|\bar{g}_{i j}\right\|^{-1}\right)$ yields following conditions

$$
\begin{equation*}
\text { (a) } \bar{g}^{\bar{i} j}+\bar{g}^{i \bar{j}}=0, \quad \text { (b) }\left|\bar{g}^{i j}\right| \neq 0 \tag{6}
\end{equation*}
$$

The equations (4) are the equations with unknown functions $\bar{g}_{i j}$ and $\psi_{i}$.
Because holomorphically projective mappings are a special case of $F$-planar mappings, the equations (4) can be reduced to a Cauchy system of differential equations [8,11]. In the following main theorem we show that there is a further simplification for some type of almost-tangent structures.

Theorem 2.1 Let $A_{n}$ be a manifold with an affine connection $\nabla$ and an almost-tangent structure $F$ satisfying

$$
\begin{equation*}
F^{2}=0, F \neq \mathrm{Id} \tag{7}
\end{equation*}
$$

and $F_{l}^{\alpha} F_{\alpha, k}^{j}$ cannot be expressed in the form

$$
\begin{equation*}
F_{l}^{\alpha} F_{\alpha, k}^{j}=\mathcal{F}^{j} a_{l k}+\mathcal{F}_{k}^{j} b^{l}+F_{l}^{j} b^{k} \tag{8}
\end{equation*}
$$

for any tensors $\mathcal{F}^{j}, \mathcal{F}_{k}^{j}, a_{l k}$ and $b^{k}$.
Then $A_{n}$ admits a holomorphically projective mapping onto a Riemannian almost-tangent space $\bar{V}_{n}$ if and only if the system of equations (4) of Cauchy type is solvable with respect to the unknown functions $\bar{g}_{i j}$, where

$$
\begin{equation*}
\varphi_{i}=\bar{g}^{\alpha \beta} T_{\alpha \beta}^{\gamma} \bar{g}_{\gamma i} . \tag{9}
\end{equation*}
$$

Here $T_{\alpha \beta}^{\gamma}$ (see (21)) are the components of a tensor $T$ which can be expressed in terms of the affine connection $\nabla$ and the almost-tangent structure $F$.

Proof. We apply covariant differentiation to the formula $\bar{g}_{i \alpha} \bar{g}^{\alpha j}=\delta_{i}^{j}: \bar{g}_{i \alpha, k} \bar{g}^{\alpha j}+\bar{g}_{i \alpha} \bar{g}^{\alpha j}{ }_{, k}=0$. We get $\bar{g}^{i j}{ }_{, k}=-\bar{g}_{\alpha \beta, k} \bar{g}^{\alpha i} \bar{g}^{\beta j}$. Applying (4) we check that

$$
\begin{equation*}
\bar{g}_{, k}^{i j}=-2 \psi_{k} \bar{g}^{i j}-\psi^{(i} \delta_{k}^{j)}-2 \psi_{\bar{k}} \bar{g}^{i \bar{j}}-\psi^{(\bar{i}} \delta_{k}^{\bar{j})}, \tag{10}
\end{equation*}
$$

where $\psi^{i} \equiv \psi_{\alpha} \bar{g}^{\alpha i}$.
The conditions (5a) are equivalent to

$$
\begin{equation*}
\bar{g}^{i \alpha} F_{\alpha}^{j}+\bar{g}^{j \alpha} F_{\alpha}^{i}=0 \tag{11}
\end{equation*}
$$

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By covariant differentiation of (11) in $A_{n}$ and applying (10) we have

$$
\begin{equation*}
\bar{g}^{i \alpha} F_{\alpha, k}^{j}+\bar{g}^{j \alpha} F_{\alpha, k}^{i}=0 \tag{12}
\end{equation*}
$$

By further differentiation and simplification we obtain

$$
\begin{gather*}
\psi^{i} F_{l, k}^{j}+\psi^{j} F_{l, k}^{i}+\delta_{l}^{i} \psi^{\alpha} F_{\alpha, k}^{j}+\delta_{l}^{j} \psi^{\alpha} F_{\alpha, k}^{i} \\
+\varphi^{i} F_{l}^{\alpha} F_{\alpha, k}^{j}+\varphi^{j} F_{l}^{\alpha} F_{\alpha, k}^{i}+F_{l}^{i} \varphi^{\alpha} F_{\alpha, k}^{j}+F_{l}^{j} \varphi^{\alpha} F_{\alpha, k}^{i}  \tag{13}\\
=\bar{g}^{i \alpha} F_{\alpha, k l}^{j}+\bar{g}^{j \alpha} F_{\alpha, k l}^{i} .
\end{gather*}
$$

Contracting equations (13) for the indices $i$ and $l$ we obtain

$$
\begin{equation*}
\psi^{\alpha} F_{\alpha, k}^{j}=\frac{1}{n+2}\left(\bar{g}^{\alpha \beta} F_{\alpha, k \beta}^{j}-\bar{g}^{j \beta} F_{\alpha, k \gamma}^{\gamma}\right) . \tag{14}
\end{equation*}
$$

We used an evident formula

$$
F_{\alpha}^{\alpha}=0, \quad F_{\beta}^{\alpha} F_{\alpha}^{\beta}=0, \quad F_{\beta}^{\alpha} F_{\alpha, k}^{\beta}=0, \quad F_{\bar{i}, k}^{h}=-F_{i, k}^{\bar{h}}
$$

which is true for almost-tangent structures.
By the aid of (14) we get from (13)

$$
\begin{equation*}
\psi^{i} F_{l, k}^{j}+\psi^{j} F_{l, k}^{i}+\varphi^{i} F_{l}^{\alpha} F_{\alpha, k}^{j}+\varphi^{j} F_{l}^{\alpha} F_{\alpha, k}^{i}+F_{l}^{i} \varphi^{\alpha} F_{\alpha, k}^{j}+F_{l}^{j} \varphi^{\alpha} F_{\alpha, k}^{i}=\bar{g}^{\alpha \beta} \stackrel{1}{T}_{\alpha \beta k l}^{i j} \tag{15}
\end{equation*}
$$

where

$$
\stackrel{1}{T}_{\alpha \beta k l}^{i j} \equiv-\delta_{\alpha}^{(i} F_{\beta, k l}^{j)}-\frac{1}{n+2}\left(\delta_{l}^{(i} F_{\alpha, k \beta}^{j)}+\delta_{l}^{(i} \delta_{\alpha}^{j)} F_{\beta, k \gamma}^{\gamma}\right) .
$$

Contracting (15) with $F_{m}^{l}$ we check that

$$
\begin{equation*}
\psi^{i} F_{m}^{l} F_{l, k}^{j}+\psi^{j} F_{m}^{l} F_{l, k}^{i}=\bar{g}^{\alpha \beta} \stackrel{1}{T}_{\alpha \beta k l}^{i j} F_{m}^{l} \tag{16}
\end{equation*}
$$

Under the condition (6) ( $\left.\Rightarrow F_{m}^{l} F_{l, k}^{j} \neq 0\right)$ we check easily, that there exist vectors $\varepsilon^{m}, \eta^{k}$ and $\theta_{j}$ such that $\varepsilon^{m} \eta^{k} \theta_{j} F_{m}^{l} F_{l, k}^{j}=1$. Contracting (16) with $\varepsilon^{m} \eta^{k} \theta_{i} \theta_{j}$ we obtain

$$
\begin{equation*}
\psi^{i} \theta_{i}=\frac{1}{2} \bar{g}^{\alpha \beta} \stackrel{1}{T}_{\alpha \beta k l}^{i j} F_{m}^{l} \varepsilon^{m} \eta^{k} \theta_{i} \theta_{j}, \tag{17}
\end{equation*}
$$

and contracting (16) with $\varepsilon^{m} \eta^{k} \theta_{j}$ we find

$$
\begin{equation*}
\psi^{i}=\bar{g}^{\alpha \beta} \stackrel{2}{T}_{\alpha \beta}^{i} \tag{18}
\end{equation*}
$$

where

$$
\stackrel{2}{T}_{\alpha \beta}^{i}=\bar{g}^{\alpha \beta} \stackrel{1}{T}_{\alpha \beta k l}^{i j} F_{m}^{l} \varepsilon^{m} \eta^{k} \theta_{j}-\frac{1}{2} \bar{g}^{\alpha \beta}\left(\stackrel{1}{T}_{\alpha \beta k l}^{\gamma j} F_{m}^{l} \varepsilon^{m} \eta^{k} \theta_{\gamma} \theta_{j}\right)\left(F_{m}^{l} F_{l, k}^{i} \varepsilon^{m} \eta^{k}\right)
$$

Applying (18) to (13) we have

$$
\begin{equation*}
\varphi^{i} F_{l}^{\alpha} F_{\alpha, k}^{j}+\varphi^{j} F_{l}^{\alpha} F_{\alpha, k}^{i}+F_{l}^{i} \varphi^{\alpha} F_{\alpha, k}^{j}+F_{l}^{j} \varphi^{\alpha} F_{\alpha, k}^{i}=\bar{g}^{\alpha \beta}{\underset{T}{3 \beta k l}}_{T_{\alpha}^{i j}} \tag{19}
\end{equation*}
$$

where

$$
\stackrel{3}{T}_{\alpha \beta k l}^{i j}=\stackrel{1}{T}_{\alpha \beta k l}^{i j}-F_{l, k}^{i} \stackrel{2}{T}_{\alpha \beta k l}^{j}-F_{l, k}^{j} \stackrel{2}{T}_{\alpha \beta k l}^{i}
$$

Because $F \neq 0$ there exist vectors $\xi^{i}$ and $\sigma_{i}$ such that $\xi^{l} \sigma_{i} F_{l}^{i}=1$. Contracting (19) with $\xi^{l} \sigma_{i} \sigma_{j}$ we check that

$$
\sigma_{i} \varphi^{\alpha} F_{\alpha, k}^{i}=\alpha F_{k}+\frac{1}{2} \bar{g}^{\alpha \beta} \stackrel{3}{T}_{\alpha \beta k l}^{i j} \xi^{l} \sigma_{i} \sigma_{j}
$$

where $\alpha=-\varphi^{i} \sigma_{i}$ and $F_{k}=F_{l}^{\alpha} F_{\alpha, k}^{i} \xi^{l} \sigma_{i}$, and after contracting (19) with $\xi^{l} \sigma_{j}$ we find

$$
\begin{equation*}
\varphi^{\alpha} F_{\alpha, k}^{i}=\varphi^{i} F_{k}+\alpha \mathcal{F}_{k}^{i}+\bar{g}^{\alpha \beta} \stackrel{4}{T}_{\alpha \beta k l}^{i} \tag{20}
\end{equation*}
$$

where the tensor $\stackrel{4}{T}_{\alpha \beta k l}^{i}$ has similarly form and $\mathcal{F}_{k}^{i}=\xi^{l}\left(F_{l}^{\alpha} F_{\alpha, k}^{i}-F_{l}^{i} F_{k}\right)$.
Applying (20) to (15) we have

$$
\begin{equation*}
\varphi^{i}\left(F_{l}^{\alpha} F_{\alpha, k}^{j}+F_{l}^{j} F_{k}\right)+\varphi^{j}\left(F_{l}^{\alpha} F_{\alpha, k}^{i}+F_{l}^{i} F_{k}\right)+\alpha\left(F_{l}^{i} \mathcal{F}_{k}^{j}+F_{l}^{j} \mathcal{F}_{k}^{i}\right)=\bar{g}^{\alpha \beta}{ }_{T}^{5_{\alpha \beta k l}^{i j}} \tag{21}
\end{equation*}
$$

The bracket on the left-hand side of (20) must be nonvanishing, otherwise there it is in contradiction with (7). From (21) applying analogically process we have that

$$
\begin{equation*}
\varphi^{i}=\alpha \mathcal{F}^{i}+\bar{g}^{\alpha \beta} \stackrel{6}{T}_{\alpha \beta}^{i} \tag{22}
\end{equation*}
$$

And after substitution (22) to (21) we obtained

$$
\begin{equation*}
\alpha\left(\mathcal{F}^{i}\left(F_{l}^{\alpha} F_{\alpha, k}^{j}+F_{l}^{j} F_{k}\right)+\mathcal{F}^{j}\left(F_{l}^{\alpha} F_{\alpha, k}^{i}+F_{l}^{i} F_{k}\right)+F_{l}^{i} \mathcal{F}_{k}^{j}+F_{l}^{j} \mathcal{F}_{k}^{i}\right)=\bar{g}^{\alpha \beta}{\underset{T}{\alpha \beta k l}}_{i j} \tag{23}
\end{equation*}
$$

If $\mathcal{F}^{i} \neq 0$ then the bracket on the left-hand side of (19) must be nonvanishing, otherwise there would be $F_{l}^{\alpha} F_{\alpha, k}^{j}=\mathcal{F}^{j} a_{l k}+\mathcal{F}_{k}^{j} b^{l}+F_{l}^{j} b^{k}$, which is in contradiction with (8). So there exists a tensor field $Q_{i j}^{k l}$ satisfying

$$
Q_{i j}^{k l}\left(\mathcal{F}^{i}\left(F_{l}^{\alpha} F_{\alpha, k}^{j}+F_{l}^{j} F_{k}\right)+\mathcal{F}^{j}\left(F_{l}^{\alpha} F_{\alpha, k}^{i}+F_{l}^{i} F_{k}\right)+F_{l}^{i} \mathcal{F}_{k}^{j}+F_{l}^{j} \mathcal{F}_{k}^{i}\right)=1
$$

Hence from (23) it follows that $\alpha=\bar{g}^{\alpha \beta} \stackrel{7}{T}_{\alpha \beta k l}^{i j} Q_{i j}^{k l}$ and further the formula (9).
This finishes the proof.

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# THE INVERSE PROBLEM RELATED TO A SECOND ORDER DYNAMICAL FORM <br> SMETANOVÁ Dana, (CZ) 


#### Abstract

The paper is devoted to a geometrical formulation of the inverse problem related to a second order dynamical form in field theory on the fibered manifolds. The geometric condition for a locally variational second order form is found. The Lepage equivalents of the second order Euler-Lagrange form are studied. The results of the paper are applicable to physical theories.


Key words and phrases. Dynamical form, Euler-Lagrange form, inverse problem, Lepage equivalents, locally variational form.
Mathematics Subject Classification. 35R30, 58E30, 70S05.

## 1 Introduction

The purpose of this paper is to announce some recent results in a geometrical formulation of the inverse problem in the calculus of variations. We study the case of a second order dynamical form in field theory on the fibered manifold.

In the second half of the 19th century the question when a system of ordinary or partial differential equations of order $r(r \geq 1)$ identifies with Euler-Lagrange equations (i.e., equations for extremals of a variational functional) was intensively studied. As the first in 1887 Helmholtz solved this problem for a system of second order ordinary differential equations. He found necessary conditions for variationality, now called Helmholtz conditions [2]. Later the Helmholtz conditions was generalized to the higher order equations and to the partial differential equations. Around 1980 the inverse variational problem began to study by methods of differential geometry and global analysis. The inverse problem was extended to study problems of a global Lagrangian and relations between properties of differential equations and differential forms. Nowadays relations between variationality and geometry of differential forms are intensively studied and explored (see e.g. [1, 6]). For ordinary variational equations (in physical

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terminology "higher-order mechanics"), the theory of Lepage $(n+1)$-forms is well-established (see book [5]). In the calculus of variations, Lepage ( $n+1$ )-forms are closed differential forms, representing Euler-Lagrange equations. They are fundamental for investigation of variational equations by means of exterior differential systems methods, with important applications in Hamilton and Hamilton-Jacobi theory and theory of integration of variational equations. For partial variational equations (in physical terminology "field theory"), the theory of Lepage $(n+1)$-forms is studied.

In [7], Lepage equivalents of a second-order Euler-Lagrange quasi-linear PDE's are characterised explicitly. A closed $(n+1)$-form uniquely determined by the Euler-Lagrange form is constructed and used to find a geometric solution of the inverse problem of the calculus of variations. In this paper we generalize part of the paper [7]. We study case of an arbitrary second-order dynamical form. A principal part of a closed $(n+1)$-form uniquely determined by the dynamical form is constructed, and a geometric formulation (as a property of the principal part of a closed $(n+1)$-form) of the inverse problem of the calculus of variations is found.

Every closed $(n+1)$-form such that the 1-contact part is a second order dynamical form admits a noninvariant decomposition $\alpha=\alpha_{E}+\phi$, where form $\alpha_{E}$ depends on the coefficients of a dynamical form and $\phi$ does not depend on the coefficients of a dynamical form. The principal part $\hat{\alpha}_{E}$ (at most 2-contact part) of such $(n+1)$-form is described. An intrinsic expression of the variationality conditions is the global defined property $p_{2} d \alpha_{E}=p_{2} \mathrm{~d} \hat{\alpha}_{E}=0$. Finally, the meaning of the Lepage equivalents $\alpha$ and $\alpha_{E}$ of a locally variational form $E$ is demonstrated.

Throughout the paper all manifolds and mappings are smooth and summation convention is used. We consider a fibered manifold (i.e., surjective submersion) $\pi: Y \rightarrow X, \operatorname{dim} X=n$, $\operatorname{dim} Y=n+m$, its $r$-jet prolongation $\pi_{r}: J^{r} Y \rightarrow X, r \geq 1$ and canonical jet projections $\pi_{r, k}: J^{r} Y \rightarrow J^{k} Y, 0 \leq k \leq r$ (with an obvious notations $J^{0} Y=Y$ ). A fibered chart on $Y$ (resp. associated fibered chart on $J^{r} Y$ ) is denoted by $(V, \psi), \psi=\left(x^{i}, y^{\sigma}\right)$, where $1 \leq i \leq n$, $1 \leq \sigma \leq m\left(\operatorname{resp} .\left(V_{r}, \psi_{r}\right), \psi_{r}=\left(x^{i}, y^{\sigma}, y_{i}^{\sigma}, \ldots, y_{i_{1} \ldots i_{r}}^{\sigma}\right)\right.$ ).

A vector field $\xi$ on $J^{r} Y$ is called $\pi_{r}$-vertical if it projects onto the zero vector field on $X$. A $q$-form $\eta$ on $J^{r} Y$ is called $\pi_{r}$-horizontal if $i_{\xi} \eta=0$ for every $\pi_{r}$-vertical vector field $\xi$ on $J^{r} Y$.

The fibered structure of $Y$ induces a morphism $h$, of exterior algebras, defined by the condition $J^{r} \gamma^{*} \eta=J^{r+1} \gamma^{*} h \eta$ for every section $\gamma$ of $\pi$, and called horizontalization. Apparently, horizontalization is an $\mathbb{R}$-linear wedge product preserving mapping such that applied to a function $f$ and to the elements of the canonical basis of 1-forms $\left(d x^{i}, d y^{\sigma}, d y_{i}^{\sigma}, \ldots, d y_{i_{1} \ldots i_{r}}^{\sigma}\right)$ on $J^{r} Y$ gives

$$
h f=f \circ \pi_{r+1, r}, h d x^{i}=d x^{i}, h d y^{\sigma}=y_{l}^{\sigma} d x^{l}, \ldots, h d y_{i_{1} \ldots i_{r}}^{\sigma}=y_{i_{1} \ldots i_{r} l}^{\sigma} d x^{l} .
$$

A $q$-form $\eta$ on $J^{r} Y$ is called contact if $h \eta=0$. A contact $q$-form $\eta$ on $J^{r} Y$ is called 1-contact if for every $\pi_{r}$-vertical vector field $\xi$ on $J^{r} Y$ the $(q-1)$-form $i_{\xi} \eta$ is horizontal. A contact $q$-form $\eta$ on $J^{r} Y$ is called $i$-contact if for every $\pi_{r}$-vertical vector field $\xi$ on $J^{r} Y$ the $(q-1)$-form $i_{\xi} \eta$ is ( $i-1$ )-contact.

Recall that every $q$-form $\eta$ on $J^{r} Y$ admits a unique (canonical) decomposition into a sum of $q$-forms on $J^{r+1} Y$ as follows:

$$
\begin{equation*}
\pi_{r+1, r}^{*} \eta=h \eta+\sum_{k=1}^{q} p_{k} \eta, \tag{1}
\end{equation*}
$$

where $h \eta$ is a horizontal form, called the horizontal part of $\eta$, and $p_{k} \eta, 1 \leq k \leq q$, is a $k$-contact part of $\eta$ (see [3]).

We use the following notations:

$$
\begin{aligned}
& \omega_{0}=d x^{1} \wedge d x^{2} \wedge \cdots \wedge d x^{n}, \omega_{i}=i_{\partial / \partial x^{i} \omega_{0}}, \omega_{i j}=i_{\partial / \partial x^{j}} \omega_{i}, \\
& \omega^{\sigma}=d y^{\sigma}-y_{j}^{\sigma} d x^{j}, \ldots, \omega_{i_{1} i_{2} \ldots i_{k}}^{\sigma}=d y_{i_{1} i_{2} \ldots i_{k}}^{\sigma}-y_{i_{1} i_{2} \ldots i_{k j}}^{\sigma} d x^{j},
\end{aligned}
$$

and

$$
d_{i}=\frac{\partial}{\partial x^{i}}+y_{i}^{\sigma} \frac{\partial}{\partial y^{\sigma}}+y_{j_{1}}^{\sigma} \frac{\partial}{\partial y_{j_{1}}^{\sigma}}+\cdots+y_{j_{1} \ldots j_{r} i}^{\sigma} \frac{\partial}{\partial y_{j_{1} \ldots j_{r}}^{\sigma}}
$$

For more details on fibered manifolds and the corresponding geometric structures we refer e.g. to [8].

A Lagrangian $\lambda$ of order $r$ is a horizontal $n$-form on $J^{r} Y$. Its expression in fibered chart is the following $\lambda=L \omega_{0}$, where $L=L\left(x^{i}, y^{\sigma}, y_{i}^{\sigma}, \ldots, y_{i_{1} \ldots i_{r}}^{\sigma}\right)$ is a Lagrange function.

A dynamical form $E$ of order $r$ is a 1-contact $(n+1)$-form on $J^{r} Y$, horizontal with respect to the projection onto $Y$. In fibered coordinates,

$$
E=E_{\sigma} \omega^{\sigma} \wedge \omega_{0}
$$

where $E_{\sigma}$ are local functions on $J^{r} Y$.
Let $\psi$ be a $(n+1)$-form on $J^{r} Y .(n+1)$-form $\hat{\psi}=p_{1} \psi+p_{2} \psi$ on $J^{r+1} Y$ is then called the principal part of the form $\psi$.

A differential $n$-form $\eta$ is called Lepage equivalent of the Lagrangian $\lambda$ (see [3]) if in the decomposition (1), $h \eta=\lambda$, and $p_{1} d \eta$ is a dynamical form; the $(n+1)$-form $E_{\lambda}=p_{1} d \eta$ is then called the Euler-Lagrange form of $\lambda$.

Consider a dynamical form $E$ on $J^{2} Y . E$ is said to be locally variational if to every point in $J^{2} Y$ exists a neighbourhood $U$, and a Lagrangian $\lambda$ on $U$, such that $\left.E\right|_{U}=E_{\lambda}$. It is known that $E$ is locally variational if and only if the components of $E$ satisfy the following identities:

$$
\begin{align*}
& \frac{\partial E_{\sigma}}{\partial y_{j k}^{\nu}}-\frac{\partial E_{\nu}}{\partial y_{j k}^{\sigma}}=0 \\
& \frac{\partial E_{\sigma}}{\partial y_{j}^{\nu}}+\frac{\partial E_{\nu}}{\partial y_{j}^{\sigma}}-2 d_{k} \frac{\partial E_{\nu}}{\partial y_{j k}^{\sigma}}=0  \tag{2}\\
& \frac{\partial E_{\sigma}}{\partial y^{\nu}}-\frac{\partial E_{\nu}}{\partial y^{\sigma}}+d_{j} \frac{\partial E_{\nu}}{\partial y_{j}^{\sigma}}-d_{j} d_{k} \frac{\partial E_{\nu}}{\partial y_{j k}^{\sigma}}=0 .
\end{align*}
$$

We recall a fundamental theorem, Krupka [4], relating locally variational forms with closed forms:

Theorem 1.1 A dynamical form $E$ is locally variational if and only if to every point in the domain of $E$ there exists a neighbourhood $W$ and an at least 2-contact form $F_{W}$ on $W$ such that the form $\alpha_{W}=E+F_{W}$ is closed.

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A $(n+1)$-form $\alpha$ is called a Lepage equivalent of $E$ if $p_{1} \alpha=E$ and $\mathrm{d} \alpha=0$. One can see immediately that if $\alpha$ is a Lepage equivalent of $E$ then, around every point, $\alpha=\mathrm{d} \eta$ where $\eta$ is a Lepage equivalent of a local Lagrangian for $E$.

The above theorem guarantees a local existence of Lepage equivalents; it does not provide us with explicit formulas for $\alpha_{W}$ by means of the components of $E$.

In the next chapter the explicit formula for the principal part of the form $\alpha_{W}$ is found for a second order dynamical form.

## 2 The inverse problem

We shall consider a second order dynamical form $E=E_{\nu} \omega^{\nu} \wedge \omega_{0}$ which coefficients $E_{\nu}$ can be non-affine in the second derivatives, i.e., its satisfy for some $\nu, \sigma, \kappa, k, l, p, q$

$$
\frac{\partial^{2} E_{\nu}}{\partial y_{k l}^{\sigma} \partial y_{p q}^{\kappa}} \neq 0,
$$

$1 \leq k, l, p, q \leq n, 1 \leq \sigma, \nu, \kappa \leq m$.
The problem is to find the geometrical property of a closed $(n+1)$-forms $\alpha$. This means that a dynamical form (1-contact part of $\alpha$ ) is locally variational, i.e. the the conditions (2) are satisfied.

We study a second order closed $(n+1)$-form $\alpha$ such that $p_{1} \alpha=E$ which is arbitrary second order dynamical form. The closedness condition on $\alpha$ means that at least some of the components of the higher-degree contact parts of $\alpha$ depend upon the coefficient of dynamical form $E_{\sigma}, 1 \leq \sigma \leq m$. This means that $\alpha$ splits into a (not necessarily invariant) sum (different from sum (1))

$$
\begin{equation*}
\alpha=\alpha_{E}+\phi, \tag{3}
\end{equation*}
$$

where $\alpha_{E}$ is determined only by the coefficients of the dynamical form, while $\phi$ does not depend upon $E$. We can easily that the principal part $\hat{\alpha}$ of $\alpha$ admits decompositions $\hat{\alpha}=p_{1} \alpha+p_{2} \alpha=$ $\hat{\alpha}_{E}+\hat{\phi}$, where $\hat{\alpha}_{E}=p_{1} \alpha_{E}+p_{2} \alpha_{E}$ is the principal part of the form $\alpha_{E}$ and $\hat{\phi}=p_{1} \phi+p_{2} \phi$ is the principal part of the form $\phi$. In the proof of the following Theorem 2.1 the explicit formulas for the principal parts $\hat{\alpha}_{E}$ and $\hat{\phi}$ are found. The form $\hat{\alpha}_{E}$ (resp. $\alpha_{E}$ ) has unique expression which depends on the coefficient of the dynamical form. The form $\hat{\phi}$ (resp. $\phi$ ) is non-unique (c.f. proof of the following theorem). We can find different forms $\hat{\alpha}$ (resp. $\alpha$ ) whose depend on choice of the form $\hat{\phi}$ (resp. $\phi$ ).

Hence, the first step to solve the inverse problem is to find the principal part $\hat{\alpha}_{E}=p_{1} \alpha_{E}+$ $p_{2} \alpha_{E}$ of the form $\alpha_{E}$.

Theorem 2.1 Let $\alpha$ be a closed $(n+1)$-form on $J^{2} Y$ such that $p_{1} \alpha=E$ where $E$ is a second order dynamical form. Let $\alpha_{E}$ be the form in the sum (3). Then the principal part $\hat{\alpha}_{E}=$ $p_{1} \alpha_{E}+p_{2} \alpha_{E}$ takes the form

$$
\begin{equation*}
\hat{\alpha}_{E}=E_{\sigma} \omega^{\sigma} \wedge \omega_{0}+\frac{1}{2} \frac{\partial E_{\sigma}}{\partial y_{j}^{\nu}} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{j}+\frac{\partial E_{\sigma}}{\partial y_{j p}^{\nu}} \omega^{\sigma} \wedge \omega_{p}^{\nu} \wedge \omega_{j} . \tag{4}
\end{equation*}
$$

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Proof. If $\alpha$ is a second order $(n+1)$-form such that $p_{1} \alpha=E$ then in fibered coordinates the principal part $\hat{\alpha}$ takes the form

$$
\begin{align*}
\hat{\alpha} & =E_{\sigma} \omega^{\sigma} \wedge \omega_{0}+A_{\sigma \nu}^{j} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{j}+B_{\sigma \nu}^{p j} \omega^{\sigma} \wedge \omega_{p}^{\nu} \wedge \omega_{j}+C_{\sigma \nu}^{k l j} \omega^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{j} \\
& +D_{\sigma \nu}^{k l j} \omega_{k}^{\sigma} \wedge \omega_{l}^{\nu} \wedge \omega_{j}+F_{\sigma \nu}^{p k l j} \omega_{p}^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{j}+G_{\sigma \nu}^{k l p q j} \omega_{k l}^{\sigma} \wedge \omega_{p q}^{\nu} \wedge \omega_{j}, \tag{5}
\end{align*}
$$

where the functions $A_{\sigma \nu}^{j}, C_{\sigma \nu}^{k l j}, D_{\sigma \nu}^{k l j}, F_{\sigma \nu}^{p k l j}, G_{\sigma \nu}^{k l p q j}$ satisfy the identities $\left(A_{\sigma \nu}^{j}\right)_{\operatorname{Sym}(\sigma \nu)}=0,\left(C_{\sigma \nu}^{k l j}\right)_{\operatorname{Alt}(k l)}=0,\left(D_{\sigma \nu}^{k l j}\right)_{\operatorname{Sym}([\sigma k][\nu]])}=0,\left(F_{\sigma \nu}^{p k l j}\right)_{\operatorname{Alt}(k l)}=0$ and $G_{\sigma \nu}^{k l p q j}=G_{\sigma \nu}^{l k p q j}=$ $G_{\sigma \nu}^{k l q p j}=-G_{\nu \sigma}^{p q k j}$.

Above and in what follows, $\operatorname{Sym}()$ and $\operatorname{Alt}()$ means complete symmetrisation and skewsymmetrisation in the indicated indices [pairs of indices], respectively.

Using the facts $\mathrm{d} \pi_{3,2}^{*} \alpha=\pi_{3,2}^{*} \mathrm{~d} \alpha$, the decomposition (1) and $\mathrm{d} \alpha=0$ we can easily see that $p_{2} \mathrm{~d} \alpha=p_{2} \mathrm{~d} \hat{\alpha}=0$. The explicit computation of $p_{2} \mathrm{~d} \hat{\alpha}$ reads

$$
\begin{align*}
p_{2} \mathrm{~d} \hat{\alpha} & =\left(\frac{\partial E_{\nu}}{\partial y^{\sigma}}+d_{i} A_{\sigma \nu}^{j}\right)_{\operatorname{Alt}(\sigma \nu)} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{0} \\
& +\left(-\frac{\partial E_{\sigma}}{\partial y_{k}^{\nu}}+2 A_{\sigma \nu}^{k}+d_{i} B_{\sigma \nu}^{k i}\right) \omega^{\sigma} \wedge \omega_{k}^{\nu} \wedge \omega_{0} \\
& +\left(-\frac{\partial E_{\sigma}}{\partial y_{k l}^{\nu}}+B_{\sigma \nu}^{k l}+d_{i} C_{\sigma \nu}^{k l i}\right)_{\operatorname{Sym}(k l)} \omega^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{0} \\
& +\left(C_{\sigma \nu}^{j k l}\right)_{\operatorname{Sym}(j k l)} \omega^{\sigma} \wedge \omega_{j k l}^{\nu} \wedge \omega_{0}+\left(F_{\sigma \nu}^{i j k l}\right)_{\operatorname{Sym}(j k l)} \omega_{i}^{\sigma} \wedge \omega_{j k l}^{\nu} \wedge \omega_{0}  \tag{6}\\
& +\left(B_{\sigma \nu}^{l k}+d_{i} D_{\sigma \nu}^{k l i}\right)_{\operatorname{Alt}([[\sigma i][\nu k])} \omega_{k}^{\sigma} \wedge \omega_{l}^{\nu} \wedge \omega_{0} \\
& +\left(C_{\sigma \nu}^{j k l}+2 D_{\sigma \nu}^{j k l}+d_{i} F_{\sigma \nu}^{j k l i}\right)_{\operatorname{Sym}(k l)} \omega_{j}^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{0} \\
& +\left(F_{\sigma \nu}^{i j k l}+d_{p} G_{\sigma \nu}^{i l j k p}\right)_{\operatorname{Sym}(i l), \operatorname{Sym}(j k), \operatorname{Alt}([\sigma i l][\nu j k])} \omega_{i l}^{\sigma} \wedge \omega_{j k}^{\nu} \wedge \omega_{0} \\
& +2\left(G_{\sigma \nu}^{i j k l p}\right)_{\operatorname{Sym}(k l p)} \omega_{i j}^{\sigma} \wedge \omega_{k l p}^{\nu l p} \wedge \omega_{0}=0 .
\end{align*}
$$

Hence, the functions $C_{\sigma \nu}^{j k l}, D_{\sigma \nu}^{j k l}, F_{\sigma \nu}^{i j k l}, G_{\sigma \nu}^{i j k l p}$ do not depend on coefficients of the dynamical form and

$$
\begin{equation*}
A_{\sigma \nu}^{j}=\frac{1}{2} \frac{\partial E_{\sigma}}{\partial y_{j}^{\nu}}+a_{\sigma \nu}^{j}, \quad B_{\sigma \nu}^{p j}=\frac{\partial E_{\sigma}}{\partial y_{j p}^{\nu}}+b_{\sigma \nu}^{p j} \tag{7}
\end{equation*}
$$

where the functions $a_{\sigma \nu}^{j}, b_{\sigma \nu}^{p j}$ do not depend on the coefficients of the dynamical form and $\left(b_{\sigma \nu}^{p j}\right)_{\operatorname{Sym}(p j)}=0$.

This means that principal part $\hat{\alpha}$ of $\alpha$ splits into a sum

$$
\begin{equation*}
\hat{\alpha}=\hat{\alpha}_{E}+\hat{\phi}, \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\alpha}_{E}=E_{\sigma} \omega^{\sigma} \wedge \omega_{0}+\frac{1}{2} \frac{\partial E_{\sigma}}{\partial y_{j}^{\nu}} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{j}+\frac{\partial E_{\sigma}}{\partial y_{j p}^{\nu}} \omega^{\sigma} \wedge \omega_{p}^{\nu} \wedge \omega_{j} . \tag{9}
\end{equation*}
$$

is the principal part of $\alpha_{E}$ in (3) and it is completely determined by the coefficients of the dynamical form and

$$
\begin{align*}
\hat{\phi} & =a_{\sigma \nu}^{j} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{j}+b_{\sigma \nu}^{p j} \omega^{\sigma} \wedge \omega_{p}^{\nu} \wedge \omega_{j}+C_{\sigma \nu}^{k l j} \omega^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{j} \\
& +D_{\sigma \nu}^{k l j} \omega_{k}^{\sigma} \wedge \omega_{l}^{\nu} \wedge \omega_{j}+F_{\sigma \nu}^{p k l j} \omega_{p}^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{j}+G_{\sigma \nu}^{k l p q j} \omega_{k l}^{\sigma} \wedge \omega_{p q}^{\nu} \wedge \omega_{j}, \tag{10}
\end{align*}
$$

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is the principal part of $\phi$ in (3) and it does not depend on $E$.
By a similar way as in the proof of the Theorem 2.1 we can find higher order contact parts of form $\alpha_{E}$ but explicit calculation is more complicated. We note that $(p+1)$-contact part of $\alpha_{E}$ depends on $p$ th partial derivatives of coefficient of the dynamical form $E_{\sigma}$.

The full structure of the form $\alpha_{E}$ for second order dynamical form which is affine (quasilinear) in second derivatives one can find in [7] (see the following remark).

Remark 2.2 Transformation properties.
Let $(V, \psi), \psi=\left(x^{i}, y^{\sigma}\right)$, and $(\bar{V}, \bar{\psi}), \bar{\psi}=\left(\bar{x}^{i}, \bar{y}^{\sigma}\right)$ be two overlapping fibered charts on $Y$. Using in the expression for $\alpha_{E}$ and $\hat{\alpha}_{E}$ transformation formulas

$$
\begin{aligned}
\bar{\omega}_{j_{i} \ldots j_{k}} & =\operatorname{det}\left(\frac{\partial \bar{x}}{\partial x}\right) \frac{\partial x^{p_{1}}}{\partial \bar{x}^{j_{1}}} \cdots \frac{\partial x^{p_{k}}}{\partial \bar{x}_{k} j_{k}} \omega_{p_{1} \ldots p_{k}}, \quad \bar{\omega}^{\sigma}=\frac{\partial y^{\sigma}}{\partial y^{\nu}} \omega^{\nu}, \quad \bar{\omega}_{j}^{\sigma}=\frac{\partial y_{j}^{\sigma}}{\partial y^{\nu}} \omega^{\nu}+\frac{\partial y_{j}^{\sigma}}{\partial y_{p}^{\nu}} \omega_{p}^{\nu}, \\
\bar{y}_{j}^{\sigma} & =\frac{\partial x^{k}}{\partial \bar{x}^{j}}\left(\frac{\partial \bar{y}^{\sigma}}{\partial x^{k}}+\frac{\partial \bar{y}^{\sigma}}{\partial y^{\rho}} y_{k}^{\rho}\right), \quad \bar{y}_{j i}^{\sigma}=\frac{\partial x^{k}}{\partial \bar{x}^{i}}\left(\frac{\partial \bar{y}_{j}^{\sigma}}{\partial x^{k}}+\frac{\partial \bar{y}_{j}^{\sigma}}{\partial y^{\rho}} y_{k}^{\rho}+\frac{\partial \bar{y}_{j}^{\sigma}}{\partial y_{p}^{\rho}} y_{p k}^{\rho}\right), \\
\bar{E}_{\sigma} & =\operatorname{det}\left(\frac{\partial x}{\partial \bar{x}}\right) \frac{\partial y^{\nu}}{\partial \bar{y}^{\sigma}} E_{\nu},
\end{aligned}
$$

and the relation

$$
\frac{\partial y_{j}^{\sigma}}{\partial \bar{y}_{k}^{\nu}} \frac{\partial \bar{y}_{k}^{\nu}}{\partial y^{\rho}}=-\frac{\partial y_{j}^{\sigma}}{\partial \bar{y}^{\nu}} \frac{\partial \bar{y}^{\nu}}{\partial y^{\rho}},
$$

we obtain the following result:
The principal part of $\alpha_{E}$ (4)

$$
\hat{\alpha}_{E}=E_{\sigma} \omega^{\sigma} \wedge \omega_{0}+\frac{1}{2} \frac{\partial E_{\sigma}}{\partial y_{j}^{\nu}} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{j}+\frac{\partial E_{\sigma}}{\partial y_{j p}^{\nu}} \omega^{\sigma} \wedge \omega_{p}^{\nu} \wedge \omega_{j} .
$$

is invariant with respect to fibered coordinate transformations. This means that the above formula defines a global differential form. However, the form $\hat{\alpha}_{E}$ is in general not closed.

Lemma 2.3 Let $E$ be a second order dynamical form. Let $\hat{\alpha}_{E}$ be the form (4). Then $\hat{\alpha}_{E}$ is invariant with respect to fibered coordinate transformations (i.e. the formula (4) defines a global differential form).

Proof. Proof of the Lemma 2.3 follows from the above remark, from transformation properties of the form $\hat{\alpha}_{E}$.

The following theorem describes the local variationality of the second order dynamical form as a property of the form $\alpha_{E}$.

Theorem 2.4 Let $E$ be a dynamical form on $J^{2} Y$. The following conditions are equivalent:
(1) $E$ is locally variational.
(2) $p_{2} \mathrm{~d} \alpha_{\mathrm{E}}=0$.

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(3) Components $E_{\sigma}$ of $E$ satisfy conditions (2).

Proof. (2) $\Rightarrow$ (3) The condition $p_{2} \mathrm{~d} \alpha_{E}=0$ reads

$$
\begin{align*}
& \frac{\partial E_{\sigma}}{\partial y^{\nu}}-\frac{\partial E_{\nu}}{\partial y^{\sigma}}-\frac{1}{2} d_{l}\left(\frac{\partial E_{\sigma}}{\partial y_{l}^{\nu}}-\frac{\partial E_{\nu}}{\partial y_{l}^{\sigma}}\right)=0 \\
& \frac{\partial E_{\sigma}}{\partial y_{j}^{\nu}}+\frac{\partial E_{\nu}}{\partial y_{j}^{\sigma}}-2 d_{l} \frac{\partial E_{\sigma}}{\partial y_{j l}^{\nu}}=0, \quad \frac{\partial E_{\sigma}}{\partial y_{j p}^{\nu}}-\frac{\partial E_{\nu}}{\partial y_{j p}^{\sigma}}=0, \tag{11}
\end{align*}
$$

The above conditions are equivalent with the conditions (2).
$(3) \Rightarrow(1)$ was proved in [4]: one has to show that if $E_{\sigma}$ satisfy (2) then $\lambda=L \omega_{0}$, where

$$
L=y^{\sigma} \int_{0}^{1} E_{\sigma}\left(x^{i}, u y^{\nu}, u y_{k}^{\nu}, u y_{k l}^{\nu}\right) d u
$$

is a local Lagrangian for $E$. This is done by a direct computation showing that the EulerLagrange expressions of $L$ are equal to the given functions $E_{\sigma}$.

Finally, $(1) \Rightarrow(2)$. If $E$ is locally variational then $E$ is the Euler-Lagrange form with coefficients satisfying (2). The conditions (2) are equivalent with (11). Hence, $p_{2} \mathrm{~d} \alpha_{E}=0$.

The above theorem provides us with a geometric meaning of the variationality conditions, as conditions, under which the $(n+1)$-form $\alpha_{E}$ is closed. Hence,

$$
p_{2} \mathrm{~d} \alpha_{E}=0
$$

is an intrinsic expression of the variationality conditions (2).
The next corollary clarify the meaning of the Lepage equivalents $\alpha$ and $\alpha_{E}$ of a locally variational form $E$. The form $\alpha$ represents class of the Lepage equivalents. The class has representatives with different properties (e.g. global differential and local differential forms). One of them is $\alpha_{E}$ which represents local Lepage equivalent which is determined only on the coefficients of the variational dynamical form.

Corollary 2.5 Let $\alpha$ be a closed $(n+1)$-form on $J^{2} Y$ such that $p_{1} \alpha=E$ where $E$ is a second order dynamical form. Let $\alpha_{E}$ be a form in the sum (3) such that $p_{2} \mathrm{~d} \alpha_{E}=0$. Then the form $\alpha$ represents the class of the Lepage equivalents of the Euler-Lagrange form $E$. Let $(V, \psi)$ be a fibered chart on $Y$ with coordinates $\left(x^{i}, y^{\sigma}\right)$. If moreover $\mathrm{d} \alpha_{E}=0$ then the $(n+1)$-form $\alpha_{E}$ determined by the Euler-Lagrange expressions of $E$ and defined on $\pi_{2,0}^{-1}(V) \subset J^{2} Y$ is a Lepage equivalent of $E$.

Proof. The proof of the assertion comes from the Theorem 2.4 and from the definition of the Lepage equivalent of the Euler-Lagrange form.

Example 2.6 On fibered manifold $\mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$, where $n \geq 4$, $m \geq 1$, we consider $a$ second order dynamical form $E$ as follows $E=\left(P_{\sigma}+Q_{\sigma \nu}^{k l} y_{k l}^{\nu}+R_{\sigma \nu \kappa}^{k l p q} y_{k l}^{\nu} y_{p q}^{\kappa}\right) \omega^{\sigma} \wedge \omega_{0}$, where the functions $P_{\sigma}, Q_{\sigma \nu}^{k l}, R_{\sigma \nu \kappa}^{k l p q}$ are constant functions and some of functions " $R$ " are non-zero. The above dynamical form represents non-affine case.

The corresponding principal part $\hat{\alpha}_{E}$ (4) is obviously

$$
\hat{\alpha}_{E}=\left(P_{\sigma}+Q_{\sigma \nu}^{k l} y_{k l}^{\nu}+R_{\sigma \nu \kappa}^{k l p q} y_{k l}^{\nu} y_{p q}^{\kappa}\right) \omega^{\sigma} \wedge \omega_{0}+\left(Q_{\sigma \nu}^{k l}+\left(R_{\sigma \nu \kappa}^{k l p q}+R_{\sigma \kappa \nu}^{p q k l}\right) y_{p q}^{\kappa}\right) \omega^{\sigma} \wedge \omega_{k l}^{\nu} \wedge \omega_{0}
$$

The variationality conditions (2) read the following coefficients conditions

$$
\begin{align*}
\left(Q_{\sigma \nu}^{k l}\right)_{\operatorname{Alt}(\sigma \nu)} & =0,\left(R_{\sigma \nu \kappa}^{k l p q}+R_{\sigma \kappa \nu}^{p q k l}\right)_{\operatorname{Sym}(k p q)}=0,  \tag{12}\\
\left(R_{\sigma \nu \kappa}^{k l p q}+R_{\sigma \kappa \nu}^{p q k l}\right)_{\operatorname{Sym}(k l p q)} & =0,\left(R_{\sigma \nu \kappa}^{k l p q}\right)_{\mathrm{Alt}(\sigma \nu)}+\left(R_{\sigma \kappa \nu}^{p q k}\right)_{\mathrm{Alt}(\sigma \kappa)}=0 .
\end{align*}
$$

The above dynamical form is variational if and only if the conditions (12) are fulfiled.
The Lagrangian $\lambda=L \omega_{0}$ corresponding to the above variational dynamical form takes the expression of the Lagrange function

$$
L=P_{\sigma} y^{\sigma}+\frac{1}{2} Q_{\sigma \nu}^{k l} y^{\sigma} y_{k l}^{\nu}+\frac{1}{3} R_{\sigma \nu \kappa}^{k l p q} y^{\sigma} y_{k l}^{\nu} y_{p q}^{\kappa}
$$

We note that the above Lagrangian is not only local but it is the global one.

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# THE REGULARIZATION PROCEDURE <br> AND THE LEGENDRE TRANSFORMATION IN EXAMPLES: DIRAC FIELD AND ELECTROMAGNETIC FIELD LAGRANGIANS. 

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#### Abstract

The aim of the paper is apply the regulatization method to the cases of concrete Lagrangians - Dirac field Lagrangian and the electromagnetic field Lagrangian. The generalized Legendre transformation is studied. Key words and phrases. Lagrangian, Lepagean form, Hamilton equations, regularity, regularizable Lagrangian, Legendre transformation, Dirac field Lagrangian, electromagnetic field Lagrangian. Mathematics Subject Classification. 70G50, 58Z05.


## 1 Introduction

In the paper we apply the regularization method (proposed by Krupková and Smetanová [3, 4]) to the cases of concrete Lagrangians. Two methods of the regularizaton are presented by examples. The Dirac field Lagrangian as Lagrangian afinne in "first derivatives" is regularizable. We use for this Lagrangian regularization method by constants. In the second case we propose the regularization by Lagrange function of the electromagnetic Lagrangian. The explicit formulas for generalized momenta and Hamiltonian are found. The equivalent Lagrangian dedonderization of the Lagrangian - (regular in the "standard" sense) is presented.

Throughout the paper all manifolds and mappins are smooth and summation convention is used. We consider a fibered manifold (i.e., surjective submersion) $\pi: Y \rightarrow X, \operatorname{dim} X=n$, $\operatorname{dim} Y=n+m$, its $r$-jet prolongation $\pi_{r}: J^{r} Y \rightarrow X, r \geq 1$ and canonical jet projections
$\pi_{r, k}: J^{r} Y \rightarrow J^{k} Y, 0 \leq k \leq r$ (with an obvious notations $J^{0} Y=Y$ ). A fibered char on $Y$ (resp. associated fibered chart on $J^{r} Y$ ) is denoted by $(V, \psi), \psi=\left(x^{i}, y^{\sigma}\right)$ (resp. $\left(V_{r}, \psi_{r}\right)$, $\left.\psi_{r}=\left(x^{i}, y^{\sigma}, y_{i}^{\sigma}, \ldots, y_{i_{1} \ldots i_{r}}^{\sigma}\right)\right)$.

A vector field $\xi$ on $J^{r} Y$ is called $\pi_{r}$-vertical (resp. $\pi_{r, k}$-vertical) if it projects onto the zero vector field on $X$ (resp. on $J^{k} Y$ ).

Recall that every $q$-form $\eta$ on $J^{r} Y$ admits a unique (canonical) decomposition into a sum of $q$-forms on $J^{r+1} Y$ as follows [1]: $\pi_{r+1, r}^{*} \eta=h \eta+\sum_{k=1}^{q} p_{k} \eta$, where $h \eta$ is a horizontal form, called the horizontal part of $\eta$, and $p_{k} \eta, 1 \leq k \leq q$, is a $k$-contact part of $\eta$.

We use the following notations: $\omega_{0}=d x^{1} \wedge d x^{2} \wedge \cdots \wedge d x^{n}, \omega_{i}=i_{\partial / \partial x^{i}} \omega_{0}, \omega_{i j}=i_{\partial / \partial x^{j}} \omega_{i}$, and $\omega^{\sigma}=d y^{\sigma}-y_{j}^{\sigma} d x^{j}, \ldots, \omega_{i_{1} i_{2} \ldots i_{k}}^{\sigma}=d y_{i_{1} i_{2} \ldots i_{k}}^{\sigma}-y_{i_{1} i_{2} \ldots i_{k} j}^{\sigma} d x^{j}$. For more details on fibered manifolds and the corresponding geometric stuctures we refer e.g. to [5].

## 2 The regularization procedure and the Legendre transformation

Definition (Krupka [1]) An n-form $\rho$ on $J^{r} Y$ is called a Lepagean equivalent of a Lagrangian $\lambda$ (resp. Lepagean $n$-form) if

1) $h \rho=\lambda, \lambda=L \omega_{0}$,
2) $(n+1)$-form $p_{1}(d \rho)$ is a $\pi_{r+1,0}$-horizontal form.

For an $r$-th order Lagrangian we have all its Lepagean equivalents of order $(2 r-1)$ characterized by the following formula

$$
\begin{equation*}
\pi_{r+1, r^{*}} \rho=\Theta+\mu, \tag{1}
\end{equation*}
$$

where $\Theta$ is a global Poincaré Cartan form associated to the Lagrangian, and $\mu$ is an arbitrary $n$-form of order of contactness $\geq 2$, i.e., such that $h(\mu)=p_{1}(\mu)=0$.

Recall that for a first order Lagrangian, $\Theta=\theta_{\lambda}$ where $\theta_{\lambda}$ is the Poincaré-Cartan form of $\lambda$, $\theta_{\lambda}=L \omega_{0}+\frac{\partial L}{\partial y_{j}^{\sigma}} \omega^{\sigma} \wedge \omega_{j}$. If $r \geq 2, \Theta$ is no more unique however, there is an invariant decomposition $\Theta=\theta_{\lambda}+d \nu$, where $\theta_{\lambda}$ is Poincaré Cartan equivalent for higher order Lagrangian and $\nu$ does not depend upon $\lambda$.

Theorem ([1]) [Let $\lambda$ be a Lagrangian on $J^{r} Y, \rho$ its Lepagean equivalent. A section $\gamma$ of $\pi$ is an extremal of $\lambda$ if and only if

$$
\begin{equation*}
J^{2 r-1} \gamma^{*} i_{J^{2 r-1} \xi} d \rho=0 \tag{2}
\end{equation*}
$$

for every $\pi$-vertical vector field $\xi$ on $Y$.]
A section $\delta$ of the fibered manifold $\pi_{2 r-1}$ is called a Hamilton extremal of $\rho$ if

$$
\begin{equation*}
\delta^{*} i_{\xi} d \rho=0 \tag{3}
\end{equation*}
$$

for every $\pi_{2 r-1}$-vertical vector field $\xi$ on $J^{2 r-1} Y$.
The equations (2) are called the Euler-Lagrange equations and (3) the Hamilton equations of $\rho$, respectively. Notice that while the Euler-Lagrange equations are uniquely determined
by $\lambda$, Hamilton equations depend upon a choice of $\mu$. Consequetly, one gets many different Hamilton theories associated to a given variational problem

Definition [2] A Lepagean form $\rho$ on $J^{1} Y$ is called regular if every its Hamilton extremal is holonomic.

### 2.1 Regularization procedure (Krupková and Smetanová [3, 4]).

In the sequel we shall consider Lepagean forms (1) where $\mu$ is 2 -contact, and $\mu=p_{2}(\beta)$, where $\beta$ is defined on $Y$ and such that $p_{i}(\beta)=0$ for all $i \geq 3$. In a fiber chart, where the Lagrangian $\lambda$ is expressed by $\lambda=L \omega_{0}$, we can write

$$
\begin{equation*}
\rho=L \omega_{0}+\frac{\partial L}{\partial y_{j}^{\sigma}} \omega^{\sigma} \wedge \omega_{j}+g_{\sigma \nu}^{i j} \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{i j} \tag{4}
\end{equation*}
$$

(summation over all sequences of indices) where the functions $g_{\sigma \nu}^{i j}$ do not depend on the $y_{l}^{\kappa}$ 's and satisfy the conditions

$$
\begin{equation*}
g_{\sigma \nu}^{i j}=-g_{\nu \sigma}^{i j}, \quad g_{\sigma \nu}^{i j}=-g_{\sigma \nu}^{j i}, \quad g_{\sigma \nu}^{i j}=g_{\nu \sigma}^{j i} . \tag{5}
\end{equation*}
$$

Note that equations (5) mean that only

$$
\binom{m}{2} \cdot\binom{n}{2}=\frac{1}{4} m n(m-1)(n-1)
$$

of the $m^{2} \cdot n^{2}$ functions $g_{\sigma \nu}^{i j}$ are independent.
Theorem Let $\lambda$ be a first-order Lagrangian, let $\lambda=L \omega_{0}$ be its expression in a fiber chart $(V, \psi), \psi=\left(x^{i}, y^{\sigma}\right)$ on $Y$. Let $\rho$ be a Lepagean equivalent of $\lambda$ of the form (4), (5). Assume that the matrix

$$
\begin{equation*}
A_{\sigma \nu}^{i j}=\left(\frac{\partial^{2} L}{\partial y_{i}^{\sigma} \partial y_{j}^{\nu}}-4 g_{\sigma \nu}^{i j}\right) \tag{6}
\end{equation*}
$$

with rows (resp. columns) labelled by the pair $(\sigma, i)$ (resp. $(\nu, j)$ ), is regular. Then $\rho$ is regular, i.e., every Hamilton extremal $\delta$ of $\rho$ is of the form $\delta=J^{1} \gamma$, where $\gamma$ is an extremal of $\lambda$.

Note that the above regularity condition (6) means that Hamilton equations and the EulerLagrange equations are equivalent.

Definition Let $W \subset Y$ be open, $W \subset V$, where $(V, \psi)$ is a fiber chart on $Y$, and let $\lambda=L \omega_{0}$ be a Lagrangian on $\pi_{1,0}^{-1}(V)$. We say that $L$ is regular over $W$ if there exist functions $g_{\sigma \nu}^{i j}$ on $W$ satisfying (5) and the condition

$$
\begin{equation*}
\operatorname{det}\left(\frac{\partial^{2} L}{\partial y_{i}^{\sigma} \partial y_{j}^{\nu}}-4 g_{\sigma \nu}^{i j}\right) \neq 0 \tag{7}
\end{equation*}
$$

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If $\lambda$ does not satisfy the standard regularity condition but is regular in the sense of the above definition, we also say that $\lambda$ is regularizable over $W$ (or, locally regularizable).

The noninvariant decomposition of Lepagean form (4), (5) reads

$$
\rho=-H \omega_{0}+p_{\sigma}^{i} d y^{\sigma} \wedge \omega_{j}+g_{\sigma \nu}^{i j} d y^{\sigma} \wedge d y^{\nu} \wedge \omega_{i j},
$$

where

$$
\begin{equation*}
p_{\sigma}^{i}=\frac{\partial L}{\partial y_{i}^{\sigma}}-4 g_{\sigma \nu}^{i j} y_{j}^{\nu}, \quad H=-L+p_{\sigma}^{i} y_{i}^{\sigma}+2 g_{\sigma \nu}^{i j} y_{i}^{\sigma} y_{j}^{\nu} \tag{8}
\end{equation*}
$$

In analogy to the standard terminology, we shall call $H$ the Hamiltonian and $p_{\sigma}^{i}$ momenta. The regularity condition (7) admits the coordinate transformation $\left(x^{i}, y^{\sigma}, y_{j}^{\sigma}\right) \rightarrow\left(x^{k}, y^{\nu}, p_{\nu}^{k}\right)$, so called Legendre transformation.

Expressing Hamilton equations in Legendre coordinates we get

$$
\begin{align*}
\frac{\partial\left(p_{\sigma}^{i} \circ \delta\right)}{\partial x^{i}} & =-\frac{\partial H}{\partial y^{\sigma}} \circ \delta+4 \frac{\partial g_{\sigma \nu}^{i j}}{\partial x^{j}} \frac{\partial\left(y^{\nu} \circ \delta\right)}{\partial x^{i}}  \tag{9}\\
& +2\left(\frac{\partial g_{\kappa \nu}^{i j}}{\partial y^{\sigma}}+\frac{\partial g_{\sigma \kappa}^{i j}}{\partial y^{\nu}}+\frac{\partial g_{\sigma \sigma}^{i j}}{\partial y^{\kappa}}\right)\left(\frac{\partial H}{\partial p_{\kappa}^{i}} \circ \delta\right)\left(\frac{\partial H}{\partial p_{\nu}^{j}} \circ \delta\right) \\
\frac{\partial\left(y^{\sigma} \circ \delta\right)}{\partial x^{i}} & =\frac{\partial H}{\partial p_{\sigma}^{i}} \circ \delta .
\end{align*}
$$

Corollary If the n -form

$$
\begin{equation*}
\eta=g_{\sigma \nu}^{i j} d y^{\sigma} \wedge d y^{\nu} \wedge \omega_{i j} \tag{10}
\end{equation*}
$$

is closed, then the above Hamilton $p_{2}$-equations (9) take the form

$$
\begin{equation*}
\frac{\partial\left(p_{\sigma}^{i} \circ \delta\right)}{\partial x^{i}}=-\frac{\partial H}{\partial y^{\sigma}} \circ \delta, \quad \frac{\partial\left(y^{\sigma} \circ \delta\right)}{\partial x^{i}}=\frac{\partial H}{\partial p_{\sigma}^{i}} \circ \delta . \tag{11}
\end{equation*}
$$

We introduced regularization as a procedure to find for a Lagrangian appropriate Hamilton equations which are equivalent with the Euler-Lagrange equations, hence represent a suitable alternative for solving the extremal problem. We can see this problem from the different point of view. For given Hamilton equations (11) we find a new Lagrangian which is regular in the Hamilton-De Donder theory and equivalent to the original one. Now the Hamilton equations are identified with Hamilton-De Donder equations for the new Lagrangian.

Proposition If the form $\eta(10)$, (5) is closed then the Lagrangian $\bar{\lambda}=\left(L-2 g_{\sigma \nu}^{i j} y_{i}^{\sigma} y_{j}^{\nu}\right) \omega_{0}$ is equivalent with Lagrangian $\lambda=L \omega_{0}$, and $d \rho=d \theta_{\bar{\lambda}}$.

Above Lagrangian $\bar{\lambda}$ is called a dedonderization of the Lagrangian $\lambda$.
Theorem Let $\lambda$ be a regularizable Lagrangian. Then for all its regularizations $\rho$ (4), (5) such that $d \eta=0(10)$, the Lagrangian $\bar{\lambda}=\lambda-h(\eta)=\left(L-2 g_{\sigma \nu}^{i j} y_{i}^{\sigma} y_{j}^{\nu}\right) \omega_{0}$ is equivalent to $\lambda$, satisfies the "standard" regularity condition $\operatorname{det}\left(\frac{\partial^{2} L}{\partial y_{i}^{\sigma} \partial y_{j}^{\nu}}\right) \neq 0$. and the Hamilton equations of $\lambda$ based upon $\rho$ coincide with Hamilton-De Donder equations based on the Poincare-Cartan form of the Lagrangian $\bar{\lambda}$.

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## 3 The regularization by constant - Dirac field Lagrangian

Dirac field Lagrangian is linear in the variables $y_{i}^{\nu}$, hence special case of affine Lagrangian. In this case we have $X=R^{4}, Y=R^{4} \times R^{2}$, i.e. $J^{1} Y=R^{4} \times R^{2} \times R^{4}$, with the global coordinates denoted by $\left(x^{\mu}, \psi, \bar{\psi}, \partial_{\mu} \psi, \partial_{\mu} \bar{\psi}\right), \mu=1,2,3,4$, Lagrangian takes the form $\lambda=L \omega_{0}$, where

$$
L=\frac{i}{2}\left(\bar{\psi} \gamma^{\mu} \partial_{\mu} \psi+\partial_{\mu} \bar{\psi} \gamma^{\mu} \psi\right)-\bar{\psi} m \psi
$$

and it is apparently degenerate in the Hamilton-De Donder sense.
The regularization procedure means that we find some Lepagean equivalent satisfying the condition (6). Note that we have only 6 independent functions exist. We can choose where the function $4 g_{12}^{12}=1,4 g_{12}^{13}=2,4 g_{12}^{14}=3,4 g_{12}^{23}=4,4 g_{12}^{24}=5,4 g_{12}^{34}=6$ as constant function.

Then the Lepagean eqiuvalent have the form

$$
\begin{aligned}
\rho & =\left[\frac { i } { 2 } \left(\bar{\psi} \gamma^{1} \partial_{1} \psi+\bar{\psi} \gamma^{2} \partial_{2} \psi+\bar{\psi} \gamma^{3} \partial_{3} \psi+\bar{\psi} \gamma^{4} \partial_{4} \psi+\partial_{1} \bar{\psi} \gamma^{1} \psi+\partial_{2} \bar{\psi} \gamma^{2} \psi+\partial_{3} \bar{\psi} \gamma^{3} \psi+\right.\right. \\
& \left.\left.+\partial_{4} \bar{\psi} \gamma^{4} \psi\right)-\bar{\psi} m \psi\right] \omega_{0}+ \\
& +\frac{i}{2} \gamma^{1} \psi \omega^{1} \wedge \omega_{1}+\frac{i}{2} \gamma^{2} \psi \omega^{1} \wedge \omega_{2}+\frac{i}{2} \gamma^{3} \psi \omega^{1} \wedge \omega_{3}+\frac{i}{2} \gamma^{4} \psi \omega^{1} \wedge \omega_{4}+ \\
& +\frac{i}{2} \gamma^{1} \bar{\psi} \omega^{2} \wedge \omega_{1}+\frac{i}{2} \gamma^{2} \bar{\psi} \omega^{2} \wedge \omega_{2}+\frac{i}{2} \gamma^{3} \bar{\psi} \omega^{2} \wedge \omega_{3}+\frac{i}{2} \gamma^{4} \bar{\psi} \omega^{2} \wedge \omega_{4}+ \\
& +\omega^{1} \wedge \omega^{2} \wedge \omega_{12}+2 \omega^{1} \wedge \omega^{2} \wedge \omega_{13}+3 \omega^{1} \wedge \omega^{2} \wedge \omega_{14}+ \\
& +4 \omega^{1} \wedge \omega^{2} \wedge \omega_{23}+5 \omega^{1} \wedge \omega^{2} \wedge \omega_{24}+6 \omega^{1} \wedge \omega^{2} \wedge \omega_{34} .
\end{aligned}
$$

The regularity condition (6) reads

$$
\operatorname{det}\left(\begin{array}{llll}
0 & 1 & 2 & 3 \\
-1 & 0 & 4 & 5 \\
-2 & -4 & 0 & 6 \\
-3 & -5 & -6 & 0
\end{array}\right) \neq 0
$$

The Hamiltonian $H=-L+p_{\sigma}^{i} y_{i}^{\sigma}+2 g_{\sigma \nu}^{i j} y_{i}^{\sigma} y_{j}^{\nu}$, where $4 g_{12}^{12}=1,4 g_{12}^{13}=2,4 g_{12}^{14}=3,4 g_{12}^{23}=$ $4,4 g_{12}^{24}=5,4 g_{12}^{34}=6$. and the momenta take the formulas

$$
\begin{array}{ll}
p_{1}^{1}=\frac{i}{2} \bar{\psi} \gamma^{1}-\partial_{2} \bar{\psi}-2 \partial_{3} \bar{\psi}-3 \partial_{4} \bar{\psi}, & p_{1}^{2}=\frac{i}{2} \bar{\psi} \gamma^{2}+\partial_{1} \bar{\psi}-4 \partial_{3} \bar{\psi}-5 \partial_{4} \bar{\psi}, \\
p_{1}^{3}=\frac{i}{2} \bar{\psi} \gamma^{3}+2 \partial_{1} \bar{\psi}+4 \partial_{2} \bar{\psi}-6 \partial_{4} \bar{\psi}, & p_{1}^{4}=\frac{i}{2} \bar{\psi} \gamma^{4}+3 \partial_{1} \bar{\psi}+5 \partial_{2} \bar{\psi}+6 \partial_{3} \bar{\psi}, \\
p_{2}^{1}=\frac{i}{2} \gamma^{1} \psi+\partial_{2} \psi+2 \partial_{3} \psi+3 \partial_{4} \psi, & p_{2}^{2}=\frac{i}{2} \gamma^{2} \psi-\partial_{1} \psi+4 \partial_{3} \psi+5 \partial_{4} \psi, \\
p_{2}^{3}=\frac{i}{2} \gamma^{3} \psi-2 \partial_{1} \psi-4 \partial_{2} \psi+6 \partial_{4} \psi, & p_{2}^{4}=\frac{i}{2} \gamma^{4} \psi-3 \partial_{1} \psi-5 \partial_{2} \psi-6 \partial_{3} \psi .
\end{array}
$$

The equivalent Lagrangian $\bar{\lambda}=\bar{L} \omega_{0}$, where

$$
\bar{L}=\frac{i}{2}\left(\bar{\psi} \gamma^{\mu} \partial_{\mu} \psi+\partial_{m} u \bar{\psi} \gamma^{\mu} \psi\right)-\bar{\psi} m \psi+\sum_{\mu \nu} u_{(\mu, \nu)} \epsilon^{\mu \nu} \partial_{\mu} \bar{\psi} \partial_{\nu} \psi
$$

where $u_{(\mu, \nu)}=u_{(\nu, \mu)}, u_{(1,2)}=1, \quad u_{(1,3)}=2, \quad u_{(1,4)}=3, \quad u_{(2,3)}=4, \quad u_{(2,4)}=5, \quad u_{(3,4)}=6$, $\epsilon^{\mu \nu}$ is the Levi-Civita symbol; is a dedonderization of the Dirac field Lagrangian, which is regular in the "standard" sense.

## 4 The regularization by the Lagrange function - electromagnetic field Lagrangian

If in particular, $n=4$, we have $m=4, X=R^{4}, Y=R^{4} \times R^{4}$. For the electromagnetic field Lagrangian

$$
\begin{equation*}
L=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \tag{12}
\end{equation*}
$$

where $F_{\mu \nu}=A_{\mu, \nu}-A_{\nu, \mu},\left(g_{\sigma \nu}\right)$ denotes the Lorentz metric $g_{\sigma \nu}=0$ for $\sigma \neq \nu,-g_{11}=g_{22}=$ $g_{33}=g_{44}=1$, and $\left.y^{\sigma}=g^{\sigma \nu} A_{\nu}=A^{\sigma}, y_{\nu}^{\sigma}=\partial A^{\sigma} / \partial x^{\nu}\right)$, the standard regularity condition gives

$$
\operatorname{det}\left(\frac{\partial^{2} L}{\partial y_{i}^{\sigma} \partial y_{j}^{\nu}}\right)=0
$$

However, this Lagrangian is regularizable, and admits many regularizations. We will study case of the Lagrangian where the Lepagean equivalent depends only on the Lagrange function, i.e. we use regularization by the Lagrange function.

Now we choose one of them the Lepagean equivalent now reads,

$$
\rho=L \omega_{0}+\frac{\partial L}{\partial y_{\alpha}^{\sigma}} \omega^{\sigma} \wedge \omega_{\alpha}+\left(\frac{\partial^{2} L}{\partial y_{\alpha}^{\sigma} \partial y_{\beta}^{\nu}}-\frac{\partial^{2} L}{\partial y_{\beta}^{\sigma} \partial y_{\alpha}^{\nu}}\right) \omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_{\alpha \beta} .
$$

and the regularity condition (7) takes the form

$$
\operatorname{det}\left(4 \frac{\partial^{2} L}{\partial y_{\beta}^{\prime} \partial y_{\alpha}^{\prime}}-3 \frac{\partial^{2} L}{\partial y_{\alpha}^{\sigma} \partial y_{\beta}^{\prime}}\right) \neq 0 .
$$

The explicit computation of the above condition leads
$\operatorname{det}\left(\begin{array}{cccccccccccccccc}0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0\end{array}\right) \neq 0$.
We can present the formulas for the momenta

$$
\begin{array}{llll}
p_{1}^{1}=4 A_{, 2}^{2}+4 A_{, 3}^{3}+4 A_{, 4}^{4}, & p_{1}^{2}=A_{, 2}^{1}-3 A_{, 1}^{2}, & p_{1}^{3}=A_{, 3}^{1}-3 A_{,,}^{3}, & p_{1}^{4}=A_{, 4}^{1}-3 A_{, 1}^{4}, \\
p_{2}^{2}=4 A_{, 1}^{1}+4 A_{, 3}^{3}+4 A_{, 4}^{4}, & p_{2}^{1}=A_{, 1}^{2}-3 A_{, 2}^{1}, & p_{2}^{3}=-A_{, 3}^{2}-3 A_{, 2}^{3}, & p_{2}^{4}=-A_{, 4}^{2}-3 A_{, 2}^{4}, \\
p_{3}^{3}=4 A_{, 1}^{1}+4 A_{, 2}^{2}+4 A_{, 4}^{4}, & p_{3}^{1}=A_{, 1}^{3}-3 A_{, 3}^{1}, & p_{3}^{2}=-A_{, 2}^{3}-3 A_{, 3}^{2}, & p_{3}^{4}=-A_{, 4}^{3}-3 A_{3,3}^{4}, \\
p_{4}^{4}=4 A_{, 1}^{1}+4 A_{, 2}^{2}+4 A_{, 3}^{3}, & p_{4}^{1}=A_{, 1}^{4}-3 A_{, 4}^{1}, & p_{4}^{2}=-A_{, 2}^{4}-3 A_{, 4}^{2}, & p_{4}^{3}=-A_{, 3}^{4}-3 A_{, 4}^{3},
\end{array}
$$

and Hamiltonian

$$
\begin{aligned}
H & =\frac{1}{2}\left(A_{, 2}^{1}\right)^{2}+\frac{1}{2}\left(A_{, 1}^{2}\right)^{2}+\frac{1}{2}\left(A_{, 3}^{1}\right)^{2}+\frac{1}{2}\left(A_{, 1}^{3}\right)^{2}+\frac{1}{2}\left(A_{, 4}^{1}\right)^{2}+\frac{1}{2}\left(A_{, 1}^{4}\right)^{2}- \\
& -\frac{1}{2}\left(A_{, 3}^{2}\right)^{2}-\frac{1}{2}\left(A_{, 2}^{3}\right)^{2}-\frac{1}{2}\left(A_{, 4}^{2}\right)^{2}-\frac{1}{2}\left(A_{, 2}^{4}\right)^{2}-\frac{1}{2}\left(A_{, 4}^{3}\right)^{2}-\frac{1}{2}\left(A_{, 3}^{4}\right)^{2}+ \\
& +4 A_{, 1}^{1} A_{, 2}^{2}+4 A_{, 1}^{1} A_{, 3}^{3}+4 A_{, 1}^{1} A_{, 4}^{4}+4 A_{, 2}^{2} A_{, 3}^{3}+4 A_{, 2}^{2} A_{, 4}^{4}+4 A_{, 3}^{3} A_{, 4}^{4}- \\
& -3 A_{, 2}^{1} A_{, 1}^{2}-3 A_{, 3}^{1} A_{, 1}^{3}-3 A_{, 4}^{1} A_{, 1}^{4}-3 A_{, 3}^{2} A_{, 2}^{3}-3 A_{, 4}^{2} A_{, 2}^{4}-3 A_{, 4}^{3} A_{, 3}^{4}
\end{aligned}
$$

We can write dedonderization of the electromagnetic field Lagrangian (12) which takes the form $\bar{\lambda}=\bar{L} \omega_{0}$, where

$$
\bar{L}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+2\left(A_{\mu}^{\mu} A_{\nu}^{\nu}-A_{\nu}^{\mu} A_{\mu}^{\nu}\right)
$$

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## GEOMETRY IN VEDIC ALTARS

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#### Abstract

Sacrificial rites were an important part of the religion in India in the Vedic era (about $1500-500 \mathrm{BC}$ ). There existed several kinds of rituals and each of them needed an altar of special shape and size. Very precise measurements were necessary so that the ritual was successful. In this period the collections of sacred texts known as Vedas were created. The Śulbasūtras are appendices to the Vedas which give rules for geometric constructions of sacrificial altars. The aim of this paper is to present some of these old rules.


Key words and phrases. Ancient Indian geometry, Sulbasutra, the Pythagoras' theorem, Pythagorean triples, geometrical construction, squaring a circle, circling a square.
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## 1 Vedic sacrifices

The Vedas are the oldest Indian sacred texts composed in the Vedic period (about 1500 - 500 $\mathrm{BC})$. In this period of time the sacrificial rituals and ceremonies had an important role in life. The ceremonies were performed on the top of the altars either in sacrificer's house or on a nearby plot of ground. There were two basic classes of sacrificial rituals - Nitya (perpetual, daily) and Kāmya (optional), see [1], [3]. The sacrifice of the first class was obligatory and it was supposed to bring happiness for the family. The sacrifice of the second class was seasonal and was performed at special times - at every new moon or full moon, at the time of the winter and summer solstices and so on. The purpose of these rituals was to give material progress, to achieve a special object. Each sacrifice had to be made on an altar of prescribed shape, size and orientation. It was stated that even a small irregularity and variation in the form and size of the altar could annul the whole ritual. So it was necessary to respect exact rules for construction of each altar.

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Later several appendices to the Vedas were created, from which the texts Kalpasūtras deal with rules for vedic sacrificial rituals. Particularly the parts named Śulbasūtras describe the rules for geometric constructions of sacrificial altars.

Thanks to this works we gain the first literary evidence of the oldest Indian mathematics.

### 1.1 Sulbasutras

Śulbasūtras are the oldest geometrical texts which represent traditional Indian mathematics developed for constructions of sacrificial altars. The word śulba means a rope or a cord which were used during a construction, the word sūtra means a thread and refers to a rule or a collection of formulas in the form of a manual.

The most important Śulbasūtras were written by Baudhāyana (about 800 BC), Āpastamba (about 600 BC ), and Kātyāyana (about 200 BC ). The Baudhāyana Śulbasūtra is the biggest and the oldest one, it is divided into three chapters and it contains 525 sutras. The $\bar{A} p a s t a m b a$ Śulbasūtra is divided into six sections, this work contains 223 sutras in twenty one chapters. The Kātyāyana Śulbasūtra is divided into two parts, the first of them is composed of 90 sutras while the second part is written in about 40 verses. Compared with the works of Baudhāyana and Āpastamba the Kātyāyana Śulbasūtra presents some interesting geometrical knowledge in a more systematic form. There exist more Śulbasūtras written by Manava, Maitrayana, Varaha and Vadhula. Several later commentaries on the Śulbasūtras are available now.

As the Śulbasūtras deal with the science of geometry and its applications, the earliest Indian name for geometry was Śulba.

### 1.2 Sacrificial altars

It has already been mentioned that there were two classes of sacrificial rituals. The perpetual fires were constructed on a smaller area of one square $v y \bar{a} m a,{ }^{1}$ the optional fires were constructed on a bigger area of $7 \frac{1}{2}$ square purusas. ${ }^{2}$ Each altar was constructed with five layers of bricks which together came up to the height of a knee and every layer contained a definite number of bricks of specified shapes. The smaller altars had one layer constructed with 21 bricks, each layer of the greater altar consisted of about 200 bricks.
 offer sacrifices in them daily. The altar of the Āhavan̄̄̄a was always square and that of the Dakșina semi-circular. The altar of the Gärhapatya had to be of the form of a square according to one school and a circle according to a different school. The area of each had to be the same and equal to one square vyāma.

One of the most elaborated altars, the Kāmya Agni had the form of a falcon (Śyena) and its area was $7 \frac{1}{2}$ square purusas. It was believed that offering a sacrifice on such an altar enable

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the soul of the supplicant to be conveyed by a falcon to heaven. The first layer of that altar consisted of 60 brick of type $a$ in each wing and the body was made from 46 bricks of type $b$, 6 of type $c$ and 24 of type $d$, see Figure 1, (borrowed from [4]).

Fire altars for other optional sacrifices had different shapes - triangle, usually an isosceles triangle (altars such as Praūga), rhombus (Ubhayatah praūga), circle (Rathacakra), isosceles trapezium (Mahāvedi, Sautramani or Śmaśāna), tortoise (Kūrma) and so on, see [1], [3]. Each of them had the same area as that of a falcon ( $7 \frac{1}{2}$ square purusas).


Figure 1: The first layer of a sacrificial altar in the shape of a falcon.

These constructions led to interesting mathematical problems like constructions of many geometrical figures and their transformations of one figure to another, the Pythagoras' theorem, discovery of Pythagorean triples.

Thus the Śulbasūtras contain:
a) construction of a line perpendicular to a given line,
b) constructions of geometrical figures - triangles, squares, rectangles, isosceles trapeziums, circles,
c) the early form of the Pythagoras' theorem,
d) constructions of the same figures with doubled, tripled or multipled areas,
e) constructions of a square equal to the sum or the difference of two unequal squares,
f) the solution of the problem of equivalence of area - squaring a circle and viceversa, transformation of a rectangle into a square and vice-versa.

Śulbasūtras do not contain any proofs of the rules which they describe. Some of the rules, such as the method of constructing a square equal in area to a given rectangle, are exact. Others, such as constructing a square equal in area to a given circle, are only approximations.

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## 2 Constructions

There are described several methods of constructing basic geometrical shapes in Śulbasūtras. There are five methods of constructing a square, some of them used a rope or a cord, others used a bamboo rod.

The following method was suggested by Āpastamba for the construction of the Mahāvedi (the great altar). The shape of the Mahāvedi is prescibed by tradition to be an isosceles trapezium which altitude is 36 padas, ${ }^{3}$ face 24 padas and base 30 padas. Āpastamba gave four methods, but all of them are in principle the same. We present only one of them, see [2].

The diagonal of a rectangle whose sides are 3 and 4 [pada] is 5 . With these increased by three times themselves [are determined] the two eastern corners of the vedi. With them increased by four times themselves [are fixed] the two western corners.


Figure 2: The construction of an isosceles trapezium.
This method (as the others) used the Pythagoras' theorem:

$$
\begin{array}{rlrl}
3^{2}+4^{2} & =5^{2} \\
(3+3 \cdot 3)^{2}+(4+3 \cdot 4)^{2} & =(5+3 \cdot 5)^{2} & & 12^{2}+16^{2}=20^{2} \\
(3+4 \cdot 3)^{2}+(4+4 \cdot 4)^{2} & =(5+4 \cdot 5)^{2} & & 15^{2}+20^{2}=25^{2}
\end{array}
$$

It is clear that the authors of the Sulbasūtras knew the Pythagoras' theorem. The Pythagoras' theorem is used frequently and there are also many examples of Pythagorean triples in the Śulbasūtras - for example $(5,12,13),(12,16,20),(8,15,17),(15,20,25),(12,35,37)$, (15, 36, 39). Besides these which are integers, we can find also a few rational triples - for example $\left(2 \frac{1}{4}, 3,3 \frac{1}{4}\right),\left(7 \frac{1}{4}, 10,12 \frac{1}{2}\right),\left(2 \frac{1}{12}, 5,5 \frac{5}{12}\right),\left(78 \frac{1}{3}, 188,203 \frac{2}{3}\right)$.

The Baudhāyana Śulbasūtra gives a special case of the Pythagoras' theorem, see [5]:
The rope which is stretched across the diagonal of a square produces an area double the size of the original square.

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This is the method of making a square with double area.
The Apastamba Ślbasūtra gives a more general version of the Pythagoras' theorem, see [2]:
The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together.

## 3 Combination of areas

Kātyāyana described a very simple method of finding a square equal to the sum of $n$ other squares of the same size, see [3]:

As many squares [of equal size] as you wish to combine into one, the transverse line be [equal to] one less than that; twice a side will be [equal to] one more than that; [thus] form a triangle. Its arrow [i.e. altitude] will do that.

That is, if $n$ is the number of equal squares to be combined together into one, form the triangle $A B C$ whose base $A B$ is of the length $(n-1)$ times the side $a$ of a given square. The sides $A C$ and $B C$ are together equal to $(n+1)$ times $a$. The constuction of a triangle according to rules in Sulbasūtras is: Draw the line $A B$ of the length $(n-1) a$. Fix two poles at $A$ and $B$. Take a cord of length $(n+1) a$. Fasten its two ends at the two poles and stretch the cord sidewise having taken it by the middle point. Let $C$ be the point reached. Bisect $A B$ at $D$ and join $C D$. Than the square on $C D$ will be equivalent to the sum of $n$ given squares.

The Pythagoras' theorem was also used. It holds for the triangle $B C D$ :

$$
\begin{aligned}
(C D)^{2} & =(B C)^{2}-(B D)^{2}=\left(\frac{n+1}{2} a\right)^{2}-\left(\frac{n-1}{2} a\right)^{2}= \\
& =\frac{n^{2}+2 n+1}{4} a^{2}-\frac{n^{2}-2 n+1}{4} a^{2}=n a^{2} .
\end{aligned}
$$



Figure 3: The square equal in area to $n$ given squares.
The construction of a square equal in area to two given unequal squares was given in most of Śulbasūtras. The following description is from Āpastamba, see [2]:

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Figure 4: The square equal in area to the sum of two squares.

Separate a parallel band of the width of the smaller square from the bigger square. The rope which is stretched diagonally across the band unites both [the squares].

Using of the Pythagoras' theorem $a^{2}+b^{2}=c^{2}$ is evident.
The problem of making a square whose area is equal to a difference of two given squares is solved similarly. The Pythagoras' theorem $a^{2}-b^{2}=c^{2}$ is used. Āpastamba wrote, see [2]:

To deduct a square from a square, cut off from the larger a [rectangular] segment with a side of the square which is to be deducted. Then draw a longer side of this segment diagonally across to the other longer side and where it falls [on the other side] cut off that portion. By this cut-off portion the deduction is finished.


Figure 5: The square equal in area to the difference of two squares.

## 4 Transformations

Āpastamba, Baudhāyana and Kātyāyana also described the method of finding a square equal in area to a given rectangle. The rule was given by Baudhāyana, see [3]:

If you wish to transform a rectangle into a square, make its breadth as the side of a square; divide the remainder into two parts and changing the place [of the farther one of them] and inverting, add it on the order side of the square. Then adding a [square] portion, fill up that [the empty space in the corner]. It has been taugth [before] how to deduct it [the added square from the full square thus formed].
Following figures demostrate this construction. The rectangle $A B C D$ is given. Let $E$ be marked on $A D$ so that $|A E|=|A B|$. Then complete the square $A B F E$. Now bisect $E D$ at $H$ and divide the rectangle $E F C D$ into two equal rectangles with the line $H G$ (Figure 6a). Now move the rectangle $H G C D$ to the position $F B I K$ and complete the square $A I L H$ (Figure 6 b ). The required square is equal in area to the difference of the squares $A I L H$ and $F K L G$. Now rotate $I L$ about $I$ so that it touches the side $B G$ at $R$, thus $|I L|=|I R|$. Now draw $R P$ parallel to $G L$ such that $P$ is on $I L$. Then $I P$ is the side of the required square equal in area to the given rectangle $A B C D$ (Figure 6c).


Figure 6: The construction of a square equal in area to a given rectangle.
If we denote $|A B|=a$ and $|B C|=b$, then the side of the small square $F K L G$ is $\frac{b-a}{2}$, the side of the large square $A I L H$ is $a+\frac{b-a}{2}=\frac{b+a}{2}$. We can see that the following identity is used $\left(\frac{b+a}{2}\right)^{2}-\left(\frac{b-a}{2}\right)^{2}=a b$.

Baudhāyana gave the following rule for transforming a square into a rectangle, see [3]:
If you wish to transform a square into a rectangle, divide it by the diagonal. Divide again one part into two, and add them suitably so as to fit the two sides [of the other half].

The square $A B C D$ is given. Divide it by the diagonals and $S$ is the point of intersection (Figure 7a). Then rotate the triangle $A B S$ into position $A D E$ and similarly the triangle $C S B$ into position $C F D$. Thus the rectangle $E A C F$ equal in area to the square $A B C D$ is formed (Figure 7b).

To transform a square into a rectangle whose side is given, Āpastamba explained, see [2]:
If you wish to transform a square into a rectangle [cut off from it a rectangular segment] by making a side as long as you wish [a side of the transformed rectangle to be] What remains in excess, should be added [to the former] as suitably as to fit.

a)

b)

Figure 7: The construction of a rectangle equal in area to a given square.

Let $A B C D$ be a given square. If the given side $b$ of the rectangle is smaller than the side $a$ of the square $A B C D$, then cut off length $b$ from the sides $A B$ and $C D$ so the rectangle $E B C F$ is gained (Figure 8a). Join $B F$ and produce it to cross $A D$ at $I$. Complete the rectangle $A B G I$ (Figure 8b). Produce $E F$ to cross $G I$ at $H$ (Figure 8c). Then $E B G H$ is the rectangle which is equivalent to the square $A B C D$ and which has a side $E B$ equal to the given length $b$.


Figure 8: The construction of a square equal in area to a given rectangle.

This method is based on the congruence of triangles. It holds

$$
\begin{aligned}
\triangle A B I & =\triangle G I B, & \text { because } B I \text { is a diagonal of the rectangle } A B G I, \\
\triangle E B F & =\triangle C F B, & \triangle D F I=\triangle H I F .
\end{aligned}
$$

As a result of that the rectangle $A E F D$ is equal in area to the rectangle $F C G H$. So the rectangle $E B G H$ is equal in area to the given square $A B C D$.

If the given side $b$ of the rectangle is longer than the side $a$ of the square $A B C D$, the procedure is similar. Produce $A D$ and $B C$ to $I$ and $G$ respectively so that $|A I|=|B G|=b$. Join $G I$ and complete the rectangle $A B G I$ (Figure 9a). Draw the diagonal $B I$ cutting $C D$ at $F$ (Figure 9b). Then $C F$ is the breadth of the tranfsormed rectangle. Through $F$ draw the straight line $E H$ parallel to $A I$ or $B G$ (Figure 9c). Then $E B G H$ is the rectangle which is equivalent to the square $A B C D$ and which has a side $B G$ equal to the given length $b$.


Figure 9: The construction of a square equal in area to a given rectangle.

This geometrical construction described above is of algebraic significance. That is an example of geometrical algebra. We find a solution of the equation

$$
b x=a^{2}
$$

where $a$ is the side of the given square, $b$ is one side of the required rectangle, $x$ is the second side of the required rectangle.

All Śulbasūtras contain a method explaining how to square a circle. Āpastamba wrote, see [2]:

Divide the [the diameter] into fifteen parts and remove two [of them] This is the gross [value of a] side of the [equivalent] square.

This is an approximate method based on a construction of a square which side $a$ is $\frac{13}{15}$ times the diameter $d$ of the given circle (Figure 10a).

a)

b)

Figure 10: The square equal in area to the given circle.
One possible reason for this method is in [3]. Draw the square $A B C D$ circumscribing the given circle and also the square $E F G H$ inscribed within it. Then the side of the square $A B C D$ is $2 r$ and its area is $4 r^{2}$. The side of the smaller square $E F G H$ is $\sqrt{2} r$ and its area is $2 r^{2}$ (Figure 10b).

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Then it holds

$$
2 r^{2}<\text { area of the circle }<4 r^{2} .
$$

An obvious approximation is

$$
\text { area of the circle }=\frac{4 r^{2}+2 r^{2}}{2}=3 r^{2}
$$

If we denote by $a$ a side of the square equal in area to the circle, we shall have approximately

$$
a^{2}=3 r^{2} \quad \Rightarrow \quad a=\sqrt{3} r
$$

The value of $\sqrt{3}$ was expressed as $\sqrt{3}=1+\frac{2}{3}+\frac{1}{15}=\frac{26}{15}$, therefore $a=\frac{26}{15} r=\frac{13}{15} d$.
The result corresponds to the value of $\pi=4\left(\frac{13}{15}\right)^{2}=\frac{676}{225} \doteq 3.00444$. It is not a very good approximation and certainly not as good as that one known earlier to the Babylonians.

The Sulbasūtras also examine the opposite problem of finding a circle equal in area to the given square. Baudhāyana, Kātyāyana and Āpastamba thought, see [3], [5]:

If you wish to circle a square, draw half its diagonal about the centre towards the east-west line; then describe a circle together with the one third of that which lies outside [the square].

Given a square $A B C D$ find the centre $S$. Rotate $S A$ to the position $S P$ such that $S P$ is perpendicular to the side $A D$. The point $O$ is the midpoint of the side $A D$. Let $Q$ be the point on $P O$ such that $|O Q|$ is one third of $|O P|$. The required circle has centre $S$ and its radius is $|S Q|$.


Figure 11: The circle equal in area to the given square.
If we denote by $a$ a side of the given square $A B C D$, then the diameter $d$ of the required circle is $d=\left(1+\frac{\sqrt{2}-1}{3}\right) a$. This method gives the following value $\pi \doteq 3.088$.

Many different values of $\pi$ appear in the Sulbasūtras, even several different ones used in one text. It is not surprising because the authors thought in terms of approximate constructions, not in terms of exact constructions with $\pi$. A given approximate construction implied some value of $\pi$.

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## 5 Conclusions

These are only some examples from the Śulbasūtras. But Śulbasūtras deal with other problems. We can find fractions and calculations with them or rational rectangular triangles. There are surds and various expressions of them. The remarkable result of the mathematics of the Śulbasūtras is a very close approximation to $\sqrt{2}$, see [3], [6]

$$
\sqrt{2}=1+\frac{1}{3}+\frac{1}{3 \cdot 4}-\frac{1}{3 \cdot 4 \cdot 34}=\frac{577}{408} \doteq 1.414215686
$$

where five decimal places are correct.
The need for finding right time of sacrifices and urge for precise construction of altars gave the first impulse for astronomical observations and accurate mathematical and geometrical discoveries.

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# SOME RESULTS ON BUNDLES OF COVELOCITIES 

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#### Abstract

For $m$-dimensional manifold $M$ and a Weil algebra $A=\mathbb{D}_{k}^{r} / I$ we investigate the spaces $T^{A *} M$ of $A$-covelocities and prove that any covelocity $T_{x}^{A} f \in T_{M}^{A *}$ is determined by its values over $k$ regular 1-jets of the form $i\left(j_{0}^{1} \varphi_{l}\right) \in i\left(P^{1} M\right) \simeq i\left(\operatorname{reg} J_{0}^{1}\left(\mathbb{R}^{k}, M\right)_{x}\right) \subseteq$ $T_{x}^{A} M$ if $i$ denotes the canonical insertion of the space of 1 -jets $J_{0}^{1}\left(\mathbb{R}^{k}, M\right)$ into $T^{A} M$. Further, we extend the rigidity result on $T^{A *} M$ obtained in [15] for $m \geq k$ to arbitrary $m$.


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## 1 Preliminaries

We give the contribution to the theory of Weil functors. We come out from the concepts of an $r$ jet, a jet space $J^{r}(M, N)$ and a bundle functor. We follow the terminology in [9]. Let us denote by $M$ an $m$-dimensional manifold and by $N$ a smooth manifold. All manifolds are considered as $\mathrm{C}^{\infty}$-manifolds. Denote by $\mathcal{M} f$ the category of smooth manifolds with smooth maps and by $\mathcal{M} f_{m}$ the category of $m$-dimensional manifolds with local diffeomorphisms. Further, consider the category $\mathcal{F M}$ of fibered manifolds with smooth fibered maps. Analogously we denote by $\mathcal{F} \mathcal{M}_{m}$ the category of fibered manifolds with $m$-dimensional bases together with the fibered maps above local diffeomorphisms. Recall that a bundle functor defined on the category $\mathcal{M} f_{m}$ is said to be a natural bundle, [9]. We essentially use jet functors $J^{r}$ defined on the category $\mathcal{M} f_{m} \times \mathcal{F} \mathcal{M}$. We define $J^{r}(M, N)$ as the space $r$-jets of smooth maps $M \rightarrow N$ to any couple $(M, N) \in \operatorname{Obj}\left(\mathcal{M} f_{m} \times \mathcal{M} f\right)$ and the map $J^{r}(g, h): J^{r}\left(M_{1}, N_{1}\right) \rightarrow J^{r}\left(M_{2}, N_{2}\right)$ defined by $j_{x}^{r} f \mapsto j_{g(x)}^{r}\left(h \circ f \circ g^{-1}\right)$ to any couple $(g, h) \in \operatorname{Morph}\left(\mathcal{M} f_{m} \times \mathcal{M} f\right)$.

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Among bundle functors defined on $\mathcal{M} f$ there is a significant class of product preserving bundle functors. The classical result of Michor and others reads that they coincide with Weil functors $T^{A}$ associated to some Weil algebra $A,[9]$. The restriction of a Weil functor to $\mathcal{M} f_{m}$ is said to be a Weil bundle. In order to resume briefly the elementary concepts from the Weil theory, consider the algebra $\mathcal{E}(k)$ of germs of smooth functions defined on $\mathbb{R}^{k}$ with the source at zero. Recall that a Weil algebra can be defined either as $A=\mathbb{R} \oplus N_{A}$ for the ideal $N_{A}$ of nilpotent elements (the so called nilpotent ideal) or as a quotient $A=\mathcal{E}(k) / I$ by an ideal $I \subset \mathcal{E}(k)$ of finite codimension. It can also be defined as a quotient $A=\mathbb{D}_{k}^{r} / J$ of the so called jet algebra $\mathbb{D}_{k}^{r}$ of polynomials of $k$ variables of order at most $r$ by some of its ideal $J$. Finally, we define width $(A)$ as $\operatorname{dim} N_{A} / N_{A}^{2}$ and $\operatorname{height}(A)$ as the minimal $r$ for which $A=\mathbb{D}_{k}^{r} / J$.

In the present paper, we use contravariant approach to the definition of Weil functors. This comes out from the $I$-factorization of germs in the following sense. Two germs germ $_{0} g: \mathbb{R}_{0}^{m} \rightarrow$ $M$ and $\operatorname{germ}_{0} h: \mathbb{R}_{0}^{m} \rightarrow M$ satisfying $g(0)=h(0)=x$ are said to be $I$-equivalent if and only if $\operatorname{germ}_{x} \gamma \circ \operatorname{germ}_{0} g-\operatorname{germ}_{x} \gamma \circ \operatorname{germ}_{0} h \in I$ for any function $\gamma: M \rightarrow \mathbb{R}$ defined near $x$. Classes of such an equivalence are denoted by $j^{A} g$, the space of them by $J^{A} M$.

For a smooth map $\varphi: M \rightarrow N$ define the map $J^{A} \varphi: J^{A} M \rightarrow J^{A} N$ as follows

$$
J^{A} \varphi\left(j^{A} g\right):=j^{A}(\varphi \circ g)
$$

We remark that a Weil functor can be also defined from the covariant point of view as a functor $T^{A}$ defined by $T^{A} M=\operatorname{Hom}\left(\mathrm{C}^{\infty}(M, \mathbb{R}), A\right)$ on objects and by $T^{A} f(H)(\varphi)=H(\varphi \circ f)$ for $H \in \operatorname{Hom}\left(\mathrm{C}^{\infty}(M, \mathbb{R}), A\right)$ and $\varphi \in \mathrm{C}^{\infty}(N, \mathbb{R})$. The identification of $J^{A} M$ with $T^{A} M$ is given by

$$
j^{A} g(\varphi)=j^{A}(\varphi \circ g)
$$

for any function $\varphi$ defined near $g(0)$. In what follows we shall use only the notation $T^{A}$ for Weil functors in spite of applying the contravariant approach to them.

As for natural transformations $\widetilde{t}_{M}: T^{B} M \rightarrow T^{A} M$, they bijectively correspond to homomorphisms $t: B \rightarrow A$. Further, homomorphisms $B \rightarrow A$ correspond bijectively to the so called $B$-admissible $A$-velocities defined in [7] as follows.

Let $A=\mathcal{E}(k) / I$ and $B=\mathcal{E}(p) / J$ be Weil algebras considered as the quotiens of the germ algebras $\mathcal{E}(k)$ and $\mathcal{E}(p)$ respectively. For a smooth map $f: \mathbb{R}_{0}^{k} \rightarrow \mathbb{R}_{0}^{p}$ an $A$-velocity $j^{A} f$ is said to be $B$-admissible if and only if

$$
\begin{equation*}
\operatorname{germ}_{0} \varphi \in J \Rightarrow \operatorname{germ}_{0}(\varphi \circ f) \in I \tag{1}
\end{equation*}
$$

Thus a natural transformation $\widetilde{t}_{M}: T^{B} M \rightarrow T^{A} M$ is assigned to a $B$-admissible $A$-velocity $j^{A} f$ by

$$
\widetilde{t}_{M}\left(j^{B} \varphi\right)=t_{M}^{j^{A} f}\left(j^{B} \varphi\right)=j^{A}(\varphi \circ f) .
$$

In particular all automorphisms are determined by those reparametrizations of $\tau^{1}, \ldots, \tau_{m}$ generating $\mathbb{D}_{m}^{r}$ which satisfy the conditions of admissibility (1).

Denote by $G_{m}^{r}$ the Lie group inv $J_{0}^{r}\left(\mathbb{R}^{m}, \mathbb{R}^{m}\right)_{0}$ consisting of zero preserving invertible $r$-jets with respect to their composition. For a Weil algebra $p: \mathbb{D}_{m}^{r} \rightarrow A=\mathbb{D}_{m}^{r} / I$ Alonso in [1] defined the subgroups $G_{A}$ and $G^{A}$ of $G_{m}^{r} \simeq \operatorname{Aut}\left(\mathbb{D}_{m}^{r}\right)$ by

$$
\begin{equation*}
G_{A}=\left\{j_{0}^{r} g \in G_{m}^{r} ; p \circ j_{0}^{r} g=p\right\}, \quad G^{A}=\left\{j_{0}^{r} g \in G_{m}^{r} ; \operatorname{Ker}\left(p \circ j_{0}^{r} g\right)=\operatorname{Ker}(p)\right\} . \tag{2}
\end{equation*}
$$

He also proved that $G_{A}$ is a normal subgroup of $G^{A}$ and the identification $G^{A} / G_{A} \simeq$ Aut $A$. Clearly, $j_{0}^{r} g$ determines an automorphism of $A$ if and only if $j_{0}^{r} g \in G^{A}$.

Besides Weil bundles there are significant natural bundles $T^{r *} M([9])$, the $r$-th order bundles of covelocities. Recall that the natural bundle $T^{r *}$ is defined by $T_{x}^{r *} M=J_{x}^{r}(M, \mathbb{R})_{0}$ on objects and by $T^{r *} g\left(j_{x}^{r} f\right)=j_{g(x)}^{r}\left(f \circ g^{-1}\right)$ on morphisms. In [15] we have presented the bundles $T^{r *} M$ in the form of $P^{r} M\left[N_{m}^{r}, \ell\right]$ where $N_{m}^{r}$ denotes the nilpotent ideal of the jet algebra $\mathbb{D}_{m}^{r}$ and the action $\ell: G_{m}^{r} \times N_{m}^{r} \rightarrow N_{m}^{r}$ on the standard fiber is defined by $\ell\left(j_{0}^{r} g, j_{0}^{r} \varphi\right)=j_{0}^{r}\left(\varphi \circ g^{-1}\right)$.

Moreover, we have generalized the space of classical $r$-th order covelocities to the space $T^{A *} M$ of $A$-covelocities $T_{x}^{A} f: T_{x}^{A} M \rightarrow T_{0}^{A} \mathbb{R} \simeq N_{A}$. In [15] we have proved for $m \geq k$ the rigidity result saying that $T^{A *} M$ with the so-called $T^{A *}$-maps defined by

$$
\begin{equation*}
T_{x}^{A *} g\left(T_{x}^{A} f\right)=T_{g(x)}^{A}\left(f \circ g^{-1}\right) \tag{3}
\end{equation*}
$$

for every local diffeomorphism $g$ form a natural bundle identified with $P^{r} M\left[N_{m}^{r}, \ell\right]$.
Apart from the spaces $T^{A *} M$ we shall investigate the spaces $T_{V, p}^{A *} M$ of $p$-vertical covelocities. Our attention will be focused on $k=\operatorname{width}(A)>m=\operatorname{dim} M$ in both cases of $T^{A *} M$ and $T_{V, p}^{A *} M$. Before resuming the basic concepts and the already achieved results from [15], recall that for $m>k$ it is usefull to consider $A=\mathbb{D}_{k}^{r} / I$ as $\mathbb{D}_{m}^{r} /\left(I \cup\left\langle\tau_{k+1}, \ldots, \tau_{m}\right\rangle\right)$ adding formally $m-k$ variables to the original ones. Thus we can suppose $m=k$ without loss of generality in the following text.

For a Lie group homomorphism $p: G_{m}^{r} \simeq \operatorname{Aut}\left(\mathbb{D}_{m}^{r}\right) \rightarrow G^{A}$ consider the left action of $p\left(G_{m}^{r}\right)$ on $T_{x}^{A} M$ defined by

$$
\begin{equation*}
\ell\left(j_{0}^{r} g, j^{A} \varphi_{x}\right)=p_{A}\left(j_{0}^{r} \hat{\varphi}_{x} \circ j_{0}^{r} g^{-1}\right) \tag{4}
\end{equation*}
$$

for any $j_{0}^{r} \hat{\varphi}_{x} \in j^{A} \varphi_{x}$ and $j_{0}^{r} g \in p\left(G_{m}^{r}\right)$. For $j_{0}^{r} \varphi_{x} \in P_{x}^{r} M \simeq \operatorname{reg}\left(T_{m}^{r}\right)_{x} M$ denote by $\mathcal{O} r b_{j_{0}^{r} \varphi_{x}}$ its $p\left(G_{m}^{r}\right)$-orbit in respect to the action of $p\left(G_{m}^{r}\right)$ from (4). Taking its corestriction to reg $T_{x}^{A} M$ defines $\mathcal{O} r b_{j^{A} \varphi_{x}}$ in the obvious way.

Consider the restrictions $X=\left.T_{x}^{A} f\right|_{\mathcal{O r b}_{j_{0}^{r} \varphi_{x}}}$ of covelocities $T_{x}^{A} f$ to the $p\left(G_{m}^{r}\right)$-orbits of regular elements of $\left(T_{m}^{r}\right)_{x} M$ and consequentely of $T_{x}^{A} M$. Such orbits will be called regular. We extend the elements $X$ to other regular $p\left(G_{m}^{r}\right)$-orbits of $T_{x}^{A} M$ as follows. For $j_{0}^{r} \psi_{x} \in \operatorname{reg}\left(T_{m}^{r}\right)_{x} M$, every element $X=\left.T_{x}^{A} f\right|_{\mathcal{O r b}_{\left(j_{0}^{r} \varphi_{x}\right)}}$ is assigned an element $Y=T_{x}^{A} \hat{f}_{\mid \mathcal{O} r b_{j_{0}^{r} \psi_{x}}}$ defined as follows

$$
\begin{equation*}
T_{x}^{A} \hat{f}\left(j_{0}^{r} \psi_{x}\right)=T_{x}^{A} f\left(j_{0}^{r} \varphi_{x}\right) \circ p\left(j_{0}^{r}\left(\varphi_{x}^{-1} \circ \psi_{x}\right)\right) \tag{5}
\end{equation*}
$$

We denote the space of such elements by $T_{V, p}^{A *} M$. In order to verify the correctness of this definition one must check that for any $j_{0}^{r} \widetilde{\psi}_{x}$ satisfying $j^{A} \psi_{x}=j^{A} \widetilde{\psi}_{x}$ it holds $T_{x}^{A} f\left(j_{0}^{r} \varphi_{x}\right) \circ$ $p\left(j_{0}^{r}\left(\varphi_{x}^{-1} \circ \psi_{x}\right)\right)=T_{x}^{A} f\left(j_{0}^{r} \varphi_{x}\right) \circ p\left(j_{0}^{r}\left(\varphi_{x}^{-1} \circ \widetilde{\psi}_{x}\right)\right)$ which is quite easy. For deteils, see [15].

For a local diffeomorphism $g: M \rightarrow N$ defined in the neighbourhood of $x \in M$ define the $\operatorname{map} T_{V, p}^{A *} g:\left(T_{V, p}^{A *}\right)_{x} M \rightarrow\left(T_{V, p}^{A *}\right)_{g(x)} N$ by

$$
\begin{equation*}
\left(T_{V, p}^{A *}\right)_{x} g\left(\left.T_{x}^{A} f\right|_{\mathcal{O r}_{p\left(G_{m}^{r}\right)}\left(j_{0}^{r} \varphi_{x}\right)}\right)=\left.T_{g(x)}^{A}\left(f \circ g^{-1}\right)\right|_{\mathcal{O}_{r b_{0}^{r}\left(g \circ \varphi_{x}\right)}} \tag{6}
\end{equation*}
$$

Let us resume of the main results achieved in [15] to be generalized in the next sections. They read as follows

Let $A=\mathbb{D}_{k}^{r} / I$ and $m \geq k$. Then it holds
(a) The spaces $T^{A *} M$ together with the maps defined by (3) form the natural bundle $P^{r} M\left[\mathbb{D}_{m}^{r}, \ell\right]$ defined on the category $\mathcal{M} f_{m}$.
(b) The spaces $T_{V, p}^{A *} M$ of $p$-vertical $A$-covelocities together with the maps $T_{V, p}^{A *} g$ defined by (6) form a natural bundle on $\mathcal{M} f_{m}$ of the form $T_{V ; p}^{A *} M=P^{r} M\left[N_{A}, \ell\right]$ with the standard fiber formed by $N_{A}$. The left action $\ell: G_{m}^{r} \times N_{A} \rightarrow N_{A}$ is defined as follows

$$
\begin{equation*}
\ell\left(j_{0}^{r} g, j^{A} \alpha\right)=p_{A}\left(j_{0}^{r} \alpha_{0} \circ\left(p\left(j_{0}^{r} g\right)\right)^{-1}\right) \tag{7}
\end{equation*}
$$

where $p_{A}: \mathbb{D}_{m}^{r} \rightarrow A$ is the canonical projection of Weil algebras and $j_{0}^{r} \alpha_{0} \in j^{A} \alpha$ is arbitrary.
In the very end of the Section we recall that the elements $T_{x}^{A} f \in T_{x}^{A *} M$ form the nilpotent ideal $N_{B}$ of a Weil algebra $B=\mathbb{D}_{m}^{r} / J$ for some ideal $J$ if we define the vector space operations on $T_{x}^{A *} M$ by $\left(T_{x}^{A} f_{1}+T_{x}^{A} f_{2}\right)\left(j^{A} \varphi_{x}\right)=T_{x}^{A} f_{1}\left(j^{A} \varphi_{x}\right)+T_{x}^{A} f_{2}\left(j^{A} \varphi_{x}\right)$ and $\left(c \cdot T_{x}^{A} f\right)\left(j^{A} \varphi_{x}\right)=c \cdot T_{x}^{A} f\left(j^{A} \varphi_{x}\right)$. The algebra multiplication is defined by $\left(T_{x}^{A} f_{1} \cdot T_{x}^{A} f_{2}\right)\left(j^{A} \varphi_{x}\right)=\mu_{A}\left(T_{x}^{A} f_{1}\left(j^{A} \varphi_{x}\right), T_{x}^{A} f_{2}\left(j^{A} \varphi_{x}\right)\right)$ where $\mu_{A}$ denotes the multiplication in $A, c \in \mathbb{R}$ and $T_{x}^{A} f_{1}\left(j^{A} \varphi_{x}\right) \in \operatorname{reg} T_{x}^{A} M$ are arbitrary. We remark that for $m \geq k$ the coincidence of $B$ with $\mathbb{D}_{m}^{r}$ was proved in [15].

For any $j_{0}^{r} \varphi_{x} \in P_{x}^{r} M$ we can define the algebra operations on the space of all $T_{x}^{A} f_{\mid \mathcal{O} r b\left(j_{0}^{r} \varphi_{x}\right)}$ restricting the operations we have just defined. To define Weil algebra operations on $\left(T_{V, p}^{A *}\right)_{x} M$ one must check that

$$
\left\{\left(\left.T_{x}^{A} f\right|_{\mathcal{O r b}_{j_{0}^{r} \varphi_{x}}},\left.T_{x}^{A} \hat{f}\right|_{\mathcal{O r b}_{j_{0} \psi_{x}}}\right) \in \mathcal{C o v}_{p, 0}^{2} M ; T_{x}^{A} \hat{f}\left(j_{0}^{r} \psi_{x}\right)=T_{x}^{A} f\left(j_{0}^{r} \varphi_{x}\right) \circ p\left(j_{0}^{r}\left(\varphi_{x}^{-1} \circ \psi_{x}\right)\right)\right\}
$$

is the congruence on $\mathcal{C o v}_{p, 0} M$ if we denote by $\mathcal{C o v}_{p, 0} M$ the space $\left\{\left.T_{x}^{A} f\right|_{\mathcal{O}_{\sigma_{j_{0}} \varphi}} ; j_{0}^{r} \varphi_{x} \in P_{x}^{r} M\right\}$. The verification of this fact is straight and simple.
$2 A$-covelocities are determined by $k$ values over elements from $P^{1} M \simeq \operatorname{reg} J_{0}^{1}\left(\mathbb{R}^{k}, M\right)$

In the present Section we prove that every covelocity $T_{x}^{A} f \in T_{x}^{A *} M$ is determined by its values over at most $k$ elements of $T_{x}^{A} M$. It was remarked in Corollary following Theorem 1 in [15] mentioned in the end of Section 1, assertion (a) that for $m \geq k$ every $A$-covelocity $T_{x}^{A} f \in T_{x}^{A *} M$ is determined by at most $k$ values of itself over some regular elements of $T_{x}^{A} M$. We present the more sophisticated formulation of this fact and extend it to the case of arbitrary $m$ independently of the mentioned Theorem.

Before the formulation of this result recall from the end of Section 1 that all $T_{x}^{A} f$ under disscussion form a Weil algebra $B$ of height $r$ independent on the choice of $M$ and $x$. In general, for every Weil algebra $B$ of height $r$ there is an subordinate Weil algebra $B_{q}$ of height $q<r$ defined as follows. If $B=\mathbb{D}_{k}^{r} / I$ then we set $B_{q}=\mathbb{D}_{k}^{r} /\left(I \cup \mu^{q+1}\right)$. Further, there is a canonical homomorphism $\pi_{q, B}: B \rightarrow B_{q}$ defined by $[p]_{I} \mapsto[p]_{I \cup \mu^{q+1}}$ and consequently we have homomorphisms $\pi_{s, B}^{q}: B_{q} \rightarrow B_{s}$ for any $s \leq q \leq r$. Further we have the natural transformation $\widetilde{\pi}_{s, B}^{q}: T^{B_{q}} M \rightarrow T^{s, B} M$ induced by the homomorphism $\pi_{s, B}^{q}$.

It is easy to see that for every local diffeomorphism $g: M \rightarrow N$, the map $T^{A *} g: T_{x}^{A *} M \rightarrow$ $T_{g(x)}^{A *} N$ from (3) determines an isomorphism of Weil algebras. This follows that our investigations can be reduced to the algebra $T_{0}^{A *} \mathbb{R}^{m} \simeq N_{B}^{m}$. Let us start with $m=k$ while the cases
$m<k$ will be disscussed later. We shall need the cyclic permutations $\sigma_{1}, \ldots, \sigma_{k} \in S_{k}$ and let us consider them as the permutation matrices which we can denote by the same symbols.

In what follows we shall need the following matrices. For any permutation matrix $\sigma$, a real $c$ and $l \in\{1, \ldots, k\}$ define the matrix $\sigma(c, l)$ by

$$
\begin{equation*}
\sigma(c, l)=d\left(1, c^{l}, c^{2 l}, \ldots, c^{(l-1) k}\right) \cdot \sigma \tag{8}
\end{equation*}
$$

where $d\left(a_{1}, \ldots, a_{k}\right)$ denotes in general the diagonal matrix consisting of $a_{1}, \ldots, a_{k}$ in the obvious way. We have the following lemma

Lemma 1 Let $c \notin\{0,1\}$ and $C_{k} \subseteq S_{k}$ denote the set of all cycles of lenght $k$. Then the matrices $\sigma(c, l)$ for $\sigma \in C_{k}$ and $l=1, \ldots, k$ form a linear basis in $g l(k, \mathbb{R})$.

Proof: It is obtained by assigning a $k^{2}$-dimensional vector to every matrix and constructing a $k^{2}$-th order matrix from them. This is a block matrix and determinants of the blocks coincide with the Van der Monde determinants of order $k$. Therefore they take non-zero value whenever $c \notin\{0,1\}$ which proves our claim.

Let us recall that $B$ denotes the Weil algebra on $T^{A *} M$. Denote by $p_{B}: \mathbb{D}_{m}^{r} \rightarrow B$ the canonical projection homomorphism. Further, let

$$
\begin{equation*}
T_{0}^{A} f=p_{B}\left(T_{m, 0}^{r} f_{(r)}+T_{m, 0}^{r} f^{(r-1)}\right)=T_{0}^{A} f_{(r)}+T_{0}^{A} f^{(r-1)}=T_{0}^{A} f_{(r)}+T_{0}^{A_{r-1}} f^{(r-1)} \tag{9}
\end{equation*}
$$

be a decomposition of $T_{0}^{A} f \in N_{B}$ such that $f_{(r)}$ is a homogenous polynomial of order $r$ and $f^{(r-1)}$ is the residual polynomial of order at most $r-1$. Clearly, $T_{0}^{A} f_{(r)} \in \operatorname{Ker} \pi_{r-1, B}^{r}$.

It is quite easy to see that under the assumption of $I \subseteq \mu^{2}$ which can be done without loss of generality we can consider $J_{0}^{1}\left(\mathbb{R}^{m}, \mathbb{R}^{m}\right)_{0}$ as a linear subspace of $T_{0}^{A} \mathbb{R}^{m}$ and consequently for $m=k$ we can consider the support of the linear $\operatorname{group} \operatorname{GL}(m, \mathbb{R})=\pi_{1}^{r}\left(\operatorname{reg} T_{0}^{A} \mathbb{R}^{m}\right)$ either as a linear subspace or as a factor linear space of $\operatorname{reg} T_{0}^{A} \mathbb{R}^{m}$.

Given $j^{A} \eta \in T_{0}^{A} \mathbb{R}^{m}$, consider a decomposition of $j^{A} \eta=T_{0}^{A} f\left(j^{A} \varphi\right)$ analogous to that of (9), i.e.

$$
\begin{equation*}
j^{A} \eta=j^{A} \eta_{(r)}+j^{A} \eta^{(r-1)} \tag{10}
\end{equation*}
$$

where $\eta_{(r)}$ is a homogenous polynomial of order $r$ and $\eta^{(r-1)}$ a polynomial of order at most $r-1$. Suppose $j^{A} \varphi \in \operatorname{reg} J_{0}^{1}\left(\mathbb{R}^{m}, \mathbb{R}^{m}\right)_{0}$. Then the coordinate formula for the composition of jets yields that $j^{A} \eta_{(r)}$ is not affected by $f^{(r-1)}$ and is determined by $f_{(r)}$. This can be expressed in coordinates by

$$
\begin{equation*}
y_{j_{1} \ldots j_{r}}=f_{l_{1} \ldots l_{r}} x_{j_{1}}^{l_{1}} \ldots x_{j_{r}}^{l_{r}} \tag{11}
\end{equation*}
$$

for some $f_{l_{1} \ldots l_{r}}$ if we denote by $y_{\alpha}$ the jet coordinates of $\eta$ and $x_{\alpha}^{i}$ those of $\varphi$. We have the following Proposition

Proposition 1 Every $T_{0}^{A} f_{(r)}$ from the decomposition (9) is determined by its values over $k$ cycles $j_{0}^{1} \sigma(1, l) \in C_{k}$, i.e. $T_{0}^{A} f_{(r)}$ is determined by $k$ values $T_{0}^{A} f_{(r)}\left(j^{A} \sigma(1, l)\right)=T_{0}^{A} f_{(r)}\left(j^{1} \sigma(1, l)\right)$.

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Proof: It follows from the preceding investigations that instead $T_{0}^{A} f_{(r)}\left(j^{A} \varphi\right)$ we can consider $T_{0}^{A} f_{(r)}\left(j_{0}^{1} \varphi\right)$. By Lemma 1, the regular matrix $j_{0}^{1} \varphi \in \operatorname{reg}\left(J_{0}^{1}\left(\mathbb{R}^{m}, \mathbb{R}^{m}\right)\right)$ is identified with a linear combination as follows

$$
\begin{equation*}
j_{0}^{1} \varphi=\sum_{\sigma \in C_{k}, l=1}^{k} p^{\sigma, l} j_{0}^{1} \sigma(c, l) \tag{12}
\end{equation*}
$$

Then $T_{0}^{A} f_{(r)}\left(j^{1} \varphi\right)$ can be expressed as

$$
\begin{equation*}
\sum_{\sigma \in C_{k}} T_{0}^{A} f_{(r)}\left(\sum_{l=1}^{k} p^{\sigma, l} j_{0}^{1} \sigma(c, l)\right) \tag{13}
\end{equation*}
$$

due to the fact that we consider cyclic permutations and the formula (11). Further, $\sum_{l=1}^{k} p^{\sigma, l} j_{0}^{1} \sigma(c, l)$ can be expressed as $D \cdot j_{0}^{1} \sigma(1,1)$ for some matrix $D$ having all elements equal to zero except those on the diagonal. Further, $D \cdot j_{0}^{1} \sigma(1, l)=j_{0}^{1} \widetilde{\sigma}(1, l) \cdot D$ where $j_{0}^{1} \widetilde{\sigma}$ is the inverse cycle to $j_{0}^{1} \sigma$. Since $D \in G^{A}$ (if we consider GL $(m, \mathbb{R})$ as a subgroup of $G_{k}^{r}$ ) every summand of (13) is determined by $T_{0}^{A} f_{(r)}\left(j_{0}^{1} \sigma(1, l)\right)$ for some $l$. This completes the proof.

Proposition 2 Let $m=\operatorname{dim} M=k=\operatorname{width} A$. Then every $T_{x}^{A} h \in T_{x}^{A *} M$ is determined by at most $k$ regular elements $j^{A} \varphi_{1}, \ldots j^{A} \varphi_{k} \in T_{x}^{A *} M$. Moreover, $T_{x}^{A} f$ depends on such elements only up to their 1 -jets $j_{0}^{1} \varphi_{1}, \ldots, j_{0}^{1} \varphi_{k}$.

Proof: In the first step we prove that for every $T_{0}^{A} f \in T_{0}^{A} \mathbb{R}^{m}$ and $j^{A} \varphi \in T_{0}^{A} \mathbb{R}^{m}$ the value $T_{0}^{A} f\left(j^{A} \varphi\right)$ is determined by the values $T_{0}^{A} f\left(j_{0}^{1} \sigma(1, l)\right)$ over $k$ cycles $j_{0}^{1} \sigma \in C_{k}$. The formula (9) yields that the map $T_{0}^{A} f^{(r-1)}=T_{0}^{A} f-T_{0}^{A} f_{(r)}$ can be considered as $T^{A_{r-1}} f^{(r-1)}$ and we can iterate the preceding Proposition which step by steps verifies the first claim. The rest of the Proposition follows from the already mentioned fact that every local diffeomorphism $g: M \rightarrow N$ induces the isomorphism $T^{A *} g: T_{x}^{A *} M \rightarrow T_{g(x)}^{A *} N$ particularly for $N=\mathbb{R}^{m}$ and $g(x)=k$.

We remark that the assertion was proved without the application of the result from [15] giving the coincidence of $T^{A *} M$ with $T_{k}^{r} M=P^{r} M\left[\mathbb{D}_{m}^{r}, \ell\right]$. We can easily extend our Proposition to the cases of $m<k$. In what follows, an element $j^{A} \varphi \in T_{x}^{A} M$ is said to be regular if and only if $j^{A} \varphi$ is over submersions.

Corollary 1 Let $m=\operatorname{dim} M<k=$ width $A$. Then every $T_{x}^{A} h \in T_{x}^{A *} M$ is determined by at most $k$ regular elements $j^{A} \varphi_{1}, \ldots j^{A} \varphi_{k} \in T_{x}^{A} M$. Moreover, $T_{x}^{A} f$ depends on such elements only up to their 1-jets $j_{0}^{1} \varphi_{1}, \ldots, j_{0}^{1} \varphi_{k}$.

Proof: The assertion is obtained immediately if we identify every $f: M \rightarrow \mathbb{R}$ with the map $\hat{f}: M \times \mathbb{R}^{k-m} \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\hat{f}(x, y)=f(x) \tag{14}
\end{equation*}
$$

for any $(x, y) \in \mathbb{R}^{m} \times \mathbb{R}^{k-m}$.

## 3 The rigidity result for $T^{A *} M$ and $m<k$

Given Weil algebra $A=\mathbb{D}_{k}^{r} / I$ satisfying width $(A)=k>m$ we extend the rigidity result concerning $T^{A *} M$ to our cases of $m$. For $k \leq m$ it was obtained in [15] and recalled in the end of Section 1, assertion (a). In order to find the Weil algebra $B$ with the nilpotent ideal formed by $T_{x}^{A *} M$ we first recall from [15] that the maps $T_{x}^{A *} g$ defined in (3) are Weil algebra isomorphisms. Thus we are going to investigate only the algebra $T_{0}^{A *} \mathbb{R}^{m}$ in order to determine $B$. Since $T^{A}$ is a bundle functor of order $r$ we have $B=\mathbb{D}_{m}^{r} / J$ for some ideal $J \subseteq \mathbb{D}_{m}^{r}$.

It follows from Corollary 1 and its proof that every $T_{0}^{A} f \in T_{0}^{A *} \mathbb{R}^{m}$ is determined by $T_{0}^{A} \hat{f}\left(j_{0}^{1} \sigma(1, l)\right)$ over $k$ cycles $j_{0}^{1} \sigma(1, l) \in C_{k}$ if $\hat{f}: \mathbb{R}^{m} \times \mathbb{R}^{k-m}$ is defined by (14). In the first step of our searching for $B$ consider the subalgebras $A_{i_{1} \ldots i_{m}}$ of $A$ corresponding to subsets $\left\{i_{1}, \ldots, i_{m}\right\} \subseteq\{1, \ldots, k\}$ of cardinality $m$ which are defined as the algebras of all $I$-classes of polynomials of order at most $r$ of variables $\tau_{i_{1}}, \ldots, \tau_{i_{m}}$ only.

Without loss of generality suppose $A=\mathcal{E}(k) / I$ in the normal form, i.e. $I \subseteq \mu_{k}^{2}$. The following Proposition yields the Weil algebra $B$.

Proposition 3 For $m<k$ the algebraic structure of $T_{0}^{A *} \mathbb{R}^{m}$ coincides with the nilpotent ideal $N_{m}^{r}$ of the Weil algebra $\mathbb{D}_{m}^{r}$.

Proof: Every $T_{0}^{A} \hat{f}: M_{x} \times \mathbb{R}^{k-m}$ actually does not depend on the second arguments (see (8)) and neither does $T_{0}^{A} \hat{f}: \mathbb{R}_{0}^{m} \times \mathbb{R}_{0}^{k-m}$. For any cycle $j_{0}^{1} \sigma=j_{0}^{1} \sigma(1, l) \in C_{k}$ the recent morphism can be restricted to $A_{\sigma(1) \ldots \sigma(m)}^{m} \times A_{\sigma(m+1) \ldots \sigma(k)}^{k-m}$. Renaming the polynomial variables $\tau_{\sigma(1)}, \ldots, \tau_{\sigma(m)}$ to $t_{1}, \ldots, t_{m}$ we can identify $\operatorname{pr}_{2} \circ\left(A_{\sigma(1) \ldots \sigma(m)}^{m} \times A_{\sigma(m+1) \ldots \sigma(k)}^{k-m}\right)$ with $B_{\sigma(1) \ldots \sigma(m)}$ which can be briefly denotes by $B_{\sigma}$ and is obtained from $A$ as follows.

Supposing $A=\mathbb{D}_{k}^{r} / I$ then the Weil algebra $B_{\sigma}$ is defined as $\mathbb{D}_{m}^{r} / J_{\sigma}$ if we define the ideal $J_{\sigma}$ by

$$
\begin{equation*}
J_{\sigma}=\left(\left(I \cup\left\{\tau_{\sigma(m+1)}, \ldots, \tau_{\sigma(k)}\right) \circ \sigma\right) \cap\left\langle\tau_{1}, \ldots, \tau_{m}\right\rangle\right. \tag{15}
\end{equation*}
$$

By the trivial renaming of polynomial variables $\tau_{1}, \ldots, \tau_{m}$ to $t_{1}, \ldots, t_{m}$ we obtain $B_{\sigma}$ as $J_{\sigma^{-}}$ classes of polynomials from $\mathbb{D}_{m}^{r}$ in variables $t_{1}, \ldots, t_{m}$.

If we evaluate $T_{0}^{A} \hat{f}$ corresponding to $T_{0}^{A} f \in T_{0}^{A *} \mathbb{R}^{m}$ in the sense of (14) over all $j_{0}^{1} \sigma=$ $j_{0}^{1} \sigma(1, l) \in C_{k}$ then every $T_{0}^{A} f$ is determined by the $k$-tuple

$$
\begin{equation*}
\left(T_{0}^{A} f\left(\operatorname{pr}_{1} \circ j_{0}^{1} \sigma(1, l)\right), \ldots, T_{0}^{A} f\left(\operatorname{pr}_{1} \circ j_{0}^{1} \sigma(1, k)\right)\right) \tag{16}
\end{equation*}
$$

If we consider all $T_{0}^{A} f$ restricted to $A_{\sigma(1), \ldots, \sigma(m)}^{m}$ only then they can be identified with elements of the form $T_{0}^{B_{\sigma}} f$ together with the $T^{B_{\sigma}{ }^{*}}$-maps induced by the obvious corestriction of $T^{A *}$-maps. By the rigidity result from [15] and recalled in the end of Section 1, assertion (a) the algebra of such elements coincides with $\mathbb{D}^{q_{\sigma}}$ if we put $q_{\sigma}=\operatorname{height}\left(A_{\sigma(1) \ldots \sigma(m)}\right)$.

Taking into account the identification (16) we obtain $J \subseteq \bigcap_{\sigma \in C_{k}} J_{\sigma}$. Since $T_{0}^{A} f$ is in general of order $r$ and every $T_{0}^{A} f$ in question is identified with (16) then $q=r$ and $J \subseteq\{0\}$ which completes the proof.

We state the rigidity result concerning $T^{A *} M$ for $m<k$ which generalizes that obtained in [15].

Theorem 1 The space of $A$-covelocities $T^{A *} M$ together with the maps $T^{A *} g$ defined in (3) forms the natural bundle $P^{r} M\left[N_{m}^{r}, \ell\right]$ with the left action $\ell: G_{m}^{r} \times N_{m}^{r} \rightarrow N_{m}^{r}$ on the standard fiber $T_{0}^{A *} \mathbb{R}^{m} \cong N_{m}^{r}$ defined by

$$
\begin{equation*}
\ell\left(j_{0}^{r} g, T_{0}^{A} \widetilde{f}\right)=T_{0}^{A}\left(\tilde{f} \circ g^{-1}\right) \tag{17}
\end{equation*}
$$

for arbitrary $T_{0}^{A} \widetilde{f} \in T_{0}^{A *} M$. In the more intrinsic way the action is defined by

$$
\begin{equation*}
\ell\left(j_{0}^{r} g, j_{0}^{r} \alpha\right)=j_{0}^{r}\left(\alpha \circ g^{-1}\right) \tag{18}
\end{equation*}
$$

for arbitrary $j_{0}^{r} \alpha \in N_{m}^{r}$.
Proof: For $k \leq m$, the result was proved in [15], Theorem 1. Although the proof is quite analogous for the other cases of $m$, we shall briefly present it. The first formula as well as the assertion follow from the general theory of natural bundles, see [9]. More exactly, we have the identification of $T_{x}^{A} f \in T_{x}^{A *} M$ with $\left\{j_{0}^{r} \varphi_{x}, T_{x}^{A} f\left(j_{0}^{r} \varphi_{x}\right)\right\} \in P^{r} M\left[T_{0}^{A *} \mathbb{R}^{m}, \ell\right]$ with the action $\ell$ defined by (17). Indeed, the correctness of this assignement is given by $\left\{j_{0}^{r} \varphi_{x} \circ j_{0}^{r} g, T_{0}^{A}(\tilde{f} \circ\right.$ $g)\}=\left\{j_{0}^{r} \varphi_{x}, T_{0}^{A} \widetilde{f}\right\}$. As for morphisms, it is easy to verify that the maps $T^{A *} g$ correspond to the morphism of associated bundles which are of the form $\left\{P^{r} g, \operatorname{id}_{T_{0}^{A *} \mathbb{R}^{m}}\right\}$. The identification $T_{0}^{A *} \mathbb{R}^{m} \cong N_{m}^{r}$ proved in Proposition 3 completes the proof of the first assertion.

In order to prove the formula (18) consider the identification $H$ of the form

$$
\begin{equation*}
T_{0}^{A} f \cong \frac{1}{\alpha!} a_{\alpha} T_{0}^{A} \operatorname{pr}_{\mathbb{R}^{m}}^{\alpha} \cong \frac{1}{\alpha!} a_{\alpha} \operatorname{pr}_{N_{A}^{m}}^{\alpha} \tag{19}
\end{equation*}
$$

This follows from the fact that $T^{A}$ preserves products and from the definition of the multiplication in $T_{0}^{A *} \mathbb{R}^{m} \cong T_{x}^{A *} M$ by the formula $T_{x}^{A} f_{1} \cdot T_{x}^{A} f_{2} \mapsto T_{x}^{A}\left(u \mapsto f_{1}(u) \cdot f_{2}(u)\right)$ for any $u \in T_{x}^{A} M$. The formula (18) will be proved if we check the the equivariancy of $H$ in respect to $\ell$ from (18), i.e.

$$
\begin{equation*}
H \circ \ell\left(j_{0}^{r} g, T_{0}^{A} \widetilde{f}\right)=\ell\left(j_{0}^{r} g, H\left(T_{0}^{A} \widetilde{f}\right)\right) \tag{20}
\end{equation*}
$$

which can be verified directly and easily in coordinates applying the definition of $\ell$.

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# MINKOWSKI DIFFERENCE IN SURFACE MODELLING 

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#### Abstract

Several known definitions of the Minkowski difference of point sets in the Euclidean space are presented in this paper, together with some of the basic properties of this operation. Examples are introduced, in which results of a particular Minkowski difference definition application and Minkowski sum and product of point sets are compared, used as tools for modelling of surface patches from curve segments determined by vector representations.


Key words. Minkowski difference, modelling of surface patches
Mathematics Subject Classification: Primary 51N25, Secondary 53A056.

## 1. Introduction

Minkowski difference of two point sets is a binary geometric operation defined on point sets in the $n$-dimensional Euclidean space. Various definitions and interpretations of this point set operation can be found in the available literature. The most common way of interpretation of Minkowski difference of sets $A$ and $B$ is the one using Minkowski sum of sets $A$ and $-B$, where $-B$ is the set symmetric to the set $B$ with respect to the reference point $O$. Other appearing definition is based on the operation of difference of the positioning vectors of all points from the given operands $A$ and $B$, which are point subsets of the Euclidean space. Another possible way how to introduce Minkowski difference is to define it as an inverse operation to the operation of Minkowski sum. Applications of Minkowski difference can be found in many areas of computer graphics, robotics, in determination of offsets and when finding trajectory of the movement of the material elaborating tools, in geometric modelling, in CAD, and others. In this paper, the algorithm is presented for calculation of Minkowski difference of two continuous manifolds that are subsets of the 3D Euclidean space, i.e. curve segments. Algorithm is based on vector representations of figures appearing as operands, while difference of the two given figures is a new figure defined by its vector equation. Interesting examples of surface patches created using this approach as modelling tool are compared to surface patches that are results of Minkowski sum and Minkowski product of the same two point sets. Using this powerful tool in geometric modelling, not only the geometric form of the figure can be defined, but its analytic representation can be received, too.

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## 2 Definition and properties of the Minkowski difference

Let $A$ and $B$ be two point sets, which are subsets of the $n$-dimensional Euclidean space $E^{n}$. Firstly, those definition of Minkowski difference of sets $A$ and $B$ is given, which is presented in [2] as a special case of Minkowski sum.

Definition 1. Minkowski difference of sets $A$ and $B$ in the Euclidean space is the set, which is the difference of all points in the set $A$ with all points in the set $B$, i.e. the set

$$
\begin{equation*}
A \ominus B=A \oplus-B=\{a-b ; a \in A, b \in B\} \tag{1}
\end{equation*}
$$

where $a-b$ is the difference of the positioning vectors of points $a, b$.
Set $A \ominus B$ is defined as Minkowski sum of set $A$ and set $-B$, which is symmetric to the set $B$ with respect to the origin of the coordinate system in the reference point, therefore it is the sum of all points in the set $A$ with all points in the set $-B$. Operation defined in this way is not the inverse operation to the operation of the Minkowski sum, anyhow it represents one of the important classification and modelling tools in computer graphics.

Several basic properties of the Minkowski difference of sets $A$ and $B$ in the Euclidean space are included, which are frequently used for its calculation.

1. If sets $A$ and $B$ are convex sets, then their Minkowski difference $A \ominus B$ is also a convex set.
2. Sets $A$ and $B$ are overlapping, if their Minkowski difference $A \ominus B$ contains the reference point, it means the origin of the coordinate system.
3. Minkowski difference is not a commutative operation, it holds: $A \ominus B \neq B \ominus A$.

Illustration of the validity of the third property can be seen in fig. 1, on examples of surface patches created as Minkowski difference of two parabolic segments $A$ and $B$ with axes parallel to the coordinate axis $z$.


Fig. 1. Minkowski difference $A \ominus B$ (left), and $B \ominus A$ (right).

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Distance $d$ of sets $A$ and $B$ can be defined by means of Minkowski difference.

$$
\begin{equation*}
d(A, B)=\min _{a \in A, b \in B}\|a-b\|=\min _{c \in A \ominus B}\|c\| \tag{2}
\end{equation*}
$$

Minkowski difference can be used for simplification of some calculation problems in more dimensional spaces by reducing dimension while transforming the problem to the space with lower dimension. Calculation of the distance of two line segments in the three-dimensional space can serve as an example. Minkowski difference of two line segments is a parallelogram; therefore the problem is reduced to the calculation of the distance of this parallelogram from the origin of the coordinate system. Projection of the coordinate system origin to the plane of the parallelogram leads to the solution, while the required distance is the distance of the projected point from the origin. Next included definition of Minkowski difference is presented in [4].

Definition 2. Minkowski difference of sets $A$ and $B$ in the Euclidean space is the set

$$
\begin{equation*}
A \ominus B=\bigcap_{b \in B} A^{b}, \tag{3}
\end{equation*}
$$

where $A^{b}$ is the set $A$ translated by vector $b$

$$
\begin{equation*}
A^{b}=\{a+b ; a \in A\} . \tag{4}
\end{equation*}
$$

Minkowski difference defined according to Definition 2. is used in computer graphics e.g. for the set layout and in placing one set into another one, but again, this binary operation is not the inverse operation to the operation of Minkowski sum.

The last presented definition of Minkowski difference of sets $A$ and $B$ is the definition as the inverse operation to Minkowski sum, available in [5].

Definition 3. Minkowski difference of sets $A$ and $B$ in the Euclidean space is the set

$$
\begin{equation*}
A \ominus B=\bigcap_{b \in B} A^{-b}, \tag{5}
\end{equation*}
$$

where $A^{-b}$ is the set $A$ translated by vector $-b$,

$$
\begin{equation*}
A^{-b}=\{a-b ; a \in A\} . \tag{6}
\end{equation*}
$$

Let $A$ and $B$ be point sets in the Euclidean space $\boldsymbol{E}^{n}$ and sets $A^{\prime}$ and $B^{\prime}$ be their complements. The following properties are valid for the Minkowski difference from the last Definition 3.

1. Minkowski difference of convex sets is again a convex set.
2. Minkowski difference of convex sets A and B can be defined by formula

$$
\begin{equation*}
A \ominus B=\left(A^{\prime} \oplus(-B)^{\prime}\right. \tag{7}
\end{equation*}
$$

3. Minkowski sum of convex sets A and B can be defined through Minkowski difference as

$$
\begin{equation*}
A \oplus B=\left(A^{\prime} \ominus(-B)\right)^{\prime} . \tag{8}
\end{equation*}
$$

4. The following relations are valid for convex sets A and B

$$
\begin{align*}
& (A \oplus B) \ominus B=A,  \tag{9}\\
& (\mathrm{~A} \ominus B) \oplus B \subset A . \tag{10}
\end{align*}
$$

5. Let the translation of the set $A$ be determined by the translation vector $\mathbf{t}_{1}$ and translation of the set $B$ by the translation vector $\mathbf{t}_{2}$. The following equation holds for Minkowski difference of the translated sets $A$ and $B$

$$
\begin{equation*}
\operatorname{Trans}\left(A, \mathbf{t}_{1}\right) \ominus \operatorname{Trans}\left(B, \mathbf{t}_{2}\right)=\operatorname{Trans}\left(A \ominus B, \mathbf{t}_{1}-\mathbf{t}_{2}\right) \tag{11}
\end{equation*}
$$

The last property implies the following consideration and preposition.

Theorem 1. Sets $\operatorname{Trans}\left(A, \mathbf{t}_{1}\right)$ and $\operatorname{Trans}\left(B, \mathbf{t}_{2}\right)$ are overlapping if and only if the set, which is Minkowski difference $A \ominus B$ contains the point with the positioning vector $\mathbf{t}_{2}-\mathbf{t}_{1}$.

## 3 Modelling of surface patches in $\mathrm{E}^{3}$

Let two curve segments $k$ and $h$ be determined by vector functions parametrized on unit intervals in the space $\boldsymbol{E}^{3}$

$$
\begin{gather*}
\mathbf{r}(u)=(x r(u), y r(u), z r(u)), u \in\langle 0,1\rangle  \tag{12}\\
\mathbf{s}(v)=(x s(v), y s(v), z s(v)), v \in\langle 0,1\rangle \tag{13}
\end{gather*}
$$

Minkowski difference of curve segments $k$ and $h$ according to the Definition 1. is a patch of translational surface $\chi$, which is created by translation of the curve $\mathbf{r}(u)$ along the curve $\mathbf{r}(v)$

$$
\begin{equation*}
k \ominus h=\chi \tag{14}
\end{equation*}
$$

and it is determined by the vector function defined on the unit square in the form

$$
\begin{equation*}
\mathbf{p}(u, v)=\mathbf{r}(u)-\mathbf{s}(v)=(x r(u)-x s(v), y r(u)-y s(v), z r(u)-z s(v)),(u, v) \in\langle 0,1\rangle^{2} . \tag{15}
\end{equation*}
$$

Surfaces in fig. 2 are examples of Minkowski difference of two ellipses compared to their Minkowski sum and product, located in the same plane

$$
\begin{equation*}
\mathbf{p}(u, v)=\left(a_{1} \cos 2 \pi u-a_{2} \cos 2 \pi v, b_{1} \sin 2 \pi u-b_{2} \sin 2 \pi v, 0\right),(u, v) \in\langle 0,1\rangle^{2} \tag{16}
\end{equation*}
$$

and in two perpendicular planes

$$
\begin{equation*}
\mathbf{p}(u, v)=\left(a_{1} \cos 2 \pi u, b_{1} \sin 2 \pi u-a_{2} \cos 2 \pi v,-b_{2} \sin 2 \pi v\right),(u, v) \in\langle 0,1\rangle^{2} \tag{17}
\end{equation*}
$$



Fig. 2. Minkowski difference - sum (top left) and product (top right) of 2 ellipses in one plane, and in 2 perpendicular planes (bottom from left).

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Surface patch (part of a plane) depicted in fig. 3 on the left is Minkowski difference of sinusoidal curve and circle, and it is determined by the vector equation

$$
\begin{equation*}
\mathbf{p}(u, v)=(2 \pi u+r \cos 2 \pi v, \sin 2 \pi u+r \sin 2 \pi v, 0),(u, v) \in\langle 0,1\rangle^{2}, \tag{18}
\end{equation*}
$$

while surface presented on the right in fig. 3 is illustration of Minkowski difference of these curve segments, with equation

$$
\begin{equation*}
\mathbf{p}(u, v)=(2 \pi u-r \cos 2 \pi v, \sin 2 \pi u-r \sin 2 \pi v, 0),(u, v) \in\langle 0,1\rangle^{2} . \tag{19}
\end{equation*}
$$

Surfaces differ only in their position in the plane $z=0$, but not in the form, which demonstrates the fact that Minkowski difference according the Definition 1. is a special case of Minkowski sum of these two sets. Anyhow, this property is not valid generally.



Fig. 3. Minkowski sum (left) and difference (right) of sinusoidal curve and circle in one plane.
Minkowski difference of sinusoidal curve and circle that are located in perpendicular planes is a surface patch determined by vector function

$$
\begin{equation*}
\mathbf{p}(u, v)=(2 \pi u-r \cos 2 \pi v,-r \sin 2 \pi v, k \sin 2 \pi u),(u, v) \in\langle 0,1\rangle^{2}, \tag{20}
\end{equation*}
$$

which is illustrated in fig. 4 , on the right (that is equal to the sum of these curves) together with the product of these curves on the left

$$
\begin{equation*}
\mathbf{p}(u, v)=(-k r \sin 2 \pi u \sin 2 \pi v,-k r \sin 2 \pi u \cos 2 \pi v, 2 r \pi u \sin 2 \pi v),(u, v) \in\langle 0,1\rangle^{2} . \tag{21}
\end{equation*}
$$



Fig. 4. Minkowski product and difference (sum) of sinusoid and circle in perpendicular planes.

Minkowski sum of a parabolic arc and a semicircle is presented in fig. 5 on the left, whose vector equation is in the form

$$
\begin{equation*}
\mathbf{p}(u, v)=\left(a u, r \cos 2 \pi v, b\left(u^{2}-u\right)+r \sin 2 \pi v\right),(u, v) \in\langle 0,1\rangle^{2}, \tag{22}
\end{equation*}
$$

while on the right in fig. 5 Minkowski difference of these curve segments is viewed, with vector equation


Fig. 5. Minkowski sum (left) and difference (right) of a parabolic arc and a semicircle.
Illustration of Minkowski product of these two curve segments, a parabolic arc and a semicircle located in perpendicular planes is in fig. 6 on left, determined parametrically by equation

$$
\begin{equation*}
\mathbf{p}(u, v)=\left(-b\left(u^{2}-u\right) \cos 2 \pi v, \text { aru } \sin 2 \pi v, \operatorname{ar} u \cos 2 \pi v\right),(u, v) \in\langle 0,1\rangle^{2} . \tag{24}
\end{equation*}
$$

On the left in fig. 6 surface created as Minkowski product of those 2 parabolic arcs is presented, whose Minkowski difference is illustrated in fig. 1, whereas its vector equation has the form

$$
\begin{equation*}
\mathbf{p}(u, v)=\left(b\left(u^{2}-u\right) \cos 2 \pi v,-a r u \sin 2 \pi v,-a r u \cos 2 \pi v\right),(u, v) \in\langle 0,1\rangle^{2} \tag{25}
\end{equation*}
$$



Fig. 6. Minkowski product of parabolic arc and semicircle (left), and of two parabolic arcs (right).

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Minkowski sum and Minkowski difference of two semicircles in perpendicular planes are presented in fig. 7, vector parametric representations are

$$
\begin{equation*}
\mathbf{p}(u, v)=\left(r_{1} \cos 2 \pi u, r_{1} \sin 2 \pi u \pm r_{2} \cos 2 \pi v, \pm r_{2} \sin 2 \pi v\right),(u, v) \in\langle 0,1\rangle^{2} \tag{26}
\end{equation*}
$$



Fig. 7. Minkowski sum (left) and difference (right) of 2 semicircles in perpendicular planes.
Minkowski sum, which equals to Minkowski difference of ellipse and sinusoidal curve that are located in two perpendicular planes and their Minkowski product are presented in fig. 8, while their vector representations are

$$
\begin{align*}
& \mathbf{p}(u, v)=(a \cos 2 \pi u, \pm 2 \pi v, b \sin 2 \pi u \pm \sin 2 \pi v),(u, v) \in\langle 0,1\rangle^{2}  \tag{27}\\
& \mathbf{p}(u, v)=(2 \pi b v \sin 2 \pi u, a \cos 2 \pi u \sin 2 \pi v, 2 \pi a v \cos 2 \pi u),(u, v) \in\langle 0,1\rangle^{2} \tag{28}
\end{align*}
$$



Fig. 8. Minkowski sum - difference (left) and product (right) of ellipse and sinusoidal curve located in perpendicular planes.

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# MODELLING OF ORTHOGONAL PROJECTIONS 

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#### Abstract

Paper brings mathematical theory for visualization of more dimensional objects by means of their parallel, orthogonal projections to the plane. Different formulas for various projection types are presented, with forms for image point coordinate calculations in particular cases. Examples of four dimensional hyper-polyhedra views in plane are introduced and explained.


Key words. orthogonal projection, orthographic view, axonometric view, 4D space, hypercube, multidimensional visualisation

Mathematics Subject Classification: Primary 51N05, 51N20; Secondary 15A04.

## 1 Introduction

This paper on different projection modelling is designed for geometry teachers working with graphical computer programs that allow modelling only of planar problems and constructions. In the sequence of 4D $\rightarrow 3 \mathrm{D} \rightarrow 2 \mathrm{D}$ dual projections, planar images of the 4-dimensional objects loose a lot of the relationships available in 4D modelling, which are well-known and prepared theoretically, but these relevant abstract concepts are not applicable for object views in lower dimensions. The situation can be compared to the case when we investigate relationships in 3D on the line after designing the sequence of its $3 \mathrm{D} \rightarrow 2 \mathrm{D} \rightarrow 1 \mathrm{D}$ views, in which almost all relevant spatial information can be lost as the line might finally appear in the 1D view simply as a single point, where all relations and properties disappear.
In order to enable modelling and visualisation of different views of more dimensional objects, so that their created mapped views might be changed, the idea of determining the particular position of the projection plane depending on parameters has arisen. Change of parameters defines a new position of the projection plane, which immediately allows a new view of the visualised object, in order to receive as much information as possible for successful reconstruction of the original object and better understanding of its spatial properties, dimensions and relations.

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## 2 Orthogonal projection from 3D to 2D

Let us choose in $\mathbf{E}^{3}$ with the orthonormal basis ( $\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}$ ) a unit vector determined by Cartesian coordinates $\mathbf{n}=(\sin \varphi \cos \psi, \sin \varphi \sin \psi, \cos \varphi)$, where $\varphi$ is the latitude and $\psi$ is the longitude, for which $0<\varphi<\pi, 0 \leq \psi<2 \pi$. Angular parameters $\varphi$ and $\psi$ are therefore spherical coordinates on the unit sphere determining the position of the vector $\mathbf{n}$ end point.
As it is generally irrelevant for a parallel projection whether the projection plane has been moved in the direction of the projection, we can choose its particular position. We define the projection plane oriented in the way that it passes through the origin of coordinates $O$ and its normal is determined by the vector $\boldsymbol{n}$. The point $A^{*}\left(x^{*}, y^{*}, z^{*}\right)$ is the orthographic projection of the point $A(x, y, z)$, and the following relations are true

$$
\begin{equation*}
\mathrm{A}^{*}=\mathrm{A}-\lambda \mathbf{n}, \mathrm{A}^{*} \cdot \mathbf{n}=(\mathrm{A}-\lambda \mathbf{n}) \cdot \mathbf{n}=0 \text {, then } \lambda=\mathrm{A} \cdot \mathbf{n} . \tag{1}
\end{equation*}
$$

The rectangular projection $\boldsymbol{e}_{3}{ }^{*}$ of the vector $\boldsymbol{e}_{3}$ satisfies the equation

$$
\begin{equation*}
\left(\boldsymbol{e}_{3}-\left(\boldsymbol{e}_{3} \cdot \boldsymbol{n}\right) \boldsymbol{n}\right) \cdot\left(\boldsymbol{e}_{3}-\left(\boldsymbol{e}_{3} \cdot \boldsymbol{n}\right) \boldsymbol{n}\right)=1-\cos ^{2} \varphi=\sin ^{2} \varphi \tag{2}
\end{equation*}
$$

let us choose a new orthonormal basis $\left(f_{1}, f_{2}, f_{3}\right)$ in the following way

$$
\begin{align*}
& \boldsymbol{f}_{3}=\boldsymbol{n} \\
& \boldsymbol{f}_{2}= \pm \boldsymbol{e}_{3}{ }^{*} / \sin \varphi=( \pm \cos \varphi \cos \psi, \pm \cos \varphi \sin \psi, \pm \sin \varphi)  \tag{3}\\
& \boldsymbol{f}_{1}=\boldsymbol{f}_{2} \times \boldsymbol{f}_{3}=( \pm \sin \psi, \pm \cos \psi, 0)
\end{align*}
$$

The transition from the base $\left(\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}\right)$ to the base $\left(\boldsymbol{f}_{1}, \boldsymbol{f}_{2}, \boldsymbol{f}_{3}\right)$ can be determined by two orthonormal square matrices

$$
\begin{align*}
& \mathbf{T}_{1}=\left(\begin{array}{ccc}
-\sin \psi & \cos \psi & 0 \\
-\cos \varphi \cos \psi & -\cos \varphi \sin \psi & \sin \varphi \\
\sin \varphi \cos \psi & \sin \varphi \sin \psi & \cos \varphi
\end{array}\right)  \tag{4}\\
& \mathbf{T}_{2}=\left(\begin{array}{ccc}
\sin \psi & -\cos \psi & 0 \\
\cos \varphi \cos \psi & \cos \varphi \sin \psi & -\sin \varphi \\
\sin \varphi \cos \psi & \sin \varphi \sin \psi & \cos \varphi
\end{array}\right) . \tag{5}
\end{align*}
$$

Let the upper index ${ }^{\mathrm{t}}$ denote the matrix transpose. For $i=1,2$ we can receive new coordinates of the object in the basis $\left(\boldsymbol{f}_{1}, \boldsymbol{f}_{2}, \boldsymbol{f}_{3}\right)$ by simple matrix multiplication

$$
\begin{equation*}
\left(x^{*}, y^{*}, z^{*}\right)^{\mathrm{t}}=\mathbf{T}_{i} \cdot(x, y, z)^{\mathrm{t}} . \tag{6}
\end{equation*}
$$

In matrices $\mathbf{T}_{i}$ the value of spherical coordinates may be $\varphi=0$ or $\varphi=\pi$. With the two angles $\varphi, \psi$ we can achieve a general position of the projection plane. All rectangular projections $3 \mathrm{D} \rightarrow 2 \mathrm{D}$ used

in throughout this article for production of illustrative views were performed with the values $\varphi=\pi / 3$ and $\psi=\pi / 5$. The first picture presents the orthogonal projection, axonometric view of the unit cube.

Fig. 1. Orthographic (axonometric) 3D view of a unit cube.
In the old base there is sufficient to choose two rotations of an object around two arbitrary coordinate axes and we can get the same relative position of the projection plane and the model. Let us select out of the six possible options a pair of rotations about coordinate axes $z$ and $x$. The first rotation is around the axis $z$ by an angle $\pi / 2 \pm \psi$. While the complex number expression is $(x+\mathbf{i} y) \mathbf{i} e^{ \pm i \psi}$, the matrix expression will be in the form

$$
\left(\begin{array}{ccc}
\mp \sin \psi & \pm \cos \psi & 0  \tag{7}\\
\mp \cos \psi & \mp \sin \psi & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

The second rotation is around the new axis $x$ by angle $\pm \varphi$. In complex numbers it is $(y+\mathbf{i z}) e^{\mp i \varphi \psi}$, while in the matrix form it appears as

$$
\left(\begin{array}{ccc}
1 & 0 & 0  \tag{9}\\
0 & \cos \varphi & \pm \sin \varphi \\
0 & \mp \sin \varphi & \cos \varphi
\end{array}\right) .
$$

In the original basis we get new coordinates of the object in the following calculation

$$
\begin{align*}
\left(\begin{array}{l}
x^{*} \\
y^{*} \\
z^{*}
\end{array}\right) & =\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \varphi & \pm \sin \varphi \\
0 & \mp \sin \varphi & \cos \varphi
\end{array}\right)\left(\begin{array}{ccc}
\mp \sin \psi & \pm \cos \psi & 0 \\
\mp \cos \psi & \mp \sin \psi & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)=  \tag{10}\\
& =\left(\begin{array}{ccc}
\mp \sin \psi & \pm \cos \psi & 0 \\
\mp \cos \varphi \cos \psi & \mp \cos \varphi \sin \psi & \pm \sin \varphi \\
\sin \varphi \cos \psi & \sin \varphi \sin \psi & \cos \varphi
\end{array}\right)\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)
\end{align*}
$$

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For $i=1,2$ we have again

$$
\begin{equation*}
\left(x^{*}, y^{*}, z^{*}\right)^{\mathrm{t}}=\mathbf{T}_{i} \cdot(x, y, z)^{\mathrm{t}} . \tag{11}
\end{equation*}
$$

In the rectangular projection to the image plane $(x, y)$ we restrict our considerations to two first new coordinates only, while the third new coordinate $z^{*}$ can serve for determination of the object visibility in the view. For $i=1,2$ matrices of projections $\mathbf{T}_{i}$ are orthonormal matrices, as for their product the equality $\mathbf{T}_{i}{ }^{-1}=\mathbf{T}_{i}{ }^{t}$ holds. We can see that also the following holds

$$
\begin{equation*}
\left(n_{1}, n_{2}, n_{3}\right)^{\mathrm{t}}=\mathbf{T}_{i}^{\mathrm{t}} \cdot(0,0,1)^{\mathrm{t}} . \tag{11}
\end{equation*}
$$

## 3 Orthogonal projection from 4D to 3D

In the Cartesian coordinates $(x, y, z, w)$ with the orthonormal basis $\left\{\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}, \boldsymbol{e}_{4}\right\}$ in 4D Euclidean space we can represent the unit vector $\boldsymbol{n},|\boldsymbol{n}|=1$, in spherical coordinates, $0 \leq \psi<2 \pi, 0 \leq \varphi \leq \pi$

$$
\begin{equation*}
n_{1}=\sin \chi \sin \varphi \cos \psi, n_{2}=\sin \chi \sin \varphi \sin \psi, n_{3}=\sin \chi \cos \psi, n_{4}=\cos \chi, \tag{12}
\end{equation*}
$$

while from this quadruple ( $n_{1}, n_{2}, n_{3}, n_{4}$ ) all three angles $\chi, \varphi, \psi$ can be uniquely determined.
Considering orthogonal projection of the space $\mathbf{E}^{4}$ to the image hyperplane passing through the origin $O$ of the coordinate system and determined by the normal vector $\boldsymbol{n}$, then for the orthographic view $A^{*}$ of the point $A$ we receive

$$
\begin{equation*}
A \rightarrow A^{*}: A^{*}=A-(A \cdot \boldsymbol{n}) \boldsymbol{n} \tag{13}
\end{equation*}
$$

The following relations hold for $i, k=1,2,3$

$$
\begin{align*}
& \boldsymbol{e}_{\mathrm{i}}^{*} \cdot \boldsymbol{e}_{\mathrm{i}}^{*}=1-n_{\mathrm{i}}^{2}, i=k  \tag{14}\\
& \boldsymbol{e}_{\mathrm{i}}^{*} \cdot \boldsymbol{e}_{\mathrm{k}}{ }^{*}=-n_{\mathrm{i}} n_{k}, i \neq k . \tag{15}
\end{align*}
$$

We introduce a new orthonormal basis $\left\{\boldsymbol{f}_{1}, \boldsymbol{f}_{2}, \boldsymbol{f}_{3}, \boldsymbol{n}\right\}$, and then we define

$$
\begin{gather*}
\rho \frac{ \pm 1}{\sqrt{1-n_{3}^{2}}}  \tag{16}\\
\boldsymbol{f}_{3}=\rho \boldsymbol{e}_{3}^{*}=\rho\left(\boldsymbol{e}_{3}-n_{3} \boldsymbol{n}\right), \text { where }-1<n_{3}<1 . \tag{17}
\end{gather*}
$$

It holds: $\left|\boldsymbol{f}_{3}\right|=1$.
Next, let us choose

$$
\begin{equation*}
\rho=\frac{1}{\sqrt{1-n_{3}^{2}}} \tag{18}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left(f_{1}, f_{2}, f_{3}, f_{4}\right)^{t}=\rho\left(-n_{1} n_{3},-n_{2} n_{3}, 1-n_{3}^{2},-n_{3} n_{4}\right)^{t} \tag{19}
\end{equation*}
$$

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Next vector $\boldsymbol{f}_{2}$ will be the direction vector of the intersection line of the plane $\left(\boldsymbol{e}_{\mathrm{i}}^{*}, \boldsymbol{e}_{2}{ }^{*}\right)$ and the plane perpendicular to $\boldsymbol{f}_{3}$, which is also passing through the origin $O$ of the coordinate system.

$$
\begin{equation*}
f_{2}=\mu_{1} e_{1}^{*}+\mu_{2} e_{2}^{*} \tag{20}
\end{equation*}
$$

with conditions: $\boldsymbol{f}_{2} \cdot \boldsymbol{f}_{3}=0,\left|\boldsymbol{f}_{2}\right|=1$.
The first condition implies for $n_{3} \neq 0$

$$
\begin{equation*}
n_{1} \mu_{1}+n_{2} \mu_{2}=0 . \tag{21}
\end{equation*}
$$

Let us choose

$$
\begin{equation*}
\mu_{1}=-\sigma n_{2}, \mu_{2}=\sigma n_{1} . \tag{22}
\end{equation*}
$$

The proportionality constant $\sigma$ can be determined from the condition

$$
\begin{equation*}
1=\boldsymbol{f}_{2} \cdot \boldsymbol{f}_{2}=\cdots=\sigma^{2}\left(n_{1}^{2}+n_{2}^{2}\right) . \tag{23}
\end{equation*}
$$

For $n_{1}{ }^{2}+n_{2}{ }^{2}>0$ we get

$$
\begin{equation*}
\sigma=\frac{ \pm 1}{\sqrt{n_{1}^{2}+n_{2}^{2}}} . \tag{24}
\end{equation*}
$$

The selection of the plus /minus sign shall determinate the orientation of the intersection. Let us choose

$$
\begin{equation*}
\sigma=\frac{1}{\sqrt{n_{1}^{2}+n_{2}^{2}}}, \tag{25}
\end{equation*}
$$

then

$$
\begin{equation*}
f_{2}=\sigma\left(n_{1} e_{2}^{*}-n_{2} e_{1}^{*}\right)=\sigma\left(n_{1} e_{2}-n_{2} e_{1}\right) . \tag{26}
\end{equation*}
$$

Vector $\boldsymbol{f}_{1}$ will be determined as the orthogonal complement

$$
f_{1}=\left|\begin{array}{cccc}
e_{1} & e_{2} & e_{3} & e_{4}  \tag{27}\\
-\sigma n_{2} & \sigma n_{1} & 0 & 0 \\
-\rho n_{1} n_{3} & -\rho n_{2} n_{3} & \rho-\rho n_{3}^{2} & -\rho n_{3} n_{4} \\
n_{1} & n_{2} & n_{3} & n_{4}
\end{array}\right|=\left|\begin{array}{cccc}
e_{1} & e_{2} & e_{3} & e_{4} \\
-\sigma n_{2} & \sigma n_{1} & 0 & 0 \\
0 & 0 & \rho & 0 \\
n_{1} & n_{2} & n_{3} & n_{4}
\end{array}\right| .
$$

This implies

$$
\begin{equation*}
f_{13}=0,\left|f_{1}\right|=1, \tag{28}
\end{equation*}
$$

or in other words, the volume of a cube with edges $\boldsymbol{f}_{2}, \boldsymbol{f}_{\mathfrak{3}}, \boldsymbol{n}$ equals one.
After transferring the coordinates to the new base $\left\{\boldsymbol{f}_{1}, \boldsymbol{f}_{2}, \boldsymbol{f}_{3}, \boldsymbol{n}\right\}$, the fourth coordinate of the orthographic views will be equal to zero, and the projection can be determined in the following matrix form

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$$
\left(\begin{array}{l}
x^{*}  \tag{29}\\
y^{*} \\
z^{*} \\
w^{*}
\end{array}\right)=\left(\begin{array}{cccc}
f_{11} & f_{12} & 0 & f_{14} \\
f_{21} & f_{22} & 0 & 0 \\
f_{31} & f_{32} & f_{33} & f_{34} \\
n_{1} & n_{2} & n_{3} & n_{4}
\end{array}\right)\left(\begin{array}{c}
x \\
y \\
z \\
w
\end{array}\right)
$$

The last coordinate $w^{*}=n_{1} x+n_{2} y+n_{3} z+n_{4} w$ can serve for determination of visibility.

## Example

Let us consider a unit tesseract (hypercube) in 4D, the coordinates of its $2^{4}$ vertices are either zeros or ones. The centre $S$ of the tesseract has coordinates ( $1 / 2,1 / 2,1 / 2,1 / 2$ ). Let us choose the unit normal vector $\mathbf{n}=\overrightarrow{O S}$. Then

$$
\begin{equation*}
\rho=\frac{2 \sqrt{3}}{3}, \sigma=\sqrt{2}, \rho \sigma=\frac{2 \sqrt{6}}{3} . \tag{30}
\end{equation*}
$$

The orthographic projection is

$$
\left(\begin{array}{l}
x^{*}  \tag{31}\\
y^{*} \\
z^{*} \\
w^{*}
\end{array}\right)=\left(\begin{array}{cccc}
\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{6}}{3} \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\
-\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{6} \\
n_{1} & n_{2} & n_{3} & n_{4}
\end{array}\right)\left(\begin{array}{l}
x \\
y \\
z \\
w
\end{array}\right) .
$$

The projections of the opposite vertices $(0,0,0,0),(1,1,1,1)$ coincide with the point $S^{*}(0,0,0)$, fig. 2 . The unit vectors in the coordinate axes will result in a tetrahedral pyramid with length $\frac{\sqrt{3}}{2}$, and angles $\alpha=\arccos (-1 / 3)$, they are actually the line segments joining the center $S^{*}$ of a regular tetrahedron with his legs. All squares of the tesseract are therefore projected as rhombi with sides of the same length $\frac{\sqrt{3}}{2}$ and with diagonals of size 1 and $\sqrt{2}$, while each rhombus - diamond has a sharp internal angle $\pi-\alpha$.


Fig. 2. Orthographic views of the unit hypercube.
Orthographic view of the unit tesseract is in our case a rhombic dodecahedron; in its centre of symmetry $S$ the projections of two opposite vertices of the tesseract coincide. In fig. 2 , on the left, there is shown the orthographic view of rhombic dodecahedron with the base in the plane $(x, y)$.
In fig. 2, on right, there is also presented the orthographic view of the tesseract, but $\overrightarrow{O S}$ is not the normal vector of the 3D projection plane. We often encounter featuring of rhombic dodecahedron in Cartesian coordinates. Eight of its vertices have coordinates $( \pm 1, \pm 1, \pm 1)$, and the remaining six vertices have coordinates $( \pm 2,0,0),(0, \pm 2,0),(0,0, \pm 2)$.

For rotation of an object in 4D space with coordinate axes $(x, y, z, w)$, we can choose three revolutionary movements:
revolution about the plane $(x, y)$ by the angle $\gamma$ is represented by matrix

$$
\mathrm{G}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{32}\\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \gamma & -\sin \gamma \\
0 & 0 & \sin \gamma & \cos \gamma
\end{array}\right)
$$

revolution about the plane $(x, w)$ by the angle $\delta$ represented by matrix

$$
\mathrm{D}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{33}\\
0 & \cos \delta & -\sin \delta & 0 \\
0 & \sin \delta & \cos \delta & 0 \\
0 & 0 & 0 & 1
\end{array}\right),
$$

revolution about the plane $(z, w)$ by the angle $\varepsilon$ represented by matrix

$$
\mathrm{E}=\left(\begin{array}{cccc}
\cos \varepsilon & -\sin \varepsilon & 0 & 0  \tag{34}\\
\sin \varepsilon & \cos \varepsilon & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

Matrix of orthographic projection can be achieved as the product of these orthonormal matrices, $\mathrm{G} \cdot \mathrm{D} \cdot \mathrm{E}$, while this transformation (selected from the $\left(\frac{4}{2}\right) \cdot 3!=36$ possibilities) will introduce the angles $\gamma, \delta, \varepsilon$. Then

$$
\begin{equation*}
\left(x^{*}, y^{*}, z^{*}, w^{*}\right)^{\mathrm{t}}=\mathbf{G} \cdot \mathbf{D} \cdot \mathbf{E} \cdot(x, y, z, w)^{\mathrm{t}} . \tag{35}
\end{equation*}
$$

Omitting the new fourth coordinate $w^{*}$, we receive the coordinates $\left(x^{*}, y^{*}, z^{*}\right)$ of the orthographic view in the hyperplane. We determine the unit normal vector in coordinates $(x, y, z, w)$ perpendicular to the new 3D projection plane. Since

$$
\begin{equation*}
\left(n_{1}, n_{2}, n_{3}, n_{4}\right)^{\mathrm{t}}=\mathbf{E}^{\mathrm{t}} \cdot \mathbf{D}^{\mathrm{t}} \cdot \mathbf{G}^{\mathrm{t}} \cdot(0,0,0,1)^{\mathrm{t}}, \tag{36}
\end{equation*}
$$

the transformation is actually inverse. Like in 3D, we should take into account the other 3 alternatives corresponding to the selection of the sign for $\rho$ and $\sigma$. From here we can express the angles $\chi, \varphi, \psi$ using angles $\gamma, \delta, \varepsilon$, for example: $\chi=\gamma, \varphi=\delta, \psi=\pi / 2-\varepsilon$. We can also determine the angles $\gamma, \delta, \varepsilon$ of rotation of the object in the selected transformation using the normal unit vector $\mathbf{n}$. In fig. 3, on the left, there is presented the orthographic view of the unit tesseract in the projection determined by parameters $\gamma=\pi / 4, \delta=\pi / 6, \varepsilon=2 \pi / 5$.
On the right, there is the popular parallel projection of the tesseract shaped as a regular octagon. The dodecahedron parallelograms are not identical rhombuses because in the 4D $\rightarrow$ 3D projection none of two opposite points of tesseract coincide.


Fig. 3. Orthographic view of the unit hypercube.

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# STUDY OF ALGEBRAIC OPERATIONS USED IN CAGD THROUGH INCIDENCE VARIETIES 

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#### Abstract

Current geometric modelling (Computer Aided Geometric Design) uses several operations defined over algebraic varieties whose output are again algebraic varieties (e.g. offsetting, operation of convolution, bisectors). Nevertheless, an uncomfortable feature of these constructions is the fact that the output variety may be, in some sense, more complicated than the input varieties - e.g. its algebraic degree is often higher, it consists of more components and mainly it does not have to be rational although the both input objects are rational. From this reason, it is necessary to leave the rational world of CAGD and use the techniques of algebraic geometry which work over the field of complex numbers and thus give global and more general results. Especially, we discuss an unifying approach based on the so called incidence varieties.


Key words and phrases. Algebraic varieties, incidence variety, convolutions, offsets, bisectors, conchoids, genus.
Mathematics Subject Classification. Primary 51N35; Secondary 14H45.

## 1 Introduction

Geometric modeling, or Computer Aided Geometric Design (CAGD) is a branch of applied mathematics that studies methods and algorithms for the mathematical description of shapes, [17]. In recent years and decades, studying various algebraic operations of shapes used in CAGD has become an active and popular research area. For instance, we recall one of the fundamental operations, offsetting, which is nothing else than convolution of hypersurfaces with a circle/sphere. Many interesting problems related to this topic have arisen, e.g. analysis of (geometric and algebraic) properties, determining number and kind of components of resulting varieties, a construction of rational parameterizations (if they exist) and formulation of suitable corresponding symbolic algorithms, cf. [10, 3, 2, 12, 13, 11, 16, 15].


Figure 1: Left: One-sided offset (magenta) of a curve $\mathcal{V}$ (blue) - arrows demonstrate the normal vector field. Right: Convolution $\mathcal{V} \star \mathcal{W}$ (magenta) of curves $\mathcal{V}$ (red) and $\mathcal{W}$ (blue).

The main aim of this paper is to study a unifying approach to various operations based on the so called incidence varieties, firstly mentioned for offsets (in a modified form) in [3] and then thoroughly studied for convolutions in [16]. We start with a brief overview of selected operations. For the sake of simplicity we restrict ourselves only to operations over planar algebraic curves.

Convolutions and offsets. One of the fundamental operations studied in geometric modelling is offsetting or, in other words, a construction of equidistant curves, see e.g. [7]. If $\mathbf{v}(s)$ is a parameterization of a curve $\mathcal{V}$, then the (two-sided) $\delta$-offset $\mathcal{O}_{\delta}$ is defined as the curve parameterized by

$$
\begin{equation*}
\mathbf{v}_{\delta}(s):=\mathbf{v}(s) \pm \delta \mathbf{n}(s) \tag{1}
\end{equation*}
$$

where $\mathbf{n}(s)$ is the unit normal vector field associated to the corresponding parameterization, see Fig. 1 (left). The operation convolution (see [16]) can be understood as a generalization of offsetting defined as follows

$$
\begin{equation*}
\mathcal{V} \star \mathcal{W}:=\left\{\mathbf{x}+\mathbf{y} \mid \mathbf{x} \in \mathcal{V}, \mathbf{y} \in \mathcal{W} \text { and } T_{\mathbf{x}} \mathcal{V} \| T_{\mathbf{y}} \mathcal{W}\right\} \tag{2}
\end{equation*}
$$

see Fig. 1 (right). It is not difficult to see that the $\delta$-offset of a given curve is nothing but the convolution of this curve with the circle centered at the origin and with the radius equal to $\delta$.


Figure 2: Left: Bisector $\mathcal{B}(\mathcal{V}, \mathcal{W})$ (magenta) of curves $\mathcal{V}$ (red) and $\mathcal{W}$ (blue). Right: Conchoid of Nicomedes (magenta).

Bisectors. Bisector $\mathcal{B}(\mathcal{V}, \mathcal{W})$ of two curves $\mathcal{V}, \mathcal{W}$ (see [6] for more details) is defined as the locus of all points halving the distance between the input curves in the following sense: A point p is a point of $\mathcal{B}(\mathcal{V}, \mathcal{W})$ if there exist points $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{W}$ such that $\mathbf{p} \in N_{\mathbf{v}} \mathcal{V} \cap N_{\mathbf{w}} \mathcal{W}$ and $|\mathrm{pv}|=|\mathrm{pw}|$, cf. Fig. 2 (left).

Conchoidal transform. Finally, we recall an operation well-known rather from the classical geometry than from geometric modelling. Nevertheless, also this operation is becoming a part of modern applications. Let be given a point $\mathbf{p}$ and a curve $\mathcal{V}$ not passing through this point. Then for each $\mathbf{v} \in \mathcal{V}$ there exists a unique line $\mathbf{p v}$. The conchoid consists of intersections of theses lines with the unit circles centered at v, cf. Fig. 2 (right) where the so called conchoid of Nicomedes is shown. If the unit circle in the definition of conchoid is replaced by a general curve $\mathcal{W}$ (i.e., the phrase 'the unit circle centered at $\mathbf{v}$ ' is replaced by 'a curve $\mathcal{W}$ translated by the position vector of $\mathbf{v}^{\prime}$ ), we obtain a binary operation called conchoidal transform and denoted by $\mathcal{C}_{\mathbf{p}}(\mathcal{V}, \mathcal{W})$.

The above mentioned operations invoke several questions whose answers are necessary for right understanding of these operations and hence subsequently also for a correct formulation of efficient algorithms:
\& Can we find any common property of all introduced operations?
Yes. Input algebraic varieties guarantee that the output objects are also algebraic varieties.
$\diamond$ What descriptions of shapes (input objects for the studied operations) are used in CAGD? Bézier, $B$-Spline, NURBS objects $\Rightarrow$ rational parameterizations $\Rightarrow$ algebraic varieties.

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$\bigcirc$ Does each algebraic variety admit a rational parameterization?
Unfortunately not. This is a rather exceptional property.
© Does an arbitrary input rational variety guarantee that applying one of above mentioned operations we arrive again at a rational variety?
Unfortunately not. This is also very exceptional case.
Thus, we have to leave a rational area of CAGD and apply the techniques of algebraic geometry, working over the field of complex numbers and thus giving global results, see e.g. $[4,5,8]$. This is a world where incidence varieties appear.

## 2 Incidence varieties

Our main aim is to obtain results of a global character (number of components, their rationality, etc.) and thus we have to work over an algebraically closed field - in what follows we will consider the field of complex numbers $\mathbb{C}$. On the other hand, since every real curve can be viewed as a complex curve with the defining polynomial possessing only real coefficients, the obtained results can be specialized for real curves, too.

A closer look at the examples given in the previous section reveals that all the operations are based on one identical scheme. Let us restrict ourselves to binary operations only. An algebraic operation $\diamond$ is given if for any two two admissible curves $\mathcal{X}, \mathcal{Y}$. we have following objects:

1. The variety $\mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{X} \times \mathcal{Y}$, such that the natural projections $\pi_{\mathcal{X}}: \mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{X}$ and $\pi_{\mathcal{Y}}: \mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y}$ are finite and dominant.
2. A rational mapping $\xi_{\mathcal{X}, \mathcal{Y}}: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{C}^{2}$.

Then the variety $\mathcal{X} \diamond \mathcal{Y}$ is defined as the closure of the set $\xi_{\mathcal{X}, \mathcal{Y}}\left(\mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y})\right)$.
Example 2.1 If the studied operation is convolution (i.e., $\diamond=\star$ ), then under an admissible variety we mean a curve which is not a line. Then the variety $\mathcal{I}^{\star}(\mathcal{X}, \mathcal{Y})$ is the closure of the set

$$
\begin{equation*}
\left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{2} \times \mathbb{C}^{2} \mid T_{\mathbf{x}} \mathcal{X} \| T_{\mathbf{y}} \mathcal{Y}\right\} \tag{3}
\end{equation*}
$$

and the mapping $\xi_{\mathcal{X}, \mathcal{Y}}$ is nothing but the sum $\mathbf{x}+\mathbf{y}$.
Definition 2.2 For two curves $\mathcal{X}, \mathcal{Y}$ and an algebraic operation $\diamond$ we define an incidence variety as $\mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y})=\mathcal{X} \times \mathcal{Y}$ together with the mappings $\pi_{\mathcal{X}}, \pi_{\mathcal{Y}}$ a $\xi_{\mathcal{X}, \mathcal{Y}}$, see the following diagram

$$
\begin{gather*}
\mathcal{X} \stackrel{\pi_{\mathcal{X}}}{\leftrightarrows} \mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\pi_{\mathcal{Y}}} \mathcal{Y} \\
\downarrow_{\mathcal{X}, \mathcal{Y}}  \tag{4}\\
\mathcal{X} \diamond \mathcal{Y}
\end{gather*}
$$

The pair of points $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}$ are said to be corresponding if $(\mathbf{x}, \mathbf{y}) \in \mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y})$.

A very important procedure when dealing with the operation $\diamond$ is the so called "arrow reversion". For instance, if there exists the mapping $\xi^{-1}$ in the diagram (4), we immediately arrive at the mappings $\mathcal{X} \diamond \mathcal{Y} \rightarrow \mathcal{X}$ and $\mathcal{X} \diamond \mathcal{Y} \rightarrow \mathcal{Y}$. The usefulness of such a construction will be evident from what follows.

Let us start with giving a condition on rational mappings to have the rational inverse. If $\phi: \mathcal{A} \rightarrow \mathcal{B}$ is a rational dominant mapping between two curves then its degree ( $\operatorname{deg} \phi$ ) is defined as the cardinality of a generic fibre, i.e., $\operatorname{deg} \phi=\sharp\left\{\phi^{-1}(\mathbf{p})\right\}$, where $\mathbf{p} \in \mathcal{B}$ is a generic point. Obviously, if there exists the rational inverse $\phi^{-1}$ (we say that $\phi$ is birational) then it has to hold $\operatorname{deg} \phi=1$. Surprisingly this condition is sufficient, too, cf [9].

Proposition 2.3 The mapping $\phi$ is birational if and only if $\operatorname{deg} \phi=1$.
Let us emphasize that the irreducibility of varieties $\mathcal{X}$ and $\mathcal{Y}$ does not guarantee that $\mathcal{X} \diamond \mathcal{Y}$ is irreducible too. If we denote the number of irreducible component of $\mathcal{A}$ by $\sharp \operatorname{IC}(\mathcal{A})$ then it is easy to see that the following theorem holds
Theorem $2.4 \sharp \operatorname{IC}(\mathcal{X} \diamond \mathcal{Y}) \leq \min \left\{\operatorname{deg} \pi_{\mathcal{X}}, \operatorname{deg} \pi_{\mathcal{Y}}\right\}$.
Remark 2.5 The degree of the projections $\pi_{\mathcal{X}}$ often depends on the variety $\mathcal{Y}$ (and analogously for the second natural projection). For instance, for conchoidal transforms it holds $\operatorname{deg} \pi_{\mathcal{X}}=$ $\operatorname{deg} \mathcal{Y}$ (cf. [1]) and for convolutions it equals to the so called convolution degree (cf. [16]).
If the operation $\diamond$ fulfills the following properties

- $\mathrm{x} \sim_{\diamond} \mathrm{y}$ and $\mathrm{y} \sim_{\diamond} \mathrm{z} \Rightarrow \mathrm{x} \sim_{\diamond} \mathrm{z}$,
- $\mathrm{x} \sim_{\diamond} \mathrm{y} \Rightarrow \mathrm{x} \sim_{\diamond} \xi(\mathrm{x}, \mathrm{y})$,
then it is possible to prove a stronger version of Theorem 2.4.
Theorem 2.6 Under the previous assumptions it holds

$$
\begin{equation*}
\sharp I C(\mathcal{X} \diamond \mathcal{Y}) \leq \operatorname{gcd}\left\{\operatorname{deg} \pi_{\mathcal{X}}, \operatorname{deg} \pi_{\mathcal{Y}}\right\} . \tag{5}
\end{equation*}
$$

Now, we focus on different types of components of $\mathcal{X} \diamond \mathcal{Y}$. Based on their nature, they can be classified into one of the three following types:

Definition 2.7 Let $\mathcal{U} \subset \mathcal{X} \diamond \mathcal{Y}$ be an irreducible component. Then we distinguish the following possibilities

$$
\left.\operatorname{deg} \xi\right|_{\xi^{-1}(\mathcal{U})}=\left\{\begin{array} { l } 
{ 1 , } \\
{ k > 1 , } \\
{ \infty , }
\end{array} \quad \Rightarrow \mathcal { U } \text { is called } \left\{\begin{array}{l}
\text { simple } \\
k \text {-special } \\
\text { degenerated }
\end{array}\right.\right.
$$

For reasonable operations, e.g. for all defined in the previous section, it holds that the components are almost always simple. This is a pleasant property as for a simple component the mapping

$$
\begin{equation*}
\left.\xi\right|_{\xi^{-1}=(\mathcal{U})}: \mathcal{I}^{\diamond}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{U} \tag{6}
\end{equation*}
$$

is birational. Thus, there exists a rational mapping from the curve $\mathcal{U}$ onto the input curves $\mathcal{X}$ and $\mathcal{Y}$. Moreover, if the variety $\mathcal{X} \diamond \mathcal{Y}$ decomposes into "enough" components, we can formulate a stronger statement.

Theorem 2.8 If $\sharp \operatorname{IC}(\mathcal{X} \diamond \mathcal{Y})=\operatorname{deg} \pi_{\mathcal{X}}$, then each simple component is birationally equivalent to $\mathcal{X}$.

Therefore, $\mathcal{X}$ is rational whenever $\mathcal{U}$ is. Unfortunately $\mathcal{X} \diamond \mathcal{Y}$ tends to be rather irreducible e.g. it was proved that conchoidal transforms and convolutions are generically irreducible, cf. [1] and [14], respectively. Thus we have to manage only with the rational mappings $\mathcal{U} \rightarrow \mathcal{X}, \mathcal{Y}$ in most cases. Then the Riemann-Hurwitz formula (cf. [9]) plays a very useful role.

Theorem 2.9 (Riemann-Hurwitz) Let $\mathcal{X}, \mathcal{Y}$ be smooth curves and $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ be a finite separable morphism. Then

$$
\begin{equation*}
2 \mathrm{~g}(\mathcal{X})-2=\operatorname{deg} \phi \cdot(2 \mathrm{~g}(\mathcal{Y})-2)+\operatorname{deg} R \tag{7}
\end{equation*}
$$

where $\operatorname{g}(\mathcal{X})$ is the genus ${ }^{1}$ of $\mathcal{X}$ and $\operatorname{deg} R$ is the degree of a ramification divisor (for demonstration/definition see Fig. 3)


Figure 3: This figure illustrates the mapping $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ of degree 3. The degree of ramification divisor can be computed by the formula $\sum_{\mathbf{x}} \mathcal{X}\left(e_{\mathbf{x}}-1\right)$.

A straightforward application of the previous theorem onto the problem of algebraic operations can be shown in the following lemma

Lemma 2.10 If $\mathcal{U}$ is a simple component of $\mathcal{X} \diamond \mathcal{Y}$ then $\mathrm{g}(\mathcal{U}) \geq \max \{\mathrm{g}(\mathcal{X}), \mathrm{g}(\mathcal{Y})\}$.
The proof of the previous lemma consists of using the Riemann-Hurwitz formula on the mappings $\phi: \mathcal{U} \rightarrow \mathcal{X}$ and $\psi: \mathcal{U} \rightarrow \mathcal{Y}$. In particular, after some computations we obtain, for instance, for the mapping $\phi$ the following equality

$$
\begin{equation*}
\mathrm{g}(\mathcal{U})=\mathrm{g}(\mathcal{X})+2(\operatorname{deg} \phi-1)(\mathrm{g} \mathcal{X}-1)+\frac{1}{2} \operatorname{deg} R . \tag{8}
\end{equation*}
$$

[^15]Since $\operatorname{deg} R \geq 0$ and $\operatorname{deg} \phi \geq 1$, the sought result follows.
Moreover, the only unknown value in (8) is in the most cases only the degree of the ramification divisor. Therefore if we are able to evaluate this degree, then we immediately arrive at a genus formula. These computations have been recently proceeded independently for offsets, convolutions and conchoidal transforms in [2], [14] and [1], respectively. For the sake of brevity, we mention only the resulting formuale and refer the readers for detailed proofs to original papers.

Theorem 2.11 Under some assumptions on curves $\mathcal{X}$ and $\mathcal{Y}$ it holds

- Offset (Arrondo-Sendra-Sendra 1999)

$$
\mathrm{g}\left(\mathcal{O}_{d}(\mathcal{Y})\right)=4 \mathrm{~g}(\mathcal{Y})+2 \operatorname{deg}(\mathcal{Y})-3
$$

- Convolution (Vršek-LÁvička 2010)

$$
\mathrm{g}(\mathcal{X} \star \mathcal{Y})=4 \mathrm{~g}(\mathcal{Y})+2 \operatorname{deg}(\mathcal{Y})-3
$$

- Conchoidal transform (Albano-Rogero 2010)

$$
\operatorname{g}(\mathcal{C}(\mathcal{X}, \mathcal{Y}))=\operatorname{deg}(\mathcal{X}) \cdot g(\mathcal{Y})+\operatorname{deg}(\mathcal{Y}) \cdot \operatorname{g}(\mathcal{X})+(\operatorname{deg}(\mathcal{X})-1)(\operatorname{deg}(\mathcal{Y})-1)
$$

## 3 Conclusion

In this contribution, we have sketched an unifying approach to distinguished algebraic operations which play an important role in geometric modelling and related applications (CAD, CAM, CAE, etc.). The presented technique (based on the methods of algebraic geometry) can serve as a first step to thorough theoretical study of these operations. We have demonstrated on several examples the results invented in recent years. It seams that the presented novel approach using the so called incidence varieties can help to understand the operations of geometric modelling from a more general point of view.

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# STATE-SPACE ANALYSIS OF $2^{\text {ND }}$ - AND $4^{\text {TH }}$ ORDER RESONANT FILTER LC AND LCLC UNDER TRANSIENT CONDITION 

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#### Abstract

The problem how to obtain sinusoidal voltage of load side at non-harmonic periodical supplying from the converters is very important in technical practices. The paper shows that there can be used either LCLC resonant filter for frequency of fundamental harmonic component, or LC filter tuned for switching frequency. Both filters have to remove higher harmonic components from the supplying voltage to reach the harmonic distortion roughly $5 \%$. Using non-symmetrical control the output voltage of inverter comprises all harmonic components, both odd and even ones. The paper deals mainly with analysis and modelling of the $4^{\text {th }}$ order LCLC filter (of the first type) under non-symmetrical supply and with comparing to the other types of filtering. Simulation results as well as experimental verification confirm good quality of output quantities of the filter, voltage and current.


Keywords. State-space modelling, Fourier series, periodical non-harmonic function, total harmonic distortion, transient analysis

Mathematics Subject Classification: Primary 42A16, 42A20; Secondary 42A24

## 1 Basic connection of single-phase inverter with output resonant filter

The single-phase voltage inverter can be realised in principle as full-bridge or half-bridge connection [1] with DC sources, Fig. 1a. For alternative AC sources we have either single-phase AC-AC converter - type of cyclo-converter (if it is a natural commutation and $f_{1}>f_{2}$ ) or single-phase matrix converter (with a forced commutation and $f_{1}>f_{2}$ or $f_{1}<f_{2}$ ), Fig. 1b.


Fig. 1. Principle schematic connections of the single-phase half-bridge voltage inverter:
the single-phase DC-AC inverter supplied from DC sources (left)
the single-phase $A C-A C$ inverter supplied from $A C$ sources (right)
In case when harmonic sinusoidal voltage of load is demanded, it is possible to use resonant AC filter tuned to basic harmonic, or filter tuned to switching frequency on converter output, Fig. 2a,b.


Fig. 2. Principle schematic connections of the single-phase voltage inverter and the output filter: the output resonant filter with basic resonance frequency (left) the output resonant filter with switching resonance frequency (right)

## 2 Transient analysis and modeling of the $4^{\text {th }}$-order $L C L C$ filter under symmetrical control

Output voltage of the inverter features wide spectrum of higher harmonic components. Fullwidth waveform is depicted in Fig. 3a. Harmonic content (odd harmonics, THD $=43.5 \%$ ) is shown in Fig. 3b.



Fig. 3. Full-width output voltage of the inverter (left) and its harmonic content without filtering (right)

Using Fourier theory one can derive relation (1) for basic harmonic amplitude of output voltage of inverter

$$
\begin{equation*}
\frac{U_{1 \mathrm{M}}(\beta)}{U}=\frac{4}{\pi} \sin (\beta / 2) \tag{1}
\end{equation*}
$$

where: $U_{1 \mathrm{M}}(\beta) \quad$ is amplitude of fundamental harmonic depending on voltage pulse width $\beta$
$U$ maximum value of inverter input $D C$ voltage
$\beta$ voltage pulse width in the range of $0-180^{\circ} \mathrm{el}$. deg., whereby

$$
\beta=\pi-\alpha, \text { and }
$$

$\alpha$ control angle oriented from end of half-period to the end of positive voltage pulse.
Considering converter scheme in Fig. 1a and $L C L C$ filter in Fig. 2a, the state-space equations can be

$$
\begin{align*}
\frac{d i_{L 1}}{d t} & =\frac{1}{L_{1}} u(t)-\frac{r_{L 1}}{L_{1}} i_{L 1}-\frac{1}{L_{1}} u_{C 1}-\frac{1}{L_{1}} u_{C 2} \\
\frac{d i_{L 2}}{d t} & =\frac{-r_{L 2}}{L_{2}} i_{L 2}+\frac{1}{L_{2}} u_{C 2} \\
\frac{d u_{C 1}}{d t} & =\frac{1}{C_{1}} i_{L 1}  \tag{2}\\
\frac{d u_{C 2}}{d t} & =\frac{1}{C_{1}} i_{L 1}-\frac{1}{C_{2}} i_{L 2}-\frac{1}{C_{2} \cdot r_{C}} u_{C 2}-\frac{1}{C_{2}} i_{L} \\
\frac{d i_{L L}}{d t} & =\frac{1}{L_{L}} u_{C 2}-\frac{R}{L_{L}} i_{L}
\end{align*}
$$

After time discretization (using implicit Euler method) of the system equations it yields

$$
\left[\begin{array}{c}
i_{L 1}(i+1)  \tag{3}\\
i_{L 2}(i+1) \\
u_{C 1}(i+1) \\
u_{C 2}(i+1) \\
i_{L L}(i+1)
\end{array}\right]=[\mathbf{J}-h \mathbf{A}]^{-1}\left[\begin{array}{c}
i_{L 1}(i) \\
i_{L 2}(i) \\
u_{C 1}(i) \\
u_{C 2}(i) \\
i_{L L}(i)
\end{array}\right]+[\mathbf{J}-h \mathbf{A}]^{-1}\left[\begin{array}{c}
\frac{1}{L} \\
0 \\
0 \\
0 \\
0
\end{array}\right] u(t) h
$$

where

$$
\mathbf{A}=\left(\begin{array}{ccccc}
\frac{-r_{L}}{L} & 0 & \frac{-1}{L} & \frac{-1}{L} & 0  \tag{4}\\
0 & \frac{-r_{L}}{L} & 0 & \frac{1}{L} & 0 \\
\frac{1}{C} & 0 & 0 & 0 & 0 \\
\frac{1}{C} & \frac{-1}{C} & 0 & \frac{-1}{C \cdot r_{C}} & \frac{-1}{C} \\
0 & 0 & 0 & \frac{1}{L_{L}} & \frac{-R}{L_{L}}
\end{array}\right)
$$

$i_{L 1}, i_{L 2} \quad$ - currents of inductors $L_{1}$ and $L_{2}$ of filter respectively
$i_{L} \quad$ - current of the load $R, L_{L}$
$u_{C 1}, u_{C 2}$ - voltage of capacitors $C_{1}$ a $C_{2}$ of $L C L C$ filter respectively
J - unit matrix
A - system matrix
$h$ - step size
$u(t) \quad$ - output voltage of the inverter

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## 3 Transient analysis and modeling of the $4^{\text {th }}$-order LCLC filter under non-symmetrical control

The real waveform of the output voltage of inverter has a wide spectrum of harmonic components. Using non-symmetrical control the output voltage of inverter (Fig. 4a) comprises all harmonic components, both odd and even ones of Fourier series as shown on Fig. 4b. The distortion is considerably high, e.g. for $2 / 3$-non-symmetrical control it is $62.5 \%$.


Fig. 4. The output voltage of the 1-phase inverter with non-symmetrical control $165 / 180^{\circ}$ el (left) and its harmonic content without filtering (right)

Using Fourier theory one can derive relation (4) for basic harmonic amplitude of output voltage of inverter

$$
\begin{equation*}
\frac{U_{1 \mathrm{M}}(\beta)}{U}=\frac{2 \sqrt{2}}{\pi} \sqrt{1-\cos (\beta / 2)} \tag{5}
\end{equation*}
$$

Considering converter scheme in Fig. 1a and $L C L C$ filter in Fig. 2a under non-symmetrical control then the state-space equations are the same as eqv. (2), (3) with symmetrical output voltage of inverter $u(t)$.

## 4 Transient analysis and modeling of the $2^{\text {nd }}$-order LC filter under bipolar PWM control

The harmonic spectrum of output voltage of inverter, which is depicted in Fig. 5a is shown in Fig. 5b.



Fig. 5. Output voltage of the 1-phase inverter under bipolar PWM control ((left) and its harmonic content (right)

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The harmonics in the inverter output voltage waveform appear as a sidebands, centered around the switching frequency and its multiples. It follows, that output voltage does not have higher harmonic components around the fundamental frequency. Now is not necessary use the output resonant filter tuned to fundamental frequency, but it can be used output resonant filter tuned to switching frequency as depicted in Fig. 2b.

Considering converter scheme in Fig. 1a and $L C$ filter in Fig. 2b then the state-space equations can be written

$$
\begin{align*}
\frac{d i_{L}}{d t} & =\frac{1}{L} u(t)-\frac{r_{L}}{L} i_{L}-\frac{1}{L} u_{C} \\
\frac{d u_{C}}{d t} & =\frac{1}{C} i_{L}-\frac{1}{C \cdot r_{C}} u_{C}-\frac{1}{C} i_{Z}  \tag{6}\\
\frac{d i_{L L}}{d t} & =\frac{1}{L_{L}} u_{C}-\frac{R}{L_{L}} i_{L L}
\end{align*}
$$

After discretization of system equations it yields

$$
\left[\begin{array}{c}
i_{L}(i+1)  \tag{7}\\
u_{C}(i+1) \\
i_{Z}(i+1)
\end{array}\right]=[\mathbf{J}-h \mathbf{A}]^{-1} *\left[\begin{array}{c}
i_{L}(i) \\
u_{C}(i) \\
i_{Z}(i)
\end{array}\right]+[\mathbf{J}-h \mathbf{A}]^{-1} *\left[\begin{array}{c}
\frac{1}{L} \\
0 \\
0
\end{array}\right] * u(t) * h
$$

where $i_{L} \quad$ - current of the inductor $L$ of $L C L C$ filter
$u_{\mathrm{C}}$ - voltage of the capacitor $C$ of $L C L C$ filter
$i_{Z} \quad$ - current of the load $R, L_{L}$

$$
\mathbf{A}=\left(\begin{array}{ccc}
\frac{-r_{L}}{L} & \frac{-1}{L} & 0  \tag{8}\\
\frac{1}{C} & \frac{-1}{C \cdot r_{C}} & \frac{-1}{C} \\
0 & \frac{1}{L_{L}} & \frac{-R}{L_{L}}
\end{array}\right)
$$

## 5 Results of simulation experiments and verifications

Following figures show both the steady-state waveforms of input and output voltages, and transients for load disconnection in the middle of half period of output filter voltage. The transients are simulated for two sets of filter parameter values: basic values of the filter parameters, i.e. resonant power is equal to the load power (so quality of the each part series- and parallel are equal one), and resonant power is equal 2-multiply of the load power (so quality of the each part seriesand parallel are equal two).


Fig. 6. Output capacitor voltage of LCLC filter with full-wide of pulse $\beta=180^{\circ} \mathrm{el}$. (left) and for load disconnection in the middle of half period of output filter voltage (right)

The output capacitor voltage of LCLC filter for load disconnection in time of maximum output filter voltage embodies overvoltage for symmetrical control (Fig. 8, Fig. 12) as well as for nonsymmetrical control (Fig. 10). The overvoltage is higher for resonant quality factor equal to two.


Fig. 7. Output capacitor voltage of LCLC filter under phase control with $\beta=165^{\circ} \mathrm{el}$. (left) and for load disconnection in the middle of half period of output filter voltage (right)



Fig. 8. Output capacitor voltage of LCLC filter under symmetrical control (left) and for load disconnection in the middle of half period of output filter voltage (right)

Transients during switch-on of the load are shown in Fig. 9; load current has no current overshoots in case of load connection.


Fig. 9. Output capacitor voltage of LCLC filter and load current for load connection
In case of simulation of the $2^{\text {nd }}$-order LC filter it is evident that output capacitor voltage of the filter for load disconnection in time of the maximum output filter voltage embodies no high overvoltage, Fig. 10a. It has only temporary growth of voltage amplitude as the saved energy of inductor is only up to $5 \%$ of load energy. This is better choice of filter application. The measured output voltage of the filter for R-L load is shown in Fig. 10b.


Fig.15. Output capacitor voltage of matrix LC filter for load disconnection in time of maximum output filter voltage (left) and measured voltage of inductive load (right)

## 6 Conclusions

A state-space transient analysis of two types of resonant filters has been done. Euler's implicit method was used for obtaining of difference equation system of the filters. Using filters the output voltage is nearly harmonic waveform without substantial distortion. It has been shown that overvoltage occurred during disconnection of the load could reach up three multiply of rated value. These facts are dangerous mainly for semiconductor switches of the inverter supplying the filters. Using second type of filter the overvoltage reaches just $5 \%$, so this type is more suitable for applications. Measured waveforms of output voltages under resistive-inductive load confirm good coincidence with simulated waveforms of the quantities.

## Appendix

The parameters of the simulation and filter components:
$U=255 \mathrm{~V} ; R=150 \Omega ; L=360 \mathrm{mH}$

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$L 1=L 2=610 \mathrm{mH} ; C 1=C 2=17 \mu \mathrm{~F}$

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# USING PARAMETERIZATION OF SURFACE AREAS FOR VOLUMES <br> BITTNEROVÁ Daniela, (CZ) 


#### Abstract

The paper follows up with the papers published in the proceedings of the Slovak-Czech-Polish Mathematical School (Ružomberok 2004) - see [1] and The International Colloquium on the Acquisition Process Management, Brno 2006, 2010 - see [2] and [3], where volumes of solids were discussed as topological problems. The new relation for a calculation of volumes of solids in $\mathbf{E}_{\mathbf{n}}$ was presented there. That relation asks a parametric description of a surface area of a given solid and then we are able to solve that problem by using basic topological properties. The surface areas of these volumes must be smooth or smooth by parts areas in Euclidean space of the corresponding dimensions. The calculations of areas and volumes could be easier in some cases, because we calculate with integrals of fewer dimensions. That paper shortly repeats all needed assumptions and also results, and presents some applications on two examples.


Key words. Volume, parameterization, surface area, smooth hypersurface

## 1 <br> Introduction

Areas and volumes of solids occur in the technical practice often. Usually, the multidimensional real integrals are used to calculations of them. Therefore, the theory of a measure and multidimensional real integrals in generally $n$-dimensional space $\mathbf{E}_{\mathbf{n}}$ are very important. In the papers [1] and [2], that problem is investigated as a problem in the topological sense and the formula for the calculation of the area (resp. volume) of the $n$-dimensional solid in the space $\mathbf{E}_{\mathbf{n}}$ is proved there. In that theory, we must find suitable parametric descriptions of the surface areas of solids for the calculation of their volumes. Then the surface areas are the smooth (respective by parts smooth) areas in Euclidean space of the corresponding dimensions. The method shows a new approach to the solving of the known problems. In the paper [3], the correspondence between the new theory in $\mathbf{E}_{2}$ and the known result of the curvilinear integral theory is presented. It is demonstrated by some examples. In that article, we present an application of it in $\mathbf{E}_{3}$. We calculate volumes of bounded closed solids in $\mathbf{E}_{\mathbf{3}}$ by using parametric descriptions of their surface areas.

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## 2 Formulation of the Problem

Let $\mathbf{x}$ be points of Euclidean $n$-dimensional space $\mathbf{E}_{\mathbf{n}}, n \geq 2$,
$x^{\alpha}, \alpha=1, \ldots, n, \quad$ be the Cartesian co-ordinates of the point $\mathbf{x} \in \mathbf{E}_{\mathbf{n}}$,
$u^{a}, a=1, \ldots, n-1$, the Cartesian co-ordinates of the point $\mathbf{u} \in \mathbf{E}_{\mathbf{n}-\mathbf{1}}$,
$\Omega \quad$ the bounded closed domain in $\mathbf{E}_{\mathbf{n}-\mathbf{1}}$,
$x^{\alpha}\left(u^{1}, \ldots, u^{n-1}\right)$ given functions defined on some domain $\mathrm{O} \subset \mathbf{E}_{\mathrm{n}-1}, \Omega \subset \mathrm{O}$, where $\alpha=1, \ldots, n$.

Let us also suppose that

- the vector function $\mathbf{x}(\mathbf{u})=\left\{x^{\alpha}\left(u^{a}\right)\right\}$ has almost everywhere in $\Omega$ the continuous partial derivatives $B_{a}^{\alpha}:=\frac{\partial x^{\alpha}}{\partial u^{a}}$ for $\alpha=1, \ldots, n, a=1, \ldots, n-1$;
- the rank of the matrix $\left(B_{a}^{\alpha}\right)_{n \times(n-1)}$ is equal to $n-1$ almost everywhere in $\Omega$;
- the subset $P^{o}:=\left\{\mathbf{x} \in \mathbf{E}_{\mathbf{n}} \mid \mathbf{x}=\mathbf{x}(\mathbf{u}), \mathbf{u} \in \operatorname{int} \Omega\right\}$
of the set $P:=\left\{\mathbf{x} \in \mathbf{E}_{\mathbf{n}} \mid \mathbf{x}=\mathbf{x}(\mathbf{u}), \mathbf{u} \in \Omega\right\}$ is a homeomorphic range of the set int $\Omega$ in $\mathbf{E}_{\mathbf{n}}$.

It follows from the given assumptions, that the set $P$ is bounded and a smooth hypersurface by parts in the space $\mathbf{E}_{\mathbf{n}}$. That manifold does not intersect by itself and that divides the space $\mathbf{E}_{\mathbf{n}}$ to two disjoint regions in $\mathbf{E}_{\mathbf{n}}$, in which one is bounded and the second one is unbounded. Then we can consider the closure $W$ of the set $P$ that is called the $\boldsymbol{n}$-dimensional solid in space $\mathbf{E}_{\mathbf{n}}, \partial W=P$ is the boundary of it. We would like to calculate the volume of it.

## 3 Formula of the Volume

Let the equations

$$
\begin{gather*}
x^{\alpha}=x^{\alpha}\left(u^{a}\right) t, \alpha=1, \ldots, n,  \tag{1}\\
x^{n+1}=v(1-t), \mathbf{u} \in \Omega, t \in\langle 0 ; 1\rangle, v \in\left\langle 0 ; x_{0}^{n+1}\right\rangle
\end{gather*}
$$

be the parametric description of a hypercone $K$ with the vertex $V=\left[0 ; x_{0}^{n+1}\right]$ (i.e. $x_{0}^{\alpha}=0$ ), that is a homeomorphic range of a hypersphere $W$ in $\mathbf{E}_{\mathbf{n}}$, then we can calculate the volume $\mu_{W}$ of the $n-$ dimensional solid $W$ in the space $\mathbf{E}_{\mathbf{n}}$ by the formula

$$
\begin{equation*}
\mu_{W}:=\frac{1}{n} \iint \ldots 反_{\Omega}|\Delta(\mathbf{u})| \mathrm{d} u^{1} \mathrm{~d} u^{2} \ldots \mathrm{~d} u^{n-1}, \tag{2}
\end{equation*}
$$

where

$$
\Delta(\mathbf{u}):=\left|\begin{array}{cccc}
x^{1}(\mathbf{u}) & x^{2}(\mathbf{u}) & \ldots & x^{n}(\mathbf{u})  \tag{3}\\
B_{1}^{1} & B_{1}^{2} & \ldots & B_{1}^{n} \\
\ldots & \ldots & \ldots & \ldots \\
B_{n-1}^{1} & B_{n-1}^{2} & \ldots & B_{n-1}^{n}
\end{array}\right| .
$$

(The proof - see D. Bittnerová - [1].)

## 4 Examples

## Example 1

Calculate the volume of the ellipsoid determined by the set

$$
\begin{equation*}
P:=\left\{\mathbf{x} \in \mathbf{E}_{\mathbf{3}} \left\lvert\, \sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1\right.\right\}, x \geq 0, y \geq 0, z \geq 0, a, b, c \in \mathbf{R}^{+} . \tag{4}
\end{equation*}
$$

## Solution:

The set $P$ is a smooth closed area in the space $\mathbf{E}_{3}$ (in the sense above). The projection of this area $P$ to the $(x, y)$ - plane is the closure of the interior of the ellipse whose implicit description is of the form

$$
\begin{equation*}
K:=\left\{(x ; y) \in \mathbf{E}_{2} \left\lvert\, \sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1\right.\right\}, x \geq 0, y \geq 0, a, b \in \mathbf{R}^{+} . \tag{5}
\end{equation*}
$$

The area $P$ is obviously described by these parametric equations

$$
\begin{equation*}
x=a \cos ^{4} \varphi \sin ^{4} v, y=b \sin ^{4} \varphi \sin ^{4} v, z=c \cos ^{4} v, \varphi \in\left\langle 0 ; \frac{\pi}{2}\right\rangle, v \in\left\langle 0 ; \frac{\pi}{2}\right\rangle . \tag{6}
\end{equation*}
$$

According to the notation given above where $u^{1}=\varphi, u^{2}=v, n=3$, there is

$$
\begin{gather*}
x^{1}\left(u^{1} ; u^{2}\right)=x(\varphi ; v)=a \cos ^{4} \varphi \sin ^{4} v,  \tag{7}\\
x^{2}\left(u^{1} ; u^{2}\right)=y(\varphi ; v)=b \sin ^{4} \varphi \sin ^{4} v, \\
x^{3}\left(u^{1} ; u^{2}\right)=z(\varphi ; v)=c \cos ^{4} v, \\
B_{1}^{1}=\frac{\partial x}{\partial \varphi}=-4 a \cos ^{3} \varphi \sin \varphi \sin ^{4} v, B_{2}^{1}=\frac{\partial x}{\partial v}=4 a \cos ^{4} \varphi \sin ^{3} v \cos v,
\end{gather*}
$$

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$$
\begin{gathered}
B_{1}^{2}=\frac{\partial y}{\partial \varphi}=4 b \sin ^{3} \varphi \cos \varphi \sin ^{4} v, B_{2}^{2}=\frac{\partial y}{\partial v}=4 b \sin ^{4} \varphi \sin ^{3} v \cos v, \\
B_{1}^{3}=\frac{\partial z}{\partial \varphi}=0, B_{2}^{3}=\frac{\partial z}{\partial v}=-4 c \cos ^{3} v \sin v .
\end{gathered}
$$

In this case, the determinant $\Delta(u)$ defined in (3) is equal to

$$
\Delta(u)=\Delta(\varphi ; v)=\left|\begin{array}{ccc}
a \cos ^{4} \varphi \sin ^{4} v, & b \sin ^{4} \varphi \sin ^{4} v, & c \cos ^{4} v \\
-4 a \cos ^{3} \varphi \sin \varphi \sin ^{4} v, & 4 b \sin ^{3} \varphi \cos \varphi \sin ^{4} v, & 0 \\
4 a \cos ^{4} \varphi \sin ^{3} v \cos v, & 4 b \sin ^{4} \varphi \sin ^{3} v \cos v, & -4 c \cos ^{3} \sin v
\end{array}\right|=(8)
$$

For calculation of the volume $\mu_{W}$ of three-dimensional solid $W$ in $\mathbf{E}_{\mathbf{3}}$, whose boundary is the smooth closed area $P$, we use the relation (2). With respect to the symmetry of this solid, we will calculate one quarter of this volume only (see limits of integral):

$$
\begin{gather*}
\mu_{W}=\frac{1}{3} \iint_{\Omega}|\Delta(\varphi ; v)| \mathrm{d} \varphi \mathrm{~d} v=\frac{16}{3} \iint_{\Omega} a b c \sin ^{3} \varphi \cos ^{3} \varphi \sin ^{7} v \cos ^{3} v \mathrm{~d} \varphi \mathrm{~d} v=  \tag{9}\\
=\frac{16}{3} a b c \int_{0}^{\frac{\pi}{2}} \sin ^{3} \varphi \cos ^{3} \varphi \mathrm{~d} \varphi \int_{0}^{\frac{\pi}{2}} \sin ^{7} v \cos ^{3} v \mathrm{~d} v=\left|s u b s t . \begin{array}{c}
\sin \varphi=t, \quad \sin v=s \\
\cos \varphi \mathrm{~d} \varphi=\mathrm{d} t, \quad \cos v \mathrm{~d} v=\mathrm{d} s
\end{array}\right|= \\
=\frac{16}{3} a b c \int_{0}^{1} t^{3}(1-t) \mathrm{d} t \cdot \int_{0}^{1} s^{7}\left(1-s^{2}\right) \mathrm{d} s=\frac{16}{3} a b c\left[\frac{t^{4}}{4}-\frac{t^{6}}{6}\right]_{0}^{1} \cdot\left[\frac{s^{8}}{8}-\frac{s^{10}}{10}\right]_{0}^{1}= \\
=\frac{16}{3} a b c\left(\frac{1}{4}-\frac{1}{6}\right)\left(\frac{1}{8}-\frac{1}{10}\right)=\frac{a b c}{90}
\end{gather*}
$$

The whole volume is equal to

$$
\begin{equation*}
\mu_{W}=\frac{a b c}{90} . \tag{10}
\end{equation*}
$$

## Example 2

Calculate the volume of the solid determined by the set

$$
\begin{equation*}
P:=\left\{\mathbf{x} \in \mathbf{E}_{\mathbf{3}} \mid\left(x^{2}+y^{2}+z^{2}\right)^{2}=z\right\} . \tag{11}
\end{equation*}
$$

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## Solution:

With respect to the symmetry, we use the parametric equations of one quarter of the set $P$ :

$$
\begin{gather*}
x=\cos \varphi \sin v \sqrt[3]{\cos v}, y=\sin \varphi \sin v \sqrt[3]{\cos v}, z=\sqrt[3]{\cos ^{4} v},  \tag{12}\\
\varphi \in\left\langle 0 ; \frac{\pi}{2}\right\rangle, v \in\left\langle 0 ; \frac{\pi}{2}\right\rangle \tag{13}
\end{gather*}
$$

According to the used denotations, there is $u^{1}=\varphi, u^{2}=v, n=3$, thus the determinant $\Delta(u)$ is equal to

$$
\begin{equation*}
\Delta(u)=\Delta(\varphi ; v)= \tag{14}
\end{equation*}
$$

$$
=\left|\begin{array}{ccc}
\cos \varphi \sin v \sqrt[3]{\cos v}, & \sin \varphi \sin v \sqrt[3]{\cos v}, & \cos ^{\frac{4}{3}} v \\
-\sin \varphi \sin v \sqrt[3]{\cos v,} & 0 \\
\cos \varphi\left(-\frac{1}{3} \sin ^{2} v \cos ^{-\frac{2}{3}} v+\cos ^{\frac{4}{3}} v\right), & \sin \varphi\left(-\frac{1}{3} \sin ^{2} v \cos ^{-\frac{2}{3}} v+\cos ^{\frac{4}{3}} v\right), & -\frac{4}{3} \cos ^{\frac{1}{3}} v \sin v
\end{array}\right|=
$$

The volume $\mu_{W}$ of the solid $W$ in $\mathbf{E}_{\mathbf{3}}$ with the boundary $P$ is:

$$
\begin{align*}
\mu_{W} & =4 \cdot \frac{1}{3} \iint_{\Omega} \Delta(\varphi ; v) \left\lvert\, \mathrm{d} \varphi \mathrm{~d} v=\frac{4}{3} \cdot \int_{0}^{\pi / 2} \mathrm{~d} \varphi \int_{0}^{\pi / 2} \sin v \cos v \mathrm{~d} v=\right.  \tag{15}\\
& =\frac{4}{3} \cdot \frac{\pi}{2} \int_{0}^{\pi / 2} \frac{\sin 2 v}{2} \mathrm{~d} v=\left[-\frac{\cos 2 v}{4}\right]_{0}^{\pi / 2}=\ldots=\frac{\pi}{3}
\end{align*}
$$

## 5 Conclusion

There exist lots of examples and technical applications where the presented method may be used. If we are able to find the suitable parameterizations of the smooth areas, the calculations of areas and volumes could be easier, because we calculate with integrals of fewer dimensions. However, to find the parameterization can be a rather complicated problem in specific cases.

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## PARALLEL MODELS OF ADAPTIVE DIFFERENTIAL EVOLUTION BASED ON MIGRATION PROCESS

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#### Abstract

The influence of parallelism on the performance of differential evolution is studied. Performance of non-parallel and two kinds of parallel topology is compared experimentally. The parallel versions were implemented on a single-processor machine and the influence of parallelism is simulated in the pseudo-parallel regime. The results show that parallel version of differential evolution with star topology exhibits the best efficiency among algorithms in comparison on the benchmark problems.


Key words and phrases. Global optimization, differential evolution, parallel migration model, migration topology, comparison on benchmark functions.
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## 1 Introduction

The global optimization problem is considered in the following form:

$$
\text { minimize } f(\boldsymbol{x}), \quad \boldsymbol{x} \in D, f(\boldsymbol{x}): D \rightarrow \mathbb{R}^{d},
$$

where $\boldsymbol{x}$ is a continuous variable with the domain $D \subseteq \mathbb{R}^{d}$ and $f(\boldsymbol{x})$ is continuous. In this paper we focus on the problems defined by specifying boundary constraints: $\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right] \times$ $\ldots \times\left[a_{d}, b_{d}\right], a_{i}<b_{i}, i=1, \ldots D$. The global minimum point $\boldsymbol{x}^{*}$ while satisfying condition $f\left(\boldsymbol{x}^{*}\right) \leq f(\boldsymbol{x})$ for $\forall \boldsymbol{x} \in D$ is the solution of the problem.

To find the global minimum of the optimization tasks are used algorithms based on heuristic. Heuristics are techniques used in the stochastic algorithms based on a skill, knowledge, chance and sense, motivated by evolution behaviour of groups of animals in nature [6]. Populationbased algorithms are efficient search methods of the global minimum in the real time. However,
their efficiency depends on settings of its control parameters. Proper setting of parallel strategy in evolutionary algorithms (EA) can bring faster search of solution, especially in difficult problems [3]. Distribution of computing power of EA enables to find a better solution of the global minimization with reducing time-demands of the search. Parallel EA are able to solve difficult optimization problems and increase reliability of the search.

## 2 Differential evolution (DE)

Differential evolution (DE) is one of the popular and efficient stochastic population-based algorithms searching the global minimum of difficult real-world problems [7, 9]. In DE are as well as in other evolutionary algorithms there are applied evolutionary operators, namely selection, mutation, and crossover. Mutation in DE uses a control parameter $F$ and control parameter $C R$ for crossover. The setting of values $F$ and $C R$ influences significantly the efficiency of the search process. In this study we use self-adaptation of parameters $F$ and $C R$ [11]. Stochastic algorithms as DE can benefit from computing power of the parallel models. The goal of this paper is to study the influence of different parallel models on the performance of DE.

Main idea of DE algorithm is shown in pseudo-code Algorithm(1). At first, there is created populations $P$ of size $N$ filled by $N$ points randomly chosen from uniform distribution from the search space $D$. A new trial point (offspring) $\boldsymbol{y}$ is made from the current point (parent) of the source population $\boldsymbol{x}_{i} \in P$ by crossover with its mutation point $\boldsymbol{u}$. A new point $\boldsymbol{y}$ is chosen into the new population $Q$ if $f(\boldsymbol{y})<f\left(\boldsymbol{x}_{i}\right)$, otherwise $\boldsymbol{x}_{i}$ is inserted into $Q$. After completion of the new population $Q$, the old population $P$ is replaced by $Q$ and the process of searching the global minimum proceeds until the stopping condition is achieved.

```
initializing population \(P=\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}, \ldots, \boldsymbol{x}_{N}\right)\);
while stopping condition not achieved do
    for \(i=1\) to \(N\) do
        generate a new trial vector \(\boldsymbol{y}\) by computing from \(P\);
        if \(f(\boldsymbol{y})<f\left(\boldsymbol{x}_{i}\right)\) then
            insert \(\boldsymbol{y}\) into \(Q\);
        else
            insert \(\boldsymbol{x}_{i}\) into \(Q\);
        end
    end
    \(P:=Q ;\)
end
```

Algorithm 1: Non-parallel DE in pseudo-code
There are several strategies how to make the trial point $\boldsymbol{y}$ from points of the old population $P[9,7]$. The strategies can use different way of mutation and crossover. In DE community, the abbreviation $\mathrm{DE} / m / a / c$ is used for the kind of strategy, where $m$ is the type of mutation, $a$ is the number of differences of randomly chosen pair of points used in mutation, and $c$ stands for type of crossover. Reliability and efficiency of differential evolution algorithm is sensitive on the choice of strategy and the values the control parameters $F$ and $C R$, usually $F \in[0.3,1]$
and $C R \in[0,1]$. The study of control parameter values settings was worked in [12].
Standard DE using only one strategy with one fixed control parameter setting often needs the time-consuming trial-and-error search of proper $F$ and $C R$ values. Adaptive setting of control parameters value and strategies based on their competition was proposed in [11]. This competitive approach to the choice of strategy and parameter' values enables robust search across relatively wide set of problems without special control parameter tuning. In this paper there are chosen adaptive DE algorithms debr18 [11] and b6e6rl [10] based on strategies DE/rand/1/bin, DE/best/2/bin, $D E /$ randRL/1/bin and $D E / r a n d R L / 1 / \exp$.

## 3 Parallel differential evolution

The main problem of optimization is the time-complexity of the global optimum point search. One way how solve this problem is using parallel models of DE. There are several approaches of parallelism that can lead to faster search of the global minimum. The most frequently used parallel models are master-slave model, migration (island) model, diffusion (cellular) model and their combination resulting in hybrid models $[2,3,5]$. This study is focused on parallel DE with application of migration model. DE time complexity is divided into several subpopulations (islands) and the individuals migrate among them. Parallel DE based on migration model is illustrated in pseudo-code Algorithm(2). The global population $P$ of $N$ individuals $\boldsymbol{x}_{i}, i=1, \ldots, N$ is uniformly divided into ( $k$ ) independent subpopulations.

```
initializing \(k\) independent subpopulations \(Q_{j}=\left(\boldsymbol{x}_{j 1}, \boldsymbol{x}_{j 2}, \ldots, \boldsymbol{x}_{j N p}\right), j=1,2, \ldots, k\);
while stopping condition not achieved do
        while migration criteria not achieved do
            for \(j=1\) to \(k\) do
                compute new generation of subpopulation \(S_{j}\) by one generation of DE
                \(Q_{j}:=S_{j} ;\)
            end
    end
    choose and migrate several individuals among subpopulations \(Q_{j}\)
end
```

Algorithm 2: Parallel DE based on migration model in pseudo-code
Thus each subpopulation contains $N p=N / k$ individuals from global population $Q$. Afterwards non-parallel DE algorithm proceeds independently and separately on each island. After ( $n d e$ ) generations migration criterion is achieved and the algorithm proceeds by migration operation. Migration provides diversity of subpopulations through exchange several individuals among selected islands ( $N p / k$ individuals in this case). Thus probability of the premature convergence is reduced. Migration model of DE has control parameters as follows:

- number of independent subpopulations (islands),
- topology of islands interconnection,
- number of individuals migrating among islands,
- number of DE generations before migration (length of epoch).

Settings of these parameters influence the efficiency and reliability of the global minimum search.

### 3.1 Topologies of island model

Interconnections among islands in migration model provide possibility to increase the subpopulations diversity and to decrease the time-demands of the search. Selection of topology in island models influences quality of the search [3, 8]. In this paper we compare two topologies of migration model with DE.

Interconnection of the first topology, named star, is shown in Figure 1. One of the subpopulations is chosen as the major subpopulation denoted $Q_{1}$. The star topology has interconnection only between the major subpopulation $Q_{1}$ and each of other subpopulations $Q_{j}(j=2, \ldots, k)$ as shown in the figure. If migration criterion is achieved all the subpopulations are sorted with respect to function values in ascending way. The sorted major subpopulation $Q_{1}$ is divided into $k$ blocks. The best block remains in the major population $Q_{1}$ and other blocks $(2 n d, 3 t h, \ldots, k t h)$ are exchanged with the best block of corresponding subpopulation $Q_{2}, \ldots, Q_{k}$.


Figure 1: A star topology of migration model
The second topology of the migration model, named ring, is depicted in Figure 2. Opposite to star topology, here are all subpopulations at the same hierarchical level. Each of independent subpopulation $Q_{i}$ has only interconnection with two neighbours $Q_{i-1}$ and $Q_{i+1}$ as shown in the figure. The $N p / k$ best individuals of $Q_{i-1}$ migrate to neighbour $Q_{i}$ and at the same time the $N p / k$ best individuals of $Q_{i}$ migrate to $Q_{i+1}$ for $i=1,2, \ldots k$.

## 4 Test functions

Six scalable functions [4] were used as a benchmark: Rosenbrock function (unimodal, non-


Figure 2: A ring topology of migration model
separable),

$$
f(\boldsymbol{x})=\sum_{j=1}^{d-1}\left[100\left(x_{j}^{2}-x_{j+1}\right)^{2}+\left(1-x_{j}\right)^{2}\right]
$$

where $x_{j} \in[-2.048,2.048], \boldsymbol{x}^{*}=(1,1, \ldots, 1), f\left(\boldsymbol{x}^{*}\right)=0$.
Schwefel function (multimodal, separable),

$$
f(\boldsymbol{x})=418.98288727 d-\sum_{j=1}^{d} x_{j} \sin \left(\sqrt{\left|x_{j}\right|}\right)
$$

where $x_{j} \in[-500,500], \boldsymbol{x}^{*}=(c, c, \ldots, c), c \doteq 420.9687, f\left(\boldsymbol{x}^{*}\right) \doteq 0$.
Because the next test functions in their standard form have the global minimum point in the middle of the search domain, we use them in a shifted version as defined bellow. First DeJong shifted function (uni-modal, separable),

$$
f(\boldsymbol{x})=\sum_{j=1}^{d}\left(x_{j}-o_{j}\right)^{2}
$$

where $x_{j} \in[-5.12,5.12], f\left(\boldsymbol{x}^{*}\right)=0, \boldsymbol{x}^{*}=\boldsymbol{o}$.
Ackley shifted function - (multimodal, separable),

$$
f(\boldsymbol{x})=-20 \exp \left(-0.02 \sqrt{\frac{1}{d} \sum_{j=1}^{d}\left(x_{j}-o_{j}\right)^{2}}\right)-\exp \left(\frac{1}{d} \sum_{j=1}^{d} \cos 2 \pi\left(x_{j}-o_{j}\right)\right)+20+\exp (1)
$$

$x_{j} \in[-30,30], f\left(\boldsymbol{x}^{*}\right)=0, \boldsymbol{x}^{*}=\boldsymbol{o}$.
Shifted Griewank function - multi-modal, non-separable,

$$
f(\boldsymbol{x})=\sum_{j=1}^{d} \frac{\left(x_{j}-o_{j}\right)^{2}}{4000}-\prod_{j=1}^{d} \cos \left(\frac{\left(x_{j}-o_{j}\right)}{\sqrt{j}}\right)+1
$$

$x_{j} \in[-400,400], f\left(\boldsymbol{x}^{*}\right)=0, \boldsymbol{x}^{*}=\boldsymbol{o}$.
Shifted Rastrigin function - multi-modal, separable,

$$
\begin{gathered}
f(\boldsymbol{x})=10 d+\sum_{j=1}^{d}\left[\left(x_{j}-o_{j}\right)^{2}-10 \cos \left(2 \pi\left(x_{j}-o_{j}\right)\right)\right] \\
x_{j} \in[-5.12,5.12], f\left(\boldsymbol{x}^{*}\right)=0, \boldsymbol{x}^{*}=\boldsymbol{o} .
\end{gathered}
$$

## 5 Experiments and results

In this study we compare six variants of DE algorithm that are labeled as shown in Table 1. In adaptive debr 18 algorithm compete 18 different combinations of DE control parameter settings, $F=0.5,0.8,1, C R=0,0.5,1$ and strategies $D E /$ rand $/ 1 /$ and $D E /$ best $/ 2 /(3 \times 3 \times 2)$, in adaptive b6e6rl compete 12 different combinations of DE control parameter settings, $F=$ $0.5,0.8, C R=0,0.5,1$ and strategy $D E / \operatorname{rand} R L / 1 /$ with binomial or exponential crossover $(2 \times 3 \times 2)$ [10].

Table 1: Variants of DE algorithm in tests

| Variant of DE | non-parallel | star-topology | ring-topology |
| :--- | :---: | :---: | :---: |
| debr18 | debr18 | debr18 star | debr18 ring |
| b6e6rl1 | b6e6rl | b6e6rl star | b6e6rl ring |

These variants of adaptive DE algorithm were tested on six test functions given above for problem dimension $d=5,10,30$. The size of global population $N$ and was set up as $N=10 \times d$ and size of all the subpopulations $N p=N / k$. The parameter $k$, denoting number of subpopulations, was set up as $k=5$ and the number of migrating individuals was set up as $m r=N p / k$. All algorithms were run one hundred times for four values of epoch parameter nde $=1,3,5,7$ and fifty times for four values nde $=10,20,30,40$. The parameters of DE competition were set up as $n_{0}=2$ and $\delta=1 /(5 H)$ where $H$ is the number of competitive adaptive strategies ( $H=12$ for $b 6 e 6 r l$ and $H=18$ for debr18).

The search process was stopped if the difference between the maximum and minimum function value of the population (or one of the subpopulations) was great, in these experiments it means $f_{\max }-f_{\min }<1 \times 10^{-7}$ or the number of function evaluations ( $f e v$ ) reached $20000 \times d$.

The reliability $R$ (percentage of successful runs) and number of the function evaluation were compared. For easier comparison of performance of DE variants, a new integrated criterion $C$ was used in (1). Ratio $d^{2} / 250$ makes the $C$ values approximately comparable for all the dimensions of the tested problems $(d=5,10,30)$.

$$
\begin{equation*}
C=\frac{R * d^{2}}{f e v * 250} \tag{1}
\end{equation*}
$$

The values of $C$ criterion of DE algorithm variant with the length of epoch providing the best performance of parallel variants are presented in Table 2.

Table 2: The best values of criterion C for each dimension and each problem

| d | problem | C-best | variant | nde |
| :---: | :--- | :---: | :---: | :---: |
| 5 | Ackley | 62 | b6e6-star | 10 |
|  | Dejong1 | 137 | b6e6-star | 1 |
|  | Griewank | 34 | b6e6-star | 1 |
|  | Rastrigin | 75 | b6e6-star | 1 |
|  | Rosenbrock | 61 | b6e6rl | - |
|  | Schwefel | 87 | b6e6-star | 1 |
| 10 | Ackley | 64 | b6e6-ring | 20,30 |
|  | Dejong1 | 134 | b6e6-star | 10 |
|  | 42 | b6e6-star | 20 |  |
|  | Rastrigin | 70 | b6e6-star | 7 |
|  | Rosenbrock | 52 | b6e6rl | - |
|  | Schwefel | 83 | b6e6-star | 10 |
| 3 | Ackley | 56 | b6e6-ring | 20 |
|  | Dejong1 | 106 | b6e6-ring | 40 |
|  | 80 | b6e6-ring | 40 |  |
|  | Rastrigin | 61 | b6e6-star | 30 |
|  | Rosenbrock | 38 | b6e6-star | 30 |
|  | Schwefel | 67 | b6e6-star | 40 |

The count of the best performance of DE variants in all tested problems together in dependence of length of epoch ( $n d e$ ) is given in Table 3. The DE variants using debr18 have never outperformed the others and that is why they do not appeared in the table.

Table 3: Summary of experimental results - counts of maximum $C$ values over all six tested DE variants and all six problems found for various length of epoch (nde).


## 6 Conclusion

Two adaptive variants of DE ( $b 6 e 6 r l$ and debr18) were compared experimentally in three types of topology (non-parallel, star, and ring) on six benchmark problems, each problem at three levels of dimension. Experimental results show:

- Adaptive variant b6e6rl outperformed the debr18 in all the tested problems.
- Benefit of parallelism is greater in the problems of higher dimension.
- Performance of parallel variants depends on the type of topology and on the frequency of migration given by the control parameter nde. The star topology in parallel variants gives better results more frequently than the ring topology.
- Parallel versions with greater length of epoch perform better for the problems of higher dimension.

Next research will be focused on the study of other types of ring topology and on the different kinds of migration.

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# COMPUTATION OF BIORTHOGONALIZATION MATRICES FOR SPLINE-WAVELET BASES ON THE INTERVAL 

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#### Abstract

The adaptation of spline-wavelet bases to the interval requires the computation of a biorthogonalization matrix, which is the matrix of inner products of primal scaling functions and generators of a dual space. Due to the low Sobolev regularity of generators of a dual space we do not obtain the sufficiently accurate entries of the biorthogonalization matrices directly by classical quadratures. In this paper, we propose a method which enables to compute the biorthogonalization matrices exactly up to the round off errors. This method includes the computation of the integrals of the product of a polynomial and a refinable function over a dyadic interval.


Key words. Spline, wavelet, interval, refinable function, integral, biorthogonalization.
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## 1 Introduction

Wavelets were first constructed on the whole real line, then they were adapted to the interval and the $n$-dimensional cube and nowadays wavelet bases are available for fairly general domains and for a wide range of applications. The construction of a wavelet basis on a general domain usually starts with the adaptation of a wavelet basis on the real line to the interval. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases on a general domain. Biorthogonal spline-wavelet bases on the unit interval were constructed in [7]. The disadvantage of them is their bad condition which causes problems in practical applications. Some modifications which lead to better conditioned bases were proposed in $[8,12]$. Our recent construction in [4] leads to optimally conditioned wavelet bases up to the order four. The adaptation of wavelet bases on the real line to the interval requires the computation of biorthogonalization matrices, i.e. the matrices of inner products of primal scaling functions and generators of a dual space. These generators are derived from

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the dual scaling functions on the real line. However the dual scaling functions have typically a low Sobolev regularity. Thus the generators of a dual space also have a low Sobolev regularity. Therefore, we do not obtain the sufficiently accurate entries of the biorthogonalization matrices directly by classical quadratures. In this paper, we propose a method which enables to compute the biorthogonalization matrices precisely up to the round off errors. The method requires the knowledge of the integrals of the product of a polynomial and the scaling function over a dyadic interval. We also propose a method for the computation of these integrals for a general refinable function with compact support.

First, we shortly describe the concept of a wavelet basis for $L^{2}(\mathbb{R})$. Let $\langle\cdot, \cdot\rangle$ be an inner product and $\|\cdot\|$ be a norm in $L^{2}(\mathbb{R})$. Let $l^{2}$ be a space of $\mathbf{v}:=\left\{v_{j, k}\right\}_{j, k \in \mathbb{Z}}$ satisfying

$$
\begin{equation*}
\|\mathbf{v}\|_{l^{2}}:=\sum_{j, k \in \mathbb{Z}}\left|\mathbf{v}_{j, k}\right|^{2}<\infty \tag{1}
\end{equation*}
$$

Definition 1.1 A function $\psi \in L^{2}(\mathbb{R})$ is called a wavelet if the family $\Psi:=\left\{\psi_{j, k}\right\}_{j, k \in \mathbb{Z}}$, where $\psi_{j, k}:=2^{j / 2} \psi\left(2^{j} \cdot-k\right)$, is a Riesz basis in $L^{2}(\mathbb{R})$, i.e. $\Psi$ is complete in $L^{2}(\mathbb{R})$ and there exist constants $c, C \in(0, \infty)$ such that

$$
\begin{equation*}
c\|\mathbf{v}\|_{l^{2}} \leq\left\|\sum_{j, k \in \mathbb{Z}} v_{j, k} \psi_{j, k}\right\| \leq C\|\mathbf{v}\|_{l^{2}}, \quad \mathbf{v} \in l^{2} . \tag{2}
\end{equation*}
$$

The functions $\psi_{j, k}$ are also called wavelets.
Let $V_{j}$ be the closure of the span of the set $\left\{\psi_{l, k}, l \leq j, k \in \mathbb{Z}\right\}$ and let us suppose that there exists a function $\phi$ such that $\Phi_{j}:=\left\{\phi_{j, k}, k \in \mathbb{Z}\right\}, \phi_{j, k}:=\phi\left(2^{j} \cdot-k\right)$, is a Riesz basis of $V_{j}$. Functions $\phi$ and $\phi_{j, k}$ are called scaling functions.

Then there exists a sequence $\left\{h_{k}\right\}_{k \in \mathbb{Z}}$ such that

$$
\begin{equation*}
\phi(x)=\sum_{k \in \mathbb{Z}} h_{k} \phi(2 x-k) \quad \text { for all } \quad x \in \mathbb{R} . \tag{3}
\end{equation*}
$$

This equation is called a refinement or a scaling equation and the coefficients $h_{k}$ are known as scaling or refinement coefficients.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi}=\left\{\psi_{j, k}\right\}_{j, k \in \mathbb{Z}}$ biorthogonal to $\Psi$, i.e.

$$
\begin{equation*}
\left\langle\psi_{i, k}, \tilde{\psi}_{j, l}\right\rangle=\delta_{i, j} \delta_{k, l}, \quad i, j, k, l \in \mathbb{Z} \tag{4}
\end{equation*}
$$

where $\delta_{i, j}$ denotes a Kronecker delta. The family $\tilde{\Psi}$ is also a Riesz basis for $L^{2}(\mathbb{R})$. The basis $\Psi$ is called a primal wavelet basis, $\tilde{\Psi}$ is called a dual wavelet basis. Dual scaling basis $\tilde{\Phi}$, dual scaling functions $\tilde{\phi}$ and $\tilde{\phi}_{j, k}$, and dual refinement coefficients $\tilde{h}_{k}$ are defined in a similar way.

We define wavelet coefficients as

$$
\begin{equation*}
g_{n}=(-1)^{n} \tilde{h}_{1-n}, \quad \tilde{g}_{n}=(-1)^{n} h_{1-n} . \tag{5}
\end{equation*}
$$

Wavelets are then given by

$$
\begin{equation*}
\psi(x)=\sum_{n \in \mathbb{Z}} g_{n} \phi(2 x-n), \quad \tilde{\psi}(x)=\sum_{n \in \mathbb{Z}} \tilde{g}_{n} \tilde{\phi}(2 x-n) \quad \text { for all } \quad x \in \mathbb{R} . \tag{6}
\end{equation*}
$$

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We say that the wavelet $\psi$ has $n$ vanishing moments, if

$$
\begin{equation*}
\int_{\mathbb{R}} x^{l} \phi(x)=0, \quad \text { for } l=0, \ldots, n-1 \tag{7}
\end{equation*}
$$

It is equivalent with the polynomial exactness of order $n$ of the dual scaling functions. It means that any polynomial up to order $n-1$ lies in $\tilde{V}_{j}$.

The concept of wavelet bases on the interval is similar. Some scaling functions and wavelets on the interval are just the restrictions of scaling functions and wavelets on the real line, only at the boundaries special functions have to be constructed, see Section 2.

In this paper, we focus on scaling functions and wavelets from [5]. Here the primal scaling functions are B-splines of order $N$ and dual scaling functions are constructed such that the primal wavelet has $\tilde{N}$ vanishing moments. It implies that the polynomial exactness of primal scaling functions is $N$ and the polynomial exactness of dual scaling functions is $\tilde{N}$. Figure 1 shows the graphs of several scaling functions denoted by ${ }_{N} \phi$ and ${ }_{N, \tilde{N}} \tilde{\phi}$ and wavelets denoted by ${ }_{N, \tilde{N}} \psi$ and ${ }_{N, \tilde{N}} \tilde{\psi}$.

Recall that the Sobolev regularity $\gamma$ of a function $f$ is defined by

$$
\begin{equation*}
\gamma:=\sup \left\{s: f \in H^{s}\right\} \tag{8}
\end{equation*}
$$

where $H^{s}$ denotes the Sobolev space. It is known that the Sobolev regularity of the primal scaling function $\phi$ is $\gamma=N-\frac{1}{2}$. The Sobolev regularity of the dual scaling functions from [5] is shown in Table 1.

Table 1: Sobolev regularity $\tilde{\gamma}$ of dual scaling functions, $N$ and $\tilde{N}$ denotes the polynomial exactness of primal and dual scaling functions, respectively.

| $N$ | $\tilde{N}$ | $\tilde{\gamma}$ | $N$ | $\tilde{N}$ | $\tilde{\gamma}$ | $N$ | $\tilde{N}$ | $\tilde{\gamma}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 2 | 0.441 | 3 | 3 | 0.175 | 4 | 6 | 0.344 |
| 2 | 4 | 1.175 | 3 | 5 | 0.793 | 4 | 8 | 0.862 |
| 2 | 6 | 1.793 | 3 | 7 | 1.344 | 4 | 10 | 1.363 |

## 2 Optimized construction of spline-wavelet bases on the interval

In this section, we briefly review the construction of a spline-wavelet basis on the interval from [3, 4]. The primal scaling bases will be the same as bases designed in [2], because they are known to be well-conditioned. Let $N$ be the desired order of polynomial exactness of the primal scaling basis and let $\mathbf{t}^{j}=\left(t_{k}^{j}\right)_{k=-N+1}^{2^{j}+N-1}$ be the Schoenberg sequence of knots:

$$
\begin{aligned}
t_{k}^{j} & =0 \text { for } k=-N+1, \ldots, 0 \\
t_{k}^{j} & =\frac{k}{2^{j}} \text { for } k=1, \ldots 2^{j}-1 \\
t_{k}^{j} & =1 \text { for } k=2^{j}, \ldots, 2^{j}+N-1
\end{aligned}
$$

Figure 1: Scaling functions and wavelets derived from B-splines [5].









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The corresponding B-splines of order $N$ are defined by

$$
\begin{equation*}
B_{k, N}^{j}(x):=\left(t_{k+N}^{j}-t_{k}^{j}\right)\left[t_{k}^{j}, \ldots, t_{k+N}^{j}\right]_{t}(t-x)_{+}^{N-1}, \quad x \in[0,1], \tag{9}
\end{equation*}
$$

where $(x)_{+}:=\max \{0, x\}$ and $\left[t_{1}, \ldots t_{N}\right]_{t} f$ is the $N$-th divided difference of $f$. The set $\Phi_{j}$ of primal scaling functions is then simply defined as

$$
\begin{equation*}
\phi_{j, k}=2^{j / 2} B_{k, N}^{j}, \quad k=-N+1, \ldots, 2^{j}-1, \quad j \geq 0 \tag{10}
\end{equation*}
$$

The desired property of the dual scaling basis $\tilde{\Phi}$ is biorthogonality to $\Phi$ and the polynomial exactness of order $\tilde{N}$. Let $\tilde{\phi}$ be a dual scaling function which was designed in [5] and which is shifted so that its support is $[-\tilde{N}+1, N+\tilde{N}-1]$. In this case $\tilde{N} \geq N$ and $\tilde{N}+N$ must be an even number. We define inner scaling functions as translations and dilations of $\tilde{\phi}$ :

$$
\begin{equation*}
\theta_{j, k}=2^{j / 2} \tilde{\phi}\left(2^{j} \cdot-k\right), \quad k=\tilde{N}-1, \ldots 2^{j}-N-\tilde{N}+1 . \tag{11}
\end{equation*}
$$

There will be two types of basis functions at each boundary. Basis functions of the first type are defined to preserve the polynomial exactness:

$$
\theta_{j, k}=\left.2^{j / 2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2}\left\langle p_{k+N-1}^{\tilde{N}-1}, \phi(\cdot-l)\right\rangle \tilde{\phi}\left(2^{j} \cdot-l\right)\right|_{[0,1]}, \quad k=1-N, \ldots, \tilde{N}-N .
$$

Here $p_{0}^{\tilde{N}-1}, \ldots, p_{\tilde{N}-1}^{\tilde{N}-1}$ is a basis of the space of all algebraic polynomials on $[0,1]$ of degree less or equal to $\tilde{N}-1$. It can be shown [4] that the resulting dual scaling basis $\tilde{\Phi}$ does not depend on the choice of a polynomial basis. The basis functions of the second type are defined as

$$
\begin{equation*}
\theta_{j, k}=\left.2^{\frac{j+1}{2}} \sum_{l=\tilde{N}-1-2 k}^{N+\tilde{N}-1} \tilde{h}_{l} \tilde{\phi}\left(2^{j+1} \cdot-2 k-l\right)\right|_{[0,1]}, \quad k=\tilde{N}-N+1, \ldots, \tilde{N}-2, \tag{12}
\end{equation*}
$$

where $\tilde{h}_{l}$ are scaling coefficients corresponding to $\tilde{\phi}$.
The boundary functions at the right boundary are defined to be symmetrical with the left boundary functions:

$$
\begin{equation*}
\theta_{j, k}=\theta_{j, 2^{j}-N+1-k}(1-\cdot), \quad k=2^{j}-N-\tilde{N}+2, \ldots, 2^{j}-1 \tag{13}
\end{equation*}
$$

Since the set $\Theta_{j}:=\left\{\theta_{j, k}: k=-3, \ldots, 2^{j}-1\right\}$ is not biorthogonal to $\Phi_{j}$, we derive a new set $\tilde{\Phi}_{j}$ from $\Theta_{j}$ by biorthogonalization. Let $\boldsymbol{\Gamma}_{j}=\left(\left\langle\phi_{j, k}, \theta_{j, l}\right\rangle\right)_{j, l=-N+1}^{2^{j}-1}$, then viewing $\tilde{\Phi}_{j}$ and $\Theta_{j}$ as column vectors we define

$$
\begin{equation*}
\tilde{\Phi}_{j}:=\boldsymbol{\Gamma}_{j}^{-T} \Theta_{j} . \tag{14}
\end{equation*}
$$

Since the construction of the corresponding wavelets is long and technical we can not provide it here. A detailed description can be found in [4].

## 3 Computation of biorthogonalization matrices

As already mentioned, due to the low Sobolev regularity of the function $\theta_{j, k}$ it can be a problem to compute the matrix $\boldsymbol{\Gamma}_{j}$ by quadrature rules. Fortunately, we are able to compute the matrix $\boldsymbol{\Gamma}_{j}$ exactly up to the round off errors. It is easy to verify that the matrix $\boldsymbol{\Gamma}_{j}$ has the structure

$$
\Gamma_{j}=\left(\begin{array}{ccc}
\boldsymbol{\Gamma}_{L} & &  \tag{15}\\
& \mathbf{I}_{j} & \\
& & \Gamma_{R}
\end{array}\right)
$$

where the matrices $\boldsymbol{\Gamma}_{L}$ and $\boldsymbol{\Gamma}_{R}$ does not depend on $j$ and $\mathbf{I}_{j}$ is the identity matrix of the size $2^{j}-N+1$. First, we compute $\boldsymbol{\Gamma}_{L}$. For $k=1-N, \ldots, \tilde{N}-2$ and $l=1-N, \ldots, \tilde{N}-N$ we have

$$
\begin{equation*}
\left\langle\phi_{j, k}, \theta_{j, l}\right\rangle=\sum_{m=-N-\tilde{N}+2}^{\tilde{N}-2} \sum_{n=0}^{\tilde{N}-1} c_{l, n}\left\langle(\cdot)^{n}, \phi(\cdot-m)\right\rangle\langle\phi(\cdot-k), \tilde{\phi}(\cdot-m)\rangle, \tag{16}
\end{equation*}
$$

$\underset{\sim}{\text { where }} c_{l, n}$ are coefficients of polynomials $p_{l}^{\tilde{N}-1}$ in (2). The function $\phi$ is piecewise polynomial and $\tilde{\phi}$ is refinable, the computation of integrals of product of a polynomial and a refinable function will be proposed in the next section. By using the refinement equation (3), we easily obtain the relations for the computation of the remaining entries of $\boldsymbol{\Gamma}_{L}$. We have for $k=-N+1, \ldots,-1$, $l=\tilde{N}-N, \ldots, \tilde{N}-2$,

$$
\begin{equation*}
\left\langle\phi_{j, k}, \theta_{j, l}\right\rangle=\sum_{m=\tilde{N}-1-2 l}^{N+\tilde{N}-1} \tilde{h}_{m}\left\langle\phi_{0, k}, \tilde{\phi}(\cdot-2 k-m)\right\rangle, \tag{17}
\end{equation*}
$$

and for $k=0, \ldots, \tilde{N}-2, l=\tilde{N}-N, \ldots, \tilde{N}-2$,

$$
\begin{equation*}
\left\langle\phi_{j, k}, \theta_{j, l}\right\rangle=2^{-1} \sum_{m=\tilde{N}-1-2 l}^{N+\tilde{N}-1} h_{2 k-2 l+m} \tilde{h}_{m} . \tag{18}
\end{equation*}
$$

Since the boundary functions at the right boundary are defined to be symmetrical with the left boundary functions, the matrix $\boldsymbol{\Gamma}_{R}$ is obtained from $\boldsymbol{\Gamma}_{L}$ by reversing the ordering of rows and columns.

## 4 Computation of the moments of scaling functions

In applications one often needs to compute the integrals involving scaling functions or wavelets. By the wavelet equation (6), the integrals of a product of some function and a wavelet can be computed as a linear combination of integrals of product of the function and scaling functions. Therefore, we consider only the integrals involving scaling functions. Since many types of scaling functions have a low Sobolev regularity, the classical quadratures such as the Simpson rule are not useful in this case. Therefore, other quadratures were designed in [1, 9, 14]. They
require the knowledge of the precise values of scaling moments, that means the integrals of the form

$$
\begin{equation*}
M_{c, d}^{i}:=\int_{c}^{d} x^{i} \phi(x) d x \tag{19}
\end{equation*}
$$

where $i=0, \ldots, M$ for some given $M$ and $\phi$ is a refinable function. If $\operatorname{supp} \phi \subset\langle c, d\rangle$, then $M_{i}:=M_{c, d}^{i}$ can be simply computed by the reccurence formula

$$
\begin{equation*}
M_{i}:=\int_{-\infty}^{\infty} x^{i} \phi(x) d x=\frac{1}{2^{i+1}-2} \sum_{l=1}^{i} \sum_{n \in \mathbb{Z}}\binom{i}{l} h_{n} n^{i-l} M_{l} . \tag{20}
\end{equation*}
$$

If supp $\phi$ is not contained in $[c, d]$, then the computation of $M_{c, d}^{i}$ is more difficult. The method of computation of precise values of $M_{c, d}^{i}$ in the case that supp $\phi$ is not a subset of $[c, d], c, d \in \mathbb{N}$, has been proposed in $[10,11]$ for the Daubechies wavelets and in [9] for more general wavelets. We should also mention the method for the computation of biorthogonalization matrices in [13]. This method enables to find the biorthogonalization matrices as a solution of the system of linear algebraic equations, but it can be used only for wavelet bases with nested boundary functions, which is not our case. In this chapter we propose a novel method for the computation of the moments $M_{c, d}^{i}$ for a refinable function $\phi$ with compact support and any dyadic points $c, d$. The advantage of our method is that it leads to solving the system of linear algebraic equation of much less size than the system in [9].

### 4.1 Computation of antiderivatives of scaling functions

Integrating the refinement equation (3), we find that an antiderivative of a scaling function $\Phi^{[1]}(x)=\int_{-\infty}^{x} \phi(s) d s$ satisfies the refinement equation

$$
\begin{equation*}
\Phi^{[1]}(x)=\sum_{k \in \mathbb{Z}} \frac{h_{k}}{2} \Phi^{[1]}(2 x-k), \quad x \in \mathbb{R} . \tag{21}
\end{equation*}
$$

Note that the function $\Phi^{[1]}$ is generally not compactly supported. Assuming that supp $\phi=[a, b]$, the function $\Phi^{[1]}$ is vanishing on $(-\infty, a]$ and $\Phi^{[1]}$ is constant on $[b, \infty)$. Due to the noncompactness of the support we could not find exact values of $\Phi^{[1]}$ at integer points by solving an eigenvalue problem as in [6]. However, we show that we are able to find these values as a solution of a system of linear algebraic equations.

Lemma 4.1 Let $\phi$ be a refinable function with compact support $[a, b]$, where $a, b \in \mathbb{Z}$, and let $\phi$ be normalized as $\int_{\mathbb{R}} \phi(x) d x=1$. Then the values of $\Phi^{[1]}$ at integers are given by

$$
\begin{array}{ll}
\Phi^{[1]}(k)=0, & k \leq a,  \tag{22}\\
\Phi^{[1]}(k)=\sum_{l=a}^{b-1} \frac{h_{2 k-l}}{2} \Phi^{[1]}(l)+\sum_{l=b}^{\infty} \frac{h_{2 k-l}}{2}, & a<k<b, \quad k \in \mathbb{Z}, \\
\Phi^{[1]}(k)=1, & b \leq k .
\end{array}
$$

Proof. It is clear that $\Phi^{[1]}(k)=0$ for $k \leq a$ and $\Phi^{[1]}(k)=\int_{a}^{b} \phi(x) d x=1$ for $k \geq b$. From the refinement equation (21) it follows that

$$
\begin{align*}
\Phi^{[1]}(k) & =\sum_{l \in \mathbb{Z}} \frac{h_{l}}{2} \Phi^{[1]}(2 k-l)=\sum_{l \in \mathbb{Z}} \frac{h_{2 k-l}}{2} \Phi^{[1]}(l)  \tag{23}\\
& =\sum_{l=a}^{b-1} \frac{h_{2 k-l}}{2} \Phi^{[1]}(l)+\sum_{l=b}^{\infty} \frac{h_{2 k-l}}{2}, \quad a<k<b, \quad k \in \mathbb{Z} .
\end{align*}
$$

Hence, we can find precise values of $\Phi^{[1]}$ at integers solving the system (23) and then applying the refinement equation $J$ times, we obtain a precise value of $\Phi^{[1]}$ at any dyadic point $\frac{k}{2^{J}}, k \in \mathbb{Z}$.

For evaluating integrals of refinable functions we will also need $\Phi^{[n+1]}(x):=\int_{-\infty}^{x} \Phi^{[n]}(s) d s$. Integrating (3), we obtain the refinement equation:

$$
\begin{equation*}
\Phi^{[n]}(x)=\sum_{k \in \mathbb{Z}} \frac{h_{k}}{2^{n}} \Phi^{[n]}(2 x-k) \tag{24}
\end{equation*}
$$

In the next theorem we show that we are able to find the values of $\Phi^{[n]}$ at integers and then at any dyadic point.

Theorem 4.2 Let $\phi$ be a refinable function with compact support $[a, b], a, b \in \mathbb{Z}$, and $\phi$ be normalized by $\int_{\mathbb{R}} \phi(x) d x=1$. Then the following relations hold:

$$
\begin{array}{ll}
\Phi^{[n]}(k)=0, & k \leq a,  \tag{25}\\
\Phi^{[n]}(k)=\sum_{l=a}^{b-1} \frac{h_{2 k-l}}{2^{n}} \Phi^{[n]}(l)+\sum_{l=b}^{\infty} \frac{h_{2 k-l}}{2^{n}} \sum_{m=1}^{n} \Phi^{[m]}(b) \frac{(l-b)^{n-m}}{(n-m)!}, & a<k<b, k \in \mathbb{Z}, \\
\Phi^{[n]}(b)=\frac{1}{2^{n}-2} \sum_{m=1}^{n-1} \frac{\Phi^{[m]}(b)}{(n-m)!} \sum_{l=a}^{b} h_{l}(b-l)^{n-m}, & b \leq k .
\end{array}
$$

Proof. We prove this theorem by induction. For $n=1$ relations (25) hold by Lemma 4.1. Now let us suppose that the above system of equations hold for $k=1, \ldots, n-1$. Then

$$
\begin{align*}
\Phi^{[n]}(b) & =\sum_{m \in \mathbb{Z}} \frac{h_{2 b-m}}{2^{n}} \Phi^{[n]}(m)=\sum_{m \in \mathbb{Z}} \frac{h_{2 b-m}}{2^{n}}\left(\Phi^{[n]}(b)+\int_{b}^{m} \Phi^{[n-1]}(x) d x\right)  \tag{26}\\
& =\sum_{m \in \mathbb{Z}} \frac{h_{2 b-m}}{2^{n}}\left(\Phi^{[n]}(b)+\int_{b}^{m} \sum_{l=1}^{n-1} \Phi^{[l]}(b) \frac{(x-b)^{n-1-l}}{(n-1-l)!} d x\right) \\
& =\sum_{m \in \mathbb{Z}} \frac{h_{2 b-m}}{2^{n}} \Phi^{[n]}(b)+\sum_{m \in \mathbb{Z}} \frac{h_{2 b-m}}{2^{n}} \sum_{l=1}^{n-1} \Phi^{[l]}(b) \frac{(m-b)^{n-l}}{(n-l)!} .
\end{align*}
$$

Integrating (3), we obtain $\sum_{m \in \mathbb{Z}} h_{m}=2$. Thus

$$
\begin{equation*}
\Phi^{[n]}(b)=\frac{1}{2^{n}-2} \sum_{m=1}^{n-1} \frac{\Phi^{[m]}(b)}{(n-m)!} \sum_{l=a}^{b} h_{l}(b-l)^{n-m} \tag{27}
\end{equation*}
$$

For $k \geq b$ we have

$$
\begin{align*}
\Phi^{[n]}(k) & =\int_{a}^{b} \Phi^{[n-1]}(x) d x+\int_{b}^{k} \Phi^{[n-1]}(x) d x  \tag{28}\\
& =\Phi^{[n]}(b)+\int_{b}^{k} \sum_{m=1}^{n-1} \Phi^{[m]}(b) \frac{(x-b)^{n-1-m}}{(n-1-m)!} d x \\
& =\Phi^{[n]}(b)+\sum_{m=1}^{n-1} \Phi^{[m]}(b) \frac{(k-b)^{n-m}}{(n-m)!} \\
& =\sum_{m=1}^{n} \Phi^{[m]}(b) \frac{(k-b)^{n-m}}{(n-m)!}
\end{align*}
$$

Now using the refinement equation (24), we obtain for $k \in \mathbb{Z}, a<k<b$,

$$
\begin{align*}
\Phi^{[n]}(k) & =\sum_{l=a}^{b} \frac{h_{l}}{2^{n}} \Phi^{[n]}(2 k-l)=\sum_{l \in \mathbb{Z}} \frac{h_{2 k-l}}{2^{n}} \Phi^{[n]}(l)  \tag{29}\\
& =\sum_{l=a}^{b-1} \frac{h_{2 k-l}}{2^{n}} \Phi^{[n]}(l)+\sum_{l=b}^{\infty} \frac{h_{2 k-l}}{2^{n}} \sum_{m=1}^{n} \Phi^{[m]}(b) \frac{(l-b)^{n-m}}{(n-m)!} .
\end{align*}
$$

Note that the size of the system of equations depends linearly on $N$ and $\tilde{N}$. The method in [9] leads to solving the systems of the size depending quadratically on $N$ and $\tilde{N}$.

### 4.2 Evaluation of moments

Now we can evaluate moments $M_{c, d}^{i}$ by the formula from the following theorem.

Theorem 4.3 Let $\phi$ be a refinable function and let $\Phi^{[n]}$ be defined as above. Then

$$
\begin{equation*}
M_{c, d}^{i}=\int_{c}^{d} x^{i} \phi(x) d x=\sum_{l=0}^{i}(-1)^{l} \frac{i!}{(i-l)!}\left(d^{i-l} \Phi^{[l+1]}(d)-c^{i-l} \Phi^{[l+1]}(c)\right) . \tag{30}
\end{equation*}
$$

Proof. We prove Theorem 4.3 by induction. Clearly for $i=0$ the relation (30) is valid. Let us now suppose that (30) holds for $0, \ldots, i$. Using integration by parts, we obtain

$$
\begin{align*}
\int_{c}^{d} x^{i+1} \phi(x) d x & =\left[x^{i+1} \Phi^{[1]}(x)\right]_{c}^{d}-(i+1) \int_{c}^{d} x^{i} \Phi^{[1]}(x) d x  \tag{31}\\
& =\left[x^{i+1} \Phi^{[1]}(x)\right]_{c}^{d}-(i+1) \sum_{l=0}^{i} \frac{(-1)^{l} i!}{(i-l)!}\left(d^{i-l} \Phi^{[l+2]}(d)-c^{i-l} \Phi^{[l+2]}(c)\right) \\
& =\left[x^{i+1} \Phi^{[1]}(x)\right]_{c}^{d}-(i+1) \sum_{l=1}^{i+1} \frac{(-1)^{l-1} i!}{(i-l+1)!}\left(d^{i+1-l} \Phi^{[l+1]}(d)-c^{i+1-l} \Phi^{[l+1]}(c)\right) \\
& =\sum_{l=0}^{i+1}(-1)^{l} \frac{(i+1)!}{(i+1-l)!}\left(d^{i+1-l} \Phi^{[l+1]}(d)-c^{i+1-l} \Phi^{[l+1]}(c)\right),
\end{align*}
$$

which proves the assertion.

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# A PRIORI ERROR ANALYSIS OF BDF-DG METHOD FOR NONLINEAR CONVECTION-DIFFUSION EQUATION 

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#### Abstract

We deal with a scalar nonstationary convection-diffusion equation with nonlinear convective as well as diffusive terms. We present a discretization of this problem by a combination of the discontinuous Galerkin method for the space semi-discretization and the backward Euler method for the time discretization. Moreover, under some assumptions on the nonlinear terms, domain partitions and the regularity of the exact solution, we introduce a priori error estimates in the discrete $L^{\infty}\left(L^{2}\right)$-norm and $L^{2}\left(H^{1}\right)$-seminorm. A set of numerical experiments verifying the theoretical results via evaluating the experimental orders of convergence with respect to mesh size and time step is presented.


Key words and phrases. Discontinuous Galerkin method, convection-diffusion equation, backward Euler method, a priori error estimates, experimental order of convergence.
Mathematics Subject Classification. 65M60, 65M15, 65M12, 65L06.

## 1 Introduction

Our aim is to develop a sufficiently robust, accurate and efficient numerical method for the solution of the viscous compressible flow problems. Due to the lack of the theory concerning an existence of the solution of the Navier-Stokes equations we consider the model problem represented by a scalar unsteady convection-diffusion equation with nonlinear convection as well as diffusion.

Among a wide class of numerical methods, the discontinuous Galerkin method (DG mehod) seems to be a promising technique for the solution of convection-diffusion problems. This DG technique is based on the idea to approximate the solution of an initial-boundary value problem by piecewise polynomial functions over a finite element mesh, without any requirement on interelement continuity, for a survey, see, e.g., [2].

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Within this paper we first consider the space semi-discretization of the model problem with the aid of three variants of DG method for stabilization of the diffusion term, see [1]. Then the resulting semi-discrete scheme is discretized in time by semi-implicit approach via the backward difference formula (BDF), the detailed treatment can be found in [6].

This article represents a generalization of research papers [3], [4], [5], [6], where the linear diffusion term was considered. Moreover, let us cite works [7], [9], where simpler forms of nonlinear diffusion were analyzed.

## 2 Problem formulation

We consider the following unsteady nonlinear convection-diffusion problem: Let $\Omega \subset \mathbb{R}^{d}$, $d=2,3$, be a bounded open polygonal (if $d=2$ ) or polyhedral (if $d=3$ ) domain with Lipschitz-continuous boundary $\partial \Omega$, and $T>0$. We seek a function $u: Q_{T}=\Omega \times(0, T) \rightarrow \mathbb{R}$ such that
(a) $\frac{\partial u}{\partial t}+\nabla \cdot \vec{f}(u)=\operatorname{div}(\mathbb{K}(u) \nabla u)+g \quad$ in $Q_{T}$,
(b) $\left.u\right|_{\partial \Omega \times(0, T)}=u_{D}$,
(c) $u(x, 0)=u^{0}(x), \quad x \in \Omega$,
where $g: Q_{T} \rightarrow \mathbb{R}, u_{D}: \partial \Omega \times(0, T) \rightarrow \mathbb{R}, u^{0}: \Omega \rightarrow \mathbb{R}$ are given functions, the vectorvalued function $\vec{f}=\left(f_{1}, \ldots, f_{d}\right): \mathbb{R} \rightarrow \mathbb{R}^{d}$ represents convective terms and the regular matrix $\mathbb{K}(u) \in \mathbb{R}^{d, d}$ (even nonsymmetric) plays a role of nonlinear anisotropic diffusive coefficients.

We generalize the approach from [9], where a nonlinear diffusion is studied in form $-\operatorname{div}(\beta(u) \nabla u)$ with a suitable scalar function $\beta(u)$. Moreover, if $\mathbb{K}(u)=\varepsilon \mathbb{I}$, where $\varepsilon$ is a positive constant and $I I \in \mathbb{R}^{d, d}$ the unit matrix, then the problem (1) - (3) reduces to the equation with linear diffusion considered in [6].

The initial boundary value problem (1) - (3) is equipped with the initial condition (3) and the Dirichlet boundary condition (2) prescribed on the whole boundary but it is also possible to consider mixed Dirichlet-Neumann boundary conditions.

In what follows, we shall assume that the continuous problem (1) - (3) has a unique sufficiently regular solution $u$. Its regularity, used in the theory of error estimates, will be specified later in Section 4.

Further, we shall introduce standard notation for function spaces and their norms $\|\cdot\|$ and seminorms $|\cdot|$. Let $k \geq 0$ be a integer and $p \in[1, \infty]$. We use the well-known Lebesgue and Sobolev spaces $L^{p}(\Omega), H^{k}(\Omega)$, Bochner spaces $L^{p}(0, T ; X)$ of functions defined in $(0, \mathrm{~T})$ with values in Banach space $X$ and the spaces $C^{k}([0, T] ; X)$ of $k$-times continuously differentiable mappings of the interval $[0, \mathrm{~T}]$ with values in $X$.

## 3 Discretization

Let $\mathcal{T}_{h}(h>0)$ be a family of the partitions of the closure $\bar{\Omega}$ of the domain $\Omega$ into a finite number of closed mutually disjoint polygons (if $\mathrm{d}=2$ ) or polyhedra (if $\mathrm{d}=3$ ) $K$. We call $\mathcal{T}_{h}=\{K\}_{K \in \mathcal{T}_{h}}$ a triangulation of $\Omega$ and do not require the conforming properties from the finite element method, i.e., more general elements (even non-convex) and the so-called hanging nodes are allowed. The symbols $h_{K}$ and $\rho_{K}$ stand for the diameter of $K$ and radius of the largest $d$-dimensional ball inscribed into $K$, respectively. By $\mathcal{F}_{h}$ we denote the smallest possible set of all open $(d-1)$-dimensional faces (open edges when $d=2$ or open faces when $d=3$ ) of all elements $K \in \mathcal{T}_{h}$. Further, we label by $\mathcal{F}_{h}^{I}$ the set of all $\Gamma \in \mathcal{F}_{h}$ that are contained in $\Omega$ (inner faces) and by $\mathcal{F}_{h}^{D}$ the set of all $\Gamma \in \mathcal{F}_{h}$ that $\Gamma \subset \partial \Omega$ (Dirichlet faces). Obviously, $\mathcal{F}_{h}=\mathcal{F}_{h}^{I} \cup \mathcal{F}_{h}^{D}$. Finally, for each $\Gamma \in \mathcal{F}_{h}$, we define a unit normal vector $\vec{n}_{\Gamma}$. We assume that $\vec{n}_{\Gamma}, \Gamma \subset \partial \Omega$, has the same orientation as the outer normal of $\partial \Omega$. For $\vec{n}_{\Gamma}, \Gamma \in \mathcal{F}_{h}^{I}$, the orientation is arbitrary but fixed for each edge.

DG method allows to treat with different polynomial degrees over elements. Therefore, we assign a local Sobolev index $s_{K} \in \mathbb{N}$ and local polynomial degree $p_{K} \in \mathbb{N}$ to each $K \in \mathcal{T}_{h}$. Then we set the vectors

$$
\begin{equation*}
\mathrm{s} \equiv\left\{s_{K}, K \in \mathcal{T}_{h}\right\}, \quad \mathrm{p} \equiv\left\{p_{K}, K \in \mathcal{T}_{h}\right\} \tag{4}
\end{equation*}
$$

Over the triangulation $\mathcal{T}_{h}$ we define the so-called broken Sobolev space corresponding to the vector s

$$
\begin{equation*}
H^{\mathrm{s}}\left(\Omega, \mathcal{T}_{h}\right) \equiv\left\{v ;\left.v\right|_{K} \in H^{s_{K}}(K) \forall K \in \mathcal{T}_{h}\right\} \tag{5}
\end{equation*}
$$

with the seminorm $|v|_{H^{s}\left(\Omega, \mathcal{T}_{h}\right)} \equiv\left(\sum_{K \in \mathcal{T}_{h}}|v|_{H^{s} K(K)}^{2}\right)^{1 / 2}$, where $|\cdot|_{H^{s} K(K)}$ denotes the standard seminorm on the Sobolev space $H^{s_{K}}(K), K \in \mathcal{T}_{h}$. Moreover, the approximate solution is sought in a space of discontinuous piecewise polynomial functions associated with the vector p by

$$
\begin{equation*}
S_{h \mathrm{p}} \equiv S_{h \mathrm{p}}\left(\Omega, \mathcal{T}_{h}\right) \equiv\left\{v ; v \in L^{2}(\Omega),\left.v\right|_{K} \in P_{p_{K}}(K) \forall K \in \mathcal{T}_{h}\right\} \tag{6}
\end{equation*}
$$

where $P_{p_{K}}(K)$ denotes the space of all polynomials of degree $\leq p_{K}$ on $K, K \in \mathcal{T}_{h}$. In order to derive a priori $h p$ error estimates we additionally assume that there exists a constant $C_{P} \geq 1$ such that $p_{K} / p_{K^{\prime}} \leq C_{P} \forall K, K^{\prime} \in \mathcal{T}_{h}$ sharing a common face.

For each $\Gamma \in \mathcal{F}_{h}^{I}$ there exist two neighbouring elements $K_{p}, K_{n} \in \mathcal{T}_{h}$ such that $\Gamma \subset \bar{K}_{p} \cap \bar{K}_{n}$. We use a convention that $K_{n}$ lies in the direction of $\vec{n}_{\Gamma}$ and $K_{p}$ in the opposite direction of $\vec{n}_{\Gamma}$. For $v \in H^{1}\left(\Omega, \mathcal{T}_{h}\right)$, we introduce notation: $\left.v\right|_{\Gamma} ^{(p)}=$ trace of $\left.v\right|_{K_{p}}$ on $\Gamma$, and $\left.v\right|_{\Gamma} ^{(n)}=$ trace of $\left.v\right|_{K_{n}}$ on $\Gamma$, for traces of $v$ on edge $\Gamma$, which are different in general. Moreover,

$$
\begin{equation*}
[v]_{\Gamma}=\left.v\right|_{\Gamma} ^{(p)}-\left.v\right|_{\Gamma} ^{(n)}, \quad\langle v\rangle_{\Gamma}=\frac{1}{2}\left(\left.v\right|_{\Gamma} ^{(p)}+\left.v\right|_{\Gamma} ^{(n)}\right), \tag{7}
\end{equation*}
$$

denote the jump and mean value of function $v$ over the edge $\Gamma$, respectively. For $\Gamma \in \mathcal{F}_{h}^{D}$ there exists an element $K_{p} \in \mathcal{T}_{h}$ such that $\Gamma \subset \overline{K_{p}} \cap \partial \Omega$. Then for $v \in S_{h p}$, we put: $\left.v\right|_{\Gamma} ^{(p)}=$ trace of $\left.v\right|_{K_{p}}$ on $\Gamma$, and $\langle v\rangle_{\Gamma}=[v]_{\Gamma}=\left.v\right|_{\Gamma} ^{(p)}$.

### 3.1 Semi-discrete DG scheme

Now, we recall the space semi-discrete DG scheme presented in [8]. The crucial item of the DG formulation is the treatment of the nonlinear diffusive term. In order to replace the interelement continuity, we add some stabilization and penalty terms into formulation of the discrete problem. A particular attention should be also paid to the nonlinear convective term, where we employ the concept of a numerical flux, known from the finite volume method.

Therefore, we say that $u_{h} \in C^{1}\left(0, T ; S_{h \mathrm{p}}\right)$ is the semi-discrete solution of problem (1) - (3) if $\left(u_{h}(0), v_{h}\right)=\left(u^{0}, v_{h}\right) \forall v_{h} \in S_{h \mathrm{p}}$ and

$$
\begin{array}{r}
\left(\frac{\partial u_{h}(t)}{\partial t}, v_{h}\right)+b_{h}\left(u_{h}(t), v_{h}\right)+a_{h}^{\Theta}\left(u_{h}(t), v_{h}\right)+\alpha J_{h}^{\sigma}\left(u_{h}(t), v_{h}\right)=l_{h}^{\Theta}\left(u_{h}(t), v_{h}\right)(t)  \tag{8}\\
\forall v_{h} \in S_{h \mathrm{p}}, \forall t \in(0, T),
\end{array}
$$

where $(\cdot, \cdot)$ denotes the $L^{2}$-scalar product and

$$
\begin{align*}
a_{h}^{\Theta}(u, v)= & \sum_{K \in \mathcal{T}_{h}} \int_{K} \mathbb{K} K(u) \nabla u \cdot \nabla v \mathrm{~d} x-\sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma}\langle\mathbb{K}(u) \nabla u \cdot \vec{n}\rangle_{\Gamma}[v]_{\Gamma} \mathrm{d} S \\
& +\Theta \sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma}\left\langle\mathbb{K}(u)^{T} \nabla v \cdot \vec{n}\right\rangle_{\Gamma}[u]_{\Gamma} \mathrm{d} S,  \tag{9}\\
b_{h}(u, v)= & -\sum_{K \in \mathcal{T}_{h}} \int_{K} \vec{f}(u) \cdot \nabla v \mathrm{~d} x+\sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma} H\left(\left.u\right|_{\Gamma} ^{(p)},\left.u\right|_{\Gamma} ^{(n)}, \vec{n}_{\Gamma}\right)[v]_{\Gamma} \mathrm{d} S,  \tag{10}\\
J_{h}^{\sigma}(u, v)= & \sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma} \sigma[u]_{\Gamma}[v]_{\Gamma} \mathrm{d} S,  \tag{11}\\
l_{h}^{\Theta}(u, v)(t)= & \int_{\Omega} g(t) v \mathrm{~d} x+\sum_{\Gamma \in \mathcal{F}_{h}^{D}} \int_{\Gamma}\left(\Theta \mathbb{K}(u)^{T} \nabla v \cdot \vec{n}_{\Gamma} u_{D}(t)+\sigma u_{D}(t) v\right) \mathrm{d} S . \tag{12}
\end{align*}
$$

According to value of parameter $\Theta$, we speak of symmetric (SIPG, $\Theta=-1$ ), incomplete (IIPG, $\Theta=0$ ) or nonsymmetric (NIPG, $\Theta=1$ ) variants of stabilization of DG method, i.e., we generally consider three variants of the diffusion form $a_{h}^{\Theta}$ and right-hand side form $l_{h}^{\Theta}$. Penalty terms are represented by $J_{h}^{\sigma}$ and the penalty parameter function $\sigma$ in (11) is defined by

$$
\left.\sigma\right|_{\Gamma}=\frac{C_{W}}{d(\Gamma)} \text { with } d(\Gamma)=\left\{\begin{array}{cl}
\min \left(h_{K_{p}} / p_{K_{p}}^{2}, h_{K_{n}} / p_{K_{n}}^{2}\right) & , \Gamma \in \mathcal{F}_{h}^{I}  \tag{13}\\
h_{K_{p}} / p_{K_{p}}^{2} & , \Gamma \in \mathcal{F}_{h}^{D}
\end{array}\right.
$$

where $C_{W}>0$ is a suitable constant depending on the used variant of scheme and on the degree of polynomial approximation. The value of multiplicative constant $\alpha$ before the penalty form $J_{h}^{\sigma}$ depends on the properties of matrix $\mathbb{K}$ and will be specified in Section 4. The symbol $\mathbb{K}(\cdot)^{T}$ stands for the transpose of matrix $\mathbb{K}(\cdot)$. As for the convective form $b_{h}$ we treat boundary terms similarly as in the finite volume method with the aid of numerical flux $H(u, v, \vec{n})$.

The problem (8) represents a system of ordinary differential equations (ODEs) for $u_{h}(t)$ which has to be discretized in time by a suitable method. Since these ODEs belong to the class
of stiff problems it is advantageous to use a semi-implicit approach which is introduced in the following subsection.

### 3.2 Semi-implicit BDF-DG scheme

The proposed approach, sidetracking the time step restriction typical for explicit schemes as well as the nonlinearity of the corresponding forms unfavourable for explicit schemes, is generally based on a suitable linearization of nonlinear forms. The linear terms are treated implicitly whereas the nonlinear ones explicitly which leads to a linear algebraic problem at each time step.

In the case of the scalar convection-diffusion problem (1) - (3), it is able to linearize only the diffusion form $a_{h}^{\Theta}(u, \cdot)$ due to its special form of the considered nonlinear diffusion - div $(\mathbb{K}(u) \nabla u)$. On the other hand, the nonlinearity appearing in the form $b_{h}(u, \cdot)$ and the used numerical flux (29) do not allow to linearize form $b_{h}$ in general that is why it has to be treated explicitly.

Now based on (9), for $\bar{u}, u, v \in H^{\mathrm{s}}\left(\Omega, \mathcal{T}_{h}\right)$ we define the linearized diffusion form

$$
\begin{align*}
a_{h L}^{\Theta}(\bar{u}, u, v) & =\sum_{K \in \mathcal{T}_{h}} \int_{K} \mathbb{K}(\bar{u}) \nabla u \cdot \nabla v \mathrm{~d} x-\sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma}\langle\mathbb{K}(\bar{u}) \nabla u \cdot \vec{n}\rangle_{\Gamma}[v]_{\Gamma} \mathrm{d} S \\
& +\Theta \sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma}\left\langle\mathbb{K}(\bar{u})^{T} \nabla v \cdot \vec{n}\right\rangle_{\Gamma}[u]_{\Gamma} \mathrm{d} S \tag{14}
\end{align*}
$$

which is linear with respect to its second and third components. Moreover it is consistent with $a_{h}^{\Theta}(\cdot, \cdot)$ by

$$
\begin{equation*}
a_{h}^{\Theta}(u, v)=a_{h L}^{\Theta}(u, u, v) \quad \forall u, v \in H^{\mathrm{s}}\left(\Omega, \mathcal{T}_{h}\right) \tag{15}
\end{equation*}
$$

In order to obtain a sufficiently accurate approximation with respect to the time coordinate we use the so-called BDF scheme for the solution of ODE problem (8). Moreover, for the nonlinear parts of $a_{h L}^{\Theta}(\cdot, \cdot, \cdot)$ we employ a suitable explicit higher order extrapolation which preserve a given order of accuracy and does not destroy the linearity of the problem at each time level, for more details see [6] and [8].

Let $0=t_{0}<t_{1}<\cdots<t_{r}=T$ be a partition of the interval $[0, T]$ and $\tau_{k} \equiv t_{k+1}-$ $t_{k}, k=0,1, \ldots, r-1$. We define the approximate solution of problem (1)-(3) as functions $u_{h}^{k} \approx u_{h}\left(t_{k}\right), t \in[0, T], k=1, \ldots, r$, satisfying the conditions
(a) $u_{h}^{k+1} \in S_{h \mathrm{p}}$,
(b) $\frac{1}{\tau_{k}}\left(\sum_{l=0}^{n} \beta_{l} u_{h}^{k+1-l}, v_{h}\right)+a_{h L}^{\Theta}\left(\sum_{l=1}^{n} \gamma_{l} u_{h}^{k+1-l}, u_{h}^{k+1}, v_{h}\right)+b_{h}\left(\sum_{l=1}^{n} \gamma_{l} u_{h}^{k+1-l}, v_{h}\right)$

$$
\begin{equation*}
+\alpha J_{h}^{\sigma}\left(u_{h}^{k+1}, v_{h}\right)=l_{h}^{\Theta}\left(\sum_{l=1}^{n} \gamma_{l} u_{h}^{k+1-l}, v_{h}\right)\left(t_{k+1}\right) \quad \forall v_{h} \in S_{h \mathrm{p}}, k=n-1, \ldots, r-1 \tag{16}
\end{equation*}
$$

(c) $u_{h}^{0}$ is $S_{h \mathrm{p}}$-approximation of $u^{0}$,
(d) $\quad u_{h}^{l} \in S_{h \mathrm{p}}, l=1, \ldots, n-1$ are given by a suitable one-step method,

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where $n \geq 1$ is the degree of the $\operatorname{BDF}$ scheme and the coefficients $\beta_{l}, l=0, \ldots, n$ and $\gamma_{l}, l=$ $1, \ldots, n$ depend on time steps $\tau_{k-l}, l=0, \ldots, n$, in general. We call this approach the $B D F-D G$ method.

The discrete problem (16), (a) - (d) is equivalent to a system of linear algebraic equations for each $t_{k+n} \in[0, T]$, which can be solved by a suitable solver, e.g. GMRES.

## 4 A priori error analysis

Our goal is now to analyze the error estimates of the approximate solution $u_{h}^{k}, k=1,2, \ldots, r$, obtained by the semi-implicit BDF-DG method (16), (b) of the first order (i.e. $n=1$ ), which really represents the semi-implicit linearization of the backward Euler method. The basic framework refers to [4], [6] and [9] with some generalization for considered problem.

In order to carry out the error analysis we need to specify the additional assumptions on mesh, numerical flux, nonlinear diffusion term and regularity of the exact solution $u$ of the continuous problem (1)-(3). Therefore, we assume that
(T) The triangulations $\mathcal{T}_{h}, h \in\left(0, h_{0}\right), h_{0}>0$, consist of a finite number of closed $d$-dimensional simplexes and/or parallelograms $K$ with mutually disjoint interiors. These elements are locally quasi-uniform, i.e., there exists a constant $C_{Q}>0$ such that $h_{K_{p}} \leq$ $C_{Q} h_{K_{n}} \forall K_{p}, K_{n} \in \mathcal{T}_{h}$ sharing face $\Gamma \in \mathcal{F}_{h}^{I}$, and shape-regular, i.e., there exists a constant $C_{S}>0$ such that $h_{K} \leq C_{S} \rho_{K} \forall K \in \mathcal{T}_{h}$.
(H) We shall assume that the numerical flux $H(u, v, \vec{n})$ is
(a) Lipschitz-continuous with respect to $u$, $v$, i.e.,

$$
\begin{equation*}
\left|H(u, v, \vec{n})-H\left(u^{*}, v^{*}, \vec{n}\right)\right| \leq C_{H}\left(\left|u-u^{*}\right|+\left|v-v^{*}\right|\right) \forall u, v, u^{*}, v^{*} \in \mathbb{R}, \forall \vec{n} \in \mathbb{R}^{d}, \tag{17}
\end{equation*}
$$

(b) consistent: $H(u, u, \vec{n})=\vec{f}(u) \cdot \vec{n} \quad \forall u \in \mathbb{R}, \forall \vec{n} \in \mathbb{R}^{d}$,
(c) conservative: $H(u, v, \vec{n})=-H(v, u,-\vec{n}) \forall u, v \in \mathbb{R}, \forall \vec{n} \in \mathbb{R}^{d}$.
(D) The diffusive matrix $\mathbb{K}(v)=\left\{k_{i j}(v)\right\}_{i, j=1}^{d}, k_{i j}(v): \mathbb{R} \rightarrow \mathbb{R}$, satisfies
(a) boundedness: $\|\mathbb{K}(v)\|_{\infty} \leq C_{U}$ and $\left\|\mathbb{K}(v)^{T}\right\|_{\infty} \leq C_{U} \forall v \in \mathbb{R}$,
(b) Lipschitz-continuity: $\left\|\mathbb{K}\left(v_{1}\right)-\mathbb{K}\left(v_{2}\right)\right\|_{\infty} \leq C_{L}\left|v_{1}-v_{2}\right| \forall v_{1}, v_{2} \in \mathbb{R}$,
(c) positive definiteness: $\mathbf{z}^{T} \mathbb{K}(v) \mathbf{z} \geq \alpha\|\mathbf{z}\|^{2}, \alpha>0, \forall v \in \mathbb{R}, \forall \mathbf{z} \in \mathbb{R}^{d}$,
where $\|\cdot\|_{\infty}$ represents the $l^{\infty}$-matrix norm, i.e., $\|\mathbb{K}\|_{\infty}=\underset{1 \leq i \leq n}{\max } \sum_{j=1}^{n}\left|k_{i j}\right|$.
( R ) The weak solution $u$ is sufficiently regular, namely
(a) $u \in L^{\infty}\left(0, T ; H^{\bar{s}}(\Omega)\right), \frac{\partial u}{\partial t} \in L^{\infty}\left(0, T ; H^{\bar{s}}(\Omega)\right), \frac{\partial^{2} u}{\partial t^{2}} \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$,
(b) $\|\nabla u(t)\|_{L^{\infty}(\Omega)} \leq C_{D} \quad$ for a.a. $t \in(0, T)$,
where $\bar{s}=\max \left\{s_{K}, s_{K} \in \mathrm{~s}\right\} \geq 2$.

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In what follows, for simplicity, we consider a uniform partition $t_{k}=k \tau, k=0,1, \ldots, r$, of the time interval $[0, T]$ with time step $\tau=T / r$, where $r>1$ is an integer. Let $e_{h}^{k}=u_{h}^{k}-u^{k}$, $k=0, \ldots, r$, be the discretization error and $\|\|\cdot\|\| \equiv\left(|\cdot|_{H^{1}\left(\Omega, \mathcal{T}_{h}\right)}^{2}+J_{h}^{\sigma}(\cdot, \cdot)\right)^{1 / 2}$. We set

$$
\begin{equation*}
\|e\|_{h, \tau, L^{\infty}\left(L^{2}\right)}^{2}=\max _{k=0, \ldots, r}\left\|e_{h}^{k}\right\|_{L^{2}(\Omega)}^{2}, \quad\|e\|_{h, \tau, L^{2}\left(H^{1}\right)}^{2}=\tau \frac{\alpha}{2} \sum_{k=0}^{r}\left\|e_{h}^{k}\right\|^{2} \tag{20}
\end{equation*}
$$

Now, we can proceed to formulation of the main result of this paper.
Theorem 4.1 Let assumptions $(T),(H),(D)$ and $(R)$ be valid. Let $u$ be the exact solution of the continuous problem (1) - (3). Let $t_{k}=k \tau, k=0,1, \ldots, r, \tau=T / r$, be a time partition of $[0, T]$ and let $u_{h}^{k}, k=0, \ldots, r$, be the approximate solution defined by (16), (a) - (d) for $n=1$ and let $\tau \leq 1 / 2$. Then there exist constants $\widetilde{C}=O(\exp (2 T(1+L / \alpha)))$ and $\widehat{C}=O(\exp (2 T(1+L / \alpha)))$ such that

$$
\begin{align*}
& \text { (a) }\|e\|_{h, \tau, L^{\infty}\left(L^{2}\right)}^{2} \leq \widetilde{C}\left(\sum_{K \in \mathcal{T}_{h}} \frac{h_{K}^{2 \mu_{K}-2}}{p_{K}^{2 s_{K}-3}}\left(h^{2}+h^{2} / \alpha+\alpha+1 / \alpha\right)+\tau^{2}(1+1 / \alpha)\right),  \tag{21}\\
& \text { (b) }\|e\|_{h, \tau, L^{2}\left(H^{1}\right)}^{2} \leq \widehat{C}\left(\sum_{K \in \mathcal{T}_{h}} \frac{h_{K}^{2 \mu_{K}-2}}{p_{K}^{2 s_{K}-4}}\left(h^{2}+h^{2} / \alpha+\alpha+1 / \alpha\right)+\tau^{2}(1+1 / \alpha)\right), \tag{22}
\end{align*}
$$

where $\mu_{K}=\min \left(p_{K}+1, s_{K}\right), K \in \mathcal{T}_{h}$, and $L$ is a constant from proof.
Proof. The main framework is based on the expression of the discretization error as $e_{h}^{k}=$ $\xi^{k}+\eta^{k}$ with $\xi^{k}=u_{h}^{k}-\Pi_{h \mathrm{p}} u^{k} \in S_{h \mathrm{p}}$ and $\eta^{k}=\Pi_{h \mathrm{p}} u^{k}-u^{k} \in H^{\mathrm{s}}\left(\Omega, \mathcal{T}_{h}\right)$, where $\Pi_{h \mathrm{p}} u^{k}$ is the $S_{h p}$-interpolation of $u^{k}=u\left(t_{k}\right), k=0, \ldots, r$. Consequently, the multiplicative trace inequality, inverse inequality and approximation properties of the space $S_{h \mathrm{p}}$, for more details see [5, Lemmas 4.2-4.4], are applied. The whole proof can be found in [8, Theorem 4.3.2].

Remark 4.2 In the case that $u$ is sufficiently regular exact solution and $p_{k}=p \forall K \in \mathcal{T}_{h}$, the estimates (21) and (22) imply that

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}=O\left(h^{p}+\tau\right), \quad\left\|u-u_{h}\right\|_{L^{2}\left(0, T ; H^{1}\left(\Omega, \mathcal{T}_{h}\right)\right)}=O\left(h^{p}+\tau\right) \quad \text { for } h \rightarrow 0+ \tag{23}
\end{equation*}
$$

## 5 Numerical examples

In this section we shall numerically verify the theoretical a priori error estimates of the presented BDF-DG scheme in the discrete $L^{\infty}\left(L^{2}\right)$-norm and $L^{2}\left(H^{1}\right)$-seminorm with respect to the time step $\tau$ and the mesh size $h$. The computational errors are evaluated at final time $T=r \tau$ in $L^{2}$-norm and $H^{1}$-seminorm, and we suppose that these errors behave according to the formulae

$$
\begin{equation*}
e_{h, \tau}^{0} \equiv\left\|e_{h}^{r}\right\|_{L^{2}(\Omega)} \approx \widetilde{D_{0}} h^{a^{0}}+\widehat{D_{0}} \tau^{b^{0}}, \quad e_{h, \tau}^{1} \equiv\left\|e_{h}^{r}\right\| \| \widetilde{D_{1}} h^{a^{1}}+\widehat{D_{1}} \tau^{b^{1}} \tag{24}
\end{equation*}
$$

The constants $\widetilde{D_{n}}, n=0,1$, are independent of $\tau$ and $\widehat{D_{n}}, n=0,1$, are independent of $h$. The values $a^{n}, b^{n}, n=0,1$, are the orders of accuracy of the method in the corresponding considered norms. We define the experimental order of convergence (EOC) by

$$
\begin{equation*}
a^{n}=\frac{\log \left(e_{h, \tau_{1}}^{n} / e_{h, \tau_{2}}^{n}\right)}{\log \left(\tau_{1} / \tau_{2}\right)} \quad \text { and } \quad b^{n}=\frac{\log \left(e_{h_{1}, \tau}^{n} / e_{h_{2}, \tau}^{n}\right)}{\log \left(h_{1} / h_{2}\right)}, \quad n=0,1 . \tag{25}
\end{equation*}
$$

We consider the 2D viscous Burgers equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}+\sum_{s=1}^{2} u \frac{\partial u}{\partial x_{s}}=\operatorname{div}(\mathbb{K}(u) \nabla u)+g \quad \text { in } \Omega \times(0, T) \tag{26}
\end{equation*}
$$

with the nonsymmetric matrix $K(w)$ in the following form

$$
\mathbb{I} K(w)=\varepsilon\left(\begin{array}{cc}
\frac{1}{2}(3+\arctan (w)) & \frac{1}{3}\left(2-\arctan ^{2}(w)\right)  \tag{27}\\
0 & 4+\arctan (w)
\end{array}\right) .
$$

We set $\varepsilon=0.02, \Omega=(0,1)^{2}, T=10$ and define the function $g$ and the initial and boundary conditions in such a way that the exact solution has the steady-state form

$$
\begin{equation*}
u\left(x_{1}, x_{2}, t\right)=2\left(1-\mathrm{e}^{-10 t}\right)\left(x_{1}^{2}+x_{2}^{2}\right) x_{1} x_{2}\left(1-x_{1}\right)\left(1-x_{2}\right) . \tag{28}
\end{equation*}
$$

In the form $b_{h}$ we use the numerical flux

$$
H\left(u_{1}, u_{2}, \vec{n}\right)= \begin{cases}\sum_{s=1}^{2} f_{s}\left(u_{1}\right) n_{s} & \text { if } D f>0  \tag{29}\\ \sum_{s=1}^{2} f_{s}\left(u_{2}\right) n_{s} & \text { if } D f \leq 0\end{cases}
$$

where $D f=\sum_{s=1}^{2} \frac{\partial f_{s}}{\partial u}\left(\left(u_{1}+u_{2}\right) / 2\right) n_{s}, \vec{n}=\left(n_{1}, n_{2}\right)$.
First, we investigate the convergence of the method with respect to $h$. Numerical experiments are carried out with the use of piecewise linear $\left(P^{1}\right)$, quadratic $\left(P^{2}\right)$ and cubic $\left(P^{3}\right)$ approximations on 6 regular triangular meshes having 128, 288, 512, 1152, 2048 and 4608 elements for SIPG, NIPG and IIPG variants. In order to guarantee the stability of the BDF-DG scheme with respect to penalty parameter $\sigma$, the parameter $C_{W}$ from (13) in each of the six cases is chosen according to [8, Table 6.3].

Table 1 shows computational errors in $L^{2}$-norm and $H^{1}$-seminorm at final time $T$ and the corresponding EOC for each IPG variant. Since $u(\cdot, \cdot, t)$ is sufficiently regular solution over $\Omega$ it follows from Remark 4.2 that the theoretical errors estimates are of order $O\left(h^{p}+\tau\right)$. On the other hand, we observe that the obtained numerical results indicate a better behaviour of EOC in $L^{2}$-norm, which is expected to be asymptotically $O\left(h^{p+1}\right)$ for $p$ odd and $O\left(h^{p}\right)$ for $p$ even in the case of NIPG and/or IIPG variant. Moreover, in the case of SIPG variant we observe optimal EOC for $p$ arbitrary. Further, the results for EOC in $H^{1}$-seminorm are in a quite good agreement with theoretical estimates, in other words, all IPG techniques produce optimal order of convergence $O\left(h^{p}\right)$.

Secondary, we verify experimentally the convergence of method in $L^{2}$-norm and $H^{1}$-seminorm with respect to the time step $\tau$. In order to restrain the discretization errors with respect to $h$, we use a fine mesh with 2048 triangles and piecewise cubic approximation $\left(P^{3}\right)$.

The computations were carried out with six different time steps $\tau_{l}$ for all three IPG variants, see Table 2. The computational error is evaluated at time $T$ in $L^{2}$-norm and $H^{1}$-seminorm, respectively. We observe that all considered IPG variants produce EOC of order $O(\tau)$ in $L^{2}$-norm and $H^{1}$-seminorm which is in good agreement with derived theoretical results.


Table 1: Convergence with respect to mesh size $h$ : Computational errors in $L^{2}$-norm $\left(e_{h, \tau}^{0}\right)$ and $H^{1}$-seminorm $\left(e_{h, \tau}^{1}\right)$ with corresponding EOC for all three IPG formulations and $P_{1}-P_{3}$

|  | SIPG |  | NIPG |  | IIPG |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $l$ | $\tau_{l}$ | $e_{h, \tau_{l}}^{0}$ | $b_{l}^{0}$ | $e_{h, \tau_{l}}^{0}$ | $b_{l}^{0}$ | $e_{h, \tau_{l}}^{0}$ | $b_{l}^{0}$ |
| 1 | $4.000 \mathrm{E}-02$ | $3.1685 \mathrm{E}-03$ | - | $3.1686 \mathrm{E}-03$ | - | $3.1687 \mathrm{E}-03$ | - |
| 2 | $2.000 \mathrm{E}-02$ | $1.6371 \mathrm{E}-03$ | 0.953 | $1.6374 \mathrm{E}-03$ | 0.953 | $1.6372 \mathrm{E}-03$ | 0.953 |
| 3 | $1.000 \mathrm{E}-02$ | $8.3140 \mathrm{E}-04$ | 0.977 | $8.3143 \mathrm{E}-04$ | 0.977 | $8.3138 \mathrm{E}-04$ | 0.977 |
| 4 | $5.000 \mathrm{E}-03$ | $4.1904 \mathrm{E}-04$ | 0.989 | $4.1901 \mathrm{E}-04$ | 0.989 | $4.1903 \mathrm{E}-04$ | 0.989 |
| 5 | $2.500 \mathrm{E}-03$ | $2.0983 \mathrm{E}-04$ | 0.998 | $2.0983 \mathrm{E}-04$ | 0.998 | $2.0983 \mathrm{E}-04$ | 0.998 |
| 6 | $1.250 \mathrm{E}-03$ | $1.0521 \mathrm{E}-04$ | 0.995 | $1.0522 \mathrm{E}-04$ | 0.996 | $1.0521 \mathrm{E}-04$ | 0.995 |
| $l$ | $\tau_{l}$ | $e_{h, \tau_{l}}^{1}$ | $b_{l}^{1}$ | $e_{h, \tau_{l}}^{1}$ | $b_{l}^{1}$ | $e_{h, \tau_{l}}^{1}$ | $b_{l}^{1}$ |
| 1 | $4.000 \mathrm{E}-02$ | $1.512 \mathrm{E}-02$ | - | $1.5152 \mathrm{E}-02$ | - | $1.5151 \mathrm{E}-02$ | - |
| 2 | $2.000 \mathrm{E}-02$ | $7.8278 \mathrm{E}-03$ | 0.953 | $7.8280 \mathrm{E}-03$ | 0.953 | $7.8278 \mathrm{E}-03$ | 0.953 |
| 3 | $1.000 \mathrm{E}-02$ | $3.9782 \mathrm{E}-03$ | 0.977 | $3.9778 \mathrm{E}-03$ | 0.977 | $3.9780 \mathrm{E}-03$ | 0.977 |
| 4 | $5.000 \mathrm{E}-03$ | $2.0051 \mathrm{E}-03$ | 0.989 | $2.0052 \mathrm{E}-03$ | 0.989 | $2.0045 \mathrm{E}-03$ | 0.989 |
| 5 | $2.500 \mathrm{E}-03$ | $1.0043 \mathrm{E}-03$ | 0.998 | $1.0042 \mathrm{E}-03$ | 0.998 | $1.0041 \mathrm{E}-03$ | 0.998 |
| 6 | $1.250 \mathrm{E}-03$ | $5.0352 \mathrm{E}-04$ | 0.995 | $5.0360 \mathrm{E}-04$ | 0.996 | $5.0351 \mathrm{E}-04$ | 0.995 |

Table 2: Convergence with respect to time step $\tau$ : Computational errors and corresponding EOC in $L^{2}$-norm $\left(e_{h, \tau}^{0}\right)$ and $H^{1}$-seminorm $\left(e_{h, \tau}^{1}\right)$ for all three IPG formulations.

## 6 Conclusion

We presented a higher order numerical scheme for the solution of scalar unsteady nonlinear convection-diffusion equation. The method, denoted by BDF-DG method, is based on the semi-discretization by the discontinuous Galerkin method in space and on the semi-implicit backward Euler method used for discretization in time. Under some additional assumptions we have derived a priori error estimates, namely $O\left(h^{p}+\tau\right)$ in the $L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$-norm and in the $L^{2}\left(0, T ; H^{1}(\Omega)\right)$-seminorm. Presented numerical examples verify the theoretical results.

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# ANALYSIS OF THE SIGNIFICANCE OF TECHNOLOGICAL PARAMETERS AT BRIQUETTING OF SELECTED TYPES OF HARDWOOD AND SOFTWOOD 

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#### Abstract

This paper describes shortly evaluation of experiment for research of parameter impact on final briquettes quality. The final briquettes quality (quality of solid high-grade biofuels) is mainly evaluated by briquettes density. Briquettes density is exactly defined by Standards for solid biofuels. At briquetting process exist parameters those with their changes we can markedly impact the final briquettes quality. One of the goals of our research was to detect the impact of some parameters on final briquettes quality. Experiment which was done on our department consists of two phases. These are closely described in this paper. From briquettes production point of view and also from briquetting machines constructions point of view is very important to know mutual interaction of these technological parameters on final briquettes quality. The main goal of this paper is to present results of analysis of the significance of the technological parameters at briquetting of selected types of hardwood and softwood.


Key words. briquetting, compacting, mathematical model, briquettes density
Mathematics Subject Classification: Primary 97M50, 62K15.

## 1 Introduction

With research of technological parameters we are dealing on our department for many years. We are trying to describe mathematically impact of technological parameters (compacting pressure, pressing temperature, material moisture, fraction size) changes on final briquette quality. Created mathematical model helps us by briquette density prediction before lonely real briquetting. We are able to say which parameters we have to set for achieving the density given by Standards. Also this mathematical model helps us by engineering of some mechanical parts of briquetting machines. Very important thing was specifying what will be the main goal of the experiment. The primary goal was to detect the impact and to analyze the significance of lonely parameters (listed above). But the conditions by briquetting process are very complicated. There are many external parameters which can influence the researched parameters impact and their mutual activity. For example type of briquetted material. It is not the same if we are briquetting some type of softwood, some type of

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hardwood or straw, or paper. For each material we need to find optimal technological parameters setting by briquetting. Everything comes from the briquetted material chemical composition. On the following picture 1 and table 1 you can see how large differences between some types of softwoods and hardwoods are. At first step we have done experiment by using one chosen type of softwood.


Table 1 Chemical composition of some domestic woods [1]


Figure 1 Main organic units in softwoods and hardwoods [1]

We choose pine sawdust for first step of experiment. We designed experimental plan for detecting of impact of changes of technological parameters at briquetting of pine sawdust. For this type of experiment was designed and made experimental pressing stend $[2,3]$ on which we were able to provide all changes of researched parameters setting according to experimental plan.

## 2 Experiment description and results

So because the impact of input (briquetted) material is very influencing, from this reason we had to divide our experiment to 2 main phases. In first phase we have done experiment according to designed levels of measured parameters (see Table 2). These levels come from our analyses, further experience and pressing stend possibilities. We realized experiment by form of full factorial experiment $2^{4}$ according to Table 2 . Goal of the experiment was to follow up the briquettes quality in dependence with pressing temperature, compacting pressure, fraction size and material moisture.

Table 2 Levels of measured parameters [3], [4]

| Pressure <br> $p(\mathrm{MPa})$ | Temperature <br> $T\left({ }^{\circ} \mathrm{C}\right)$ | Size <br> $(\mathrm{mm})$ | Moisture <br> $w_{r}(\%)$ |
| :---: | :---: | :---: | :---: |
| $95-159$ | $85-115$ | $1-4$ | $8-12$ |

Briquettes quality was evaluated by briquettes density as is given in EU Standards about solid biofuels. According to EU Standards briquette have very good quality if density is from 1 to 1,4

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$\mathrm{kg} / \mathrm{dm}^{3}$. In every setting we pressed 7 briquettes. We measured briquette's dimension, length and weight. These measured values were the base for density calculation. Briquettes density values were processed by various mathematical and statistical methods (e.g. Bartlett's Test, ANOVA, etc...) with the help of software Stathgraphic S Plus. For closely determination of parameters impact and also impact of their mutual interaction we used method of parameters effect [5]. Results you can see on the following Figure 2.


Figure 2 Individual parameters impact - Paret's effects diagram [3], [4]

Method of parameters effect determined that the biggest impact have pressing temperature and material moisture. Also their interaction has very significant impact on final briquette density. This is very important and helpful result. Pressing temperature is influencing the lignin plastification in cellular structures of material. Lignin is in briquette as nature glue. Helps to briquettes be stronger and to have higher density. Pressing temperature significance is important also from the machines engineering point of view. If we are able to provide higher pressing temperature we can use not so high compacting pressure (as usual). This is influencing the final price of briquetting machine and of course final price of hour production. Because higher pressing temperature is cheaper to provide as higher compacting pressure. But very important is also the interaction between temperature and material moisture. When is the pressed material moisture very low and vice-versa very high that means out of some optimal interval ( $8-15 \%$ ), material particulars aren't consistence and briquettes is falling to pieces. At lower moisture values material isn't sintered what is needed for consistence of material particulars into the briquette. Researches and experiment proved that moisture have impact also on lignin plastification. As we wrote above pressing temperature is influencing the lignin plastification, therefore is also very important mutual interaction between temperature and moisture content. Temperature of lignin softening is depending also on type of wood from which is isolated and also on isolation method. Temperature of lignin phase switching form solid to plastic state is direct proportion to its molecular weight and indirect proportion to its moisture content. The next step was execution of selection of variables for mathematical model. For this selection we used three widely known criterions: index of multi-launching determination ( $R^{2}$ ), Akaike's criterion (AICc) and Root mean squared error criterion (RMSE). We choose model with containing of all

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parameters and also with their most important mutual interactions (see following Formula 1). This model had the best results after selection by criterions.

$$
\begin{equation*}
\rho=f(A, B, C, D, A B, A C, B C, A B C) \tag{1}
\end{equation*}
$$

We were able to design the mathematical model with help of software SAS and JMP 8 and also we were able to calculate or estimate the regression parameters values. The final designed form of mathematical model for the first phase you can see in following Formula 2, where the " $\rho$ " is briquette density. This model is valid only for briquetting of pine sawdust and is valid only by parameters intervals listed in Table 2.

$$
\begin{equation*}
\rho=e^{\binom{4,98371-0,0261781 \cdot p-0,0410292 \cdot T-0,620594 \cdot w_{r}-0,015446 \cdot L+0,000228845 \cdot p \cdot T+}{0,0031851 \cdot p \cdot w_{r}+0,00528717 \cdot T \cdot w_{r}-0,0000273004 \cdot p \cdot T \cdot w_{r}}}\left(\mathrm{~kg} \cdot \mathrm{dm}^{-3}\right) \tag{2}
\end{equation*}
$$

With this model we obtained tool for effective and quickly prediction of final briquettes density values, pressing temperature values, compacting pressure values, material moisture values and fraction size values.

Second phase of experiment was very similar. Also the experiment procedure was very similar but the differences were used material (pine and oak) and extended parameters intervals where was experiment done. Comparison of these phases from parameters intervals point of view you can see in following Figure 3 and Table 3.


Figure 3 Comparison of experiment phases from parameters intervals point of view [4], [5]
Table 3 Levels of measured parameters in second phase [4], [5]

| Pressure <br> $p(\mathrm{MPa})$ | Temperature <br> $T\left({ }^{\circ} \mathrm{C}\right)$ | Size <br> $(\mathrm{mm})$ | Moisture <br> $w_{r}(\%)$ |
| :---: | :---: | :---: | :---: |
| $63-191$ | $55-130$ | $0,5-4$ | $5-15$ |

The second phase of experiment was done with the pine sawdust and also with the oak sawdust. The goal of the second phase was to design the mathematical model which will describe behavior of each material (pine, oak) at compacting process. But these models will valid in extended parameters intervals as the mathematical model (Formula 2).
Similarly as in first phase we set compacting pressure "p", pressing temperature "T", material moisture " $w_{r}$ " and fraction size "L" as independent variables. Briquette density " $\rho$ " was set as dependent variable.
At first was needed to make Goodness-of-Fit test by Shapir-Wilk W test. This test was realized for each level of each parameter. On the following Figures 4 and 5 you can see outputs of these analyses for one level of parameters at pine sawdust briquetting and for one level of parameters at oak sawdust briquetting. As you can see - measured data for given level of parameters approaching to the line of normality. Also the histogram shows for normality of measured data. A result of Shapir-Wilk W test of normality was that we did not reject the testing hypothesis.


Figure 4 Output example of Goodness-of-Fit test for given level of parameters at pine sawdust [6]


Figure 5 Output example of Goodness-of-Fit test for given level of parameters at oak sawdust [6]

In the next step we analyzed measured data from outliers data point of view. We had to determine if we had some extreme data in our file of data. For testing we used Dean-Dixon non-parametric test of outliers and Grubbs test. If some data are outlying we need to exclude them from measured data for following processing. Also these tests were done for each level of measured data. From file of measured data at pine sawdust we excluded some values from following data processing. At oak sawdust we didn't had any outlying data.
We also tested equal variances in groups of measured data because it is very important for following processing. The most famous tests for equal variances testing are Bartlett's test, BrownForsythe test, Leven and O'Brien tests. We used these tests through much known statistical software SAS JMP 8. We tested also softwood (pine sawdust) and also hardwood (oak sawdust). Results of these tests said us that we could accept the hypothesis about variances equal at $95 \%$ interval of probability also for pine and also for oak sawdust.
Analysis of variance examines interaction between dependent variable and one or more independent variables. File of data was divided into the groups according parameters changes and was tested if differences between group's averages were random or statistically significant. If the differences are

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significant, parameter is statistically significant - we can say between dependent variable and parameter exist some interaction. This analysis was done with software SAS JMP 8. You can see on following figure that parameters which are overhung the given interval (two full vertical lines). These parameters have significance impact on final briquette quality from pine sawdust.


Figure 6 Analysis of significance of individual parameters - pine sawdust [6]
With help of software SAS JMP 8 we have done estimation of regression parameters, evaluation and generated all possible models. We knew also all parameters of each generated model. For final design of mathematical model and regression parameters selection was used index of multilaunching determination ( $R^{2}$ ), Akaike's criterion (AICc), Malowo's criterion ( Cp ) and Root mean squared error criterion (RMSE). We choose model containing of all significance parameters and also with their most important mutual interactions (see following Formula 3). This model had the best results after selection by criterions. This model is valid only for pine sawdust briquetting.

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$$
\begin{align*}
& \rho=0,833+0,000646 * p+0,00403 * T-0,02144 * w_{r}-0,01226 * L-0,000045 *(p-130,2) *(T-103,5)+ \\
& +0,000249 *(p-130,2) *\left(w_{r}-9,83\right)+0,00133 *(T-103,5) *\left(w_{r}-9,83\right)- \\
& -0,000022 *(p-130,2) *(T-103,5) *\left(w_{r}-9,83\right)-0,000005 *(p-130,2) *(T-103,5) *\left(w_{r}-9,83\right) *(L-2,03)+ \\
& +0,00171 *\left(w_{r}-9,83\right)^{2}
\end{align*}
$$

Also the same procedure was done with measured data from oak sawdust briquetting. Analysis of significance you can see following Figure 7. And below the Figure you can see Formula 4 - final form of mathematical model for oak sawdust.


Figure 7 Analysis of significance of individual parameters - oak sawdust [6]

$$
\begin{align*}
& \rho=0,142+0,00112 * p+0,00596 * T-0,039 * w_{r}+0,00095 *(T-103,5) *\left(w_{r}-9,83\right)+0,15732 * L- \\
& -0,000004 *(p-130,2) *(T-103,5) *\left(w_{r}-9,83\right) *(L-2,03)-0,00003 *(T-103,5)^{3}+ \\
& +0,02894 *(L-2,03)^{2}-0,06399 *(L-2,03)^{3} \tag{4}
\end{align*}
$$



Figure 8 Graphic comparisons of measured density values and calculated density values - pine sawdust


Figure 9 Graphic comparisons of measured density values and calculated density values - oak sawdust

On Figures 8 and 9 you can see simple graphic comparisons of measured density values and calculated density values for pine and also for oak sawdust. These comparisons confirmed that our experiment and measured data processing were correct. With these designed mathematical models

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we obtained the tool for briquette density prediction and by engineering of some important parts of compacting machines.

## 3 Conclusion

The main aim of the experiment was to detect and identify the effect rate of monitored parameters on the final briquettes quality evaluated by briquettes density. By the individual steps we discovered that the most significant effect on briquettes quality has pressing temperature and then material moisture and mutual interaction of these two parameters. The results our hypothesis that compacting pressure, which may seem to be a parameter having the biggest effect on the final briquettes quality, is minor in analyze of effects on briquettes quality. With usage of mathematical and statistical tools we were able to design mathematical model of single axis pressing of pine and oak sawdust. From results of our experiment you can see that pressing temperature and material moisture are most significant parameters also at briquetting of softwoods and also at briquetting of hardwoods. Temperature and moisture are influencing the lignin plastification. Lignin is at biomass compacting as a nature glue and from briquette density and strength point of view is very important component.

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# ANT COLONY OPTIMIZER WITH APPLICATION TO THE VEHICLE ROUTING PROBLEM 

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#### Abstract

This paper introduces a new open source Ant Colony Optimization framework for parallel environment, named ACOptim. The aim of the proposed system is to provide environment for studying behaviour of various strategies of synchronization and communication among colonies, population creation, different termination criteria, etc. ACOptim is coded in C++ and uses POSIX threads keeping in mind portability and object oriented principles. In the proposed paper we show an application of ACOptim to solving large Vehicle Routing Problems. We present and analyze results of experiments achieved by different configurations.


Key words and phrases. Ant Colony Optimization, Parallel Metaheuristic, Vehicle Routing Problem, POSIX threads.
Mathematics Subject Classification. Parallel computation 65Y05; Transportation 90B06; Parallel algorithms 68 W 10 .

## 1 Introduction

An optimization of combinatorial problems related to logistics and transportation systems is nowadays very important. One of the most challenging problems related to transportation is the Vehicle Routing Problem (VRP)introduced by Dantzig [9]. This NP-hard optimization problem and it's variants can be applied to several situations in transportation, telecommunication and logistics. In practice, it is often required to receive quality solutions of such problems in relatively short time, leading so to requirements of using metaheuristics and computations in parallel environments. It is especially important in the case, when exact optimal solutions (for larger instances of the VRP) are not calculable by current computing power, since the time of computation increases exponentially. For these reasons, many scientists are looking for improvements in new parallel execution models of various metaheuristics. Recently, it

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was shown that nontrivial communication and synchronization strategies between calculation processes have significant impact on quality of received solutions.

This paper focuses on application of new approaches in parallelization of the Ant Colony Optimization (ACO) and presents the framework which helps in study of parallel ACO behaviour. It is structured into four sections. The first section gives reader short introduction to the ACO and brings short formulation of the VRP problem. In the second section we describe shortly parallelization strategies for metaheuristics, and compare several software frameworks suitable for solving of the ACO. Section 3 describes architecture and basic features of the ACOptim framework and it's application to the VRP. It presents computational results gained with several parallelization configurations. The paper concludes with several remarks and outlooks concerning the future work.

### 1.1 A short introduction to ACO

Ant Colony Optimization is a relatively new metaheuristic approach in calculating combinatorial problems including the VRP, first published in [12]. Basic principle is given by observation of real ants behaviour, where ants cooperate by leaving pheromone on the ground by moving from food sources to the nest. In this way better routes are identified by another ants during the path selection process. Assuming that ant $k$ stays in the vertex $i$, then probability of selecting $j$ as the next vertex in route, is given by following rule:

$$
p_{i j}^{k}= \begin{cases}\frac{\left[\tau_{i j}\right]^{\alpha}\left[\eta_{i j}\right]^{\beta}}{\sum_{l \in N_{i}^{N}}^{\left[\tau_{i i}\right]^{[ }\left[\eta_{i l}\right]^{\beta}}} & \text { if } j \in N_{i}^{k}, \\ 0 & \text { otherwise }\end{cases}
$$

where $N_{i}^{k}$ is set of feasible neighborhood of the vertex $i, \tau_{i j}$ is amount of pheromone and $\eta_{i j}=1 / d_{i j}$ is heuristic information, where $d_{i j}$ is the distance between $i$ and $j$. Parameters $\alpha$ and $\beta$ specify the relative influence of the pheromone and the distance during the decision process. Each ant remembers the passed route in its local memory. The amount of the pheromone on the ground evaporates in time.

There are several models of artificial ants and their behaviour. For more detailed review reader should see [14]. In our paper we use ACOptim framework to the algorithm $A S_{\text {rank }}$ [3], which provides good quality solutions of the VRP in a relatively short calculation time [11]. This method combines basic Ant System and Elitists Ant System into a single method. The specific behaviour of $A S_{\text {rank }}$ method allows only ants which belong to the elite to affect global pheromone at the end of each iteration.

### 1.2 Formulation of the VRP

The symmetric VRP [6] is defined on a complete, not oriented graph $G=(V, E)$, where set $V=$ $\{0, \ldots, n\}$ defines vertices. Each vertex $i \in V \backslash\{0\}$ represents customer with positive demand $q_{i}$, where 0 stands for the depot and $q_{0}=0$. Let each edge $e \in E=\{(i, j): i, j \in V, i<j\}$ has assigned transportation costs $c_{i j}>=0$ (usually $c_{i j}=d_{i j}$ ). There are available $m$ vehicles, each with capacity $Q$, in depot. The goal is to specify $m$ routes with minimal transportation costs,
with following constrains: each route starts and ends in depot, each customer is served exactly once, there are not overloaded vehicles. Solution can be represented as set of arcs which share only one vertex - the depot.

For description of problem it is necessary to introduce binary decision variable $x_{i j}^{k}$ with following representation:

$$
x_{i j}^{k}= \begin{cases}1 & \text { if vehicle } k \text { follows vertex } j \text { immediately after } i, \\ 0 & \text { otherwise }\end{cases}
$$

The problem is defined as:

$$
\begin{equation*}
\operatorname{minf}(x)=\sum_{i=0}^{n} \sum_{\substack{j=0 \\ j \neq i}}^{n} \sum_{k=1}^{m} c_{i j} x_{i j}^{k} \tag{1}
\end{equation*}
$$

with following constrains:

$$
\begin{array}{rlr}
\sum_{\substack{i=0}}^{n} \sum_{\substack{j=1 \\
j \neq i}}^{n} x_{i j}^{k} d_{j} \leq Q & 1 \leq k \leq m \\
\sum_{\substack{i=0 \\
i \neq j}}^{n} x_{i j}^{k}-\sum_{\substack{l=0 \\
l \neq j}}^{n} x_{j l}^{k} & =0 & 1 \leq k \leq m, 0 \leq j \leq n \\
\sum_{\substack{i=0 \\
i \neq j}}^{n} \sum_{k=1}^{m} x_{i j}^{k} & =1 & \\
\sum_{\substack{i \in S}} \sum_{\substack{j \in S \\
j \neq i}} x_{i j}^{k} \leq|S|-1 & \forall S \subseteq\{1, \ldots, n\}, 1 \leq k \leq m,|S| \geq 2 \\
x_{i j}^{k} & \in\{0,1\} & 1 \leq k \leq m, 0 \leq i \leq n, 0 \leq j \leq n, i \neq j
\end{array}
$$

First, it is ensured that no vehicle is overloaded by constrains (2). Following constrains (3) ensure, that each customer visited by vehicle is also left by the same vehicle. Constrains (4) ensure that each customer is visited only once and the depot is left by all used vehicles. Finally, constrains (5) are used to eliminate sub-tours, followed by binary constrains (6).

There are several applications of the ACO metaheuristics to the VRP in the literature, first proposed by [4]. Construction of solutions is usually based on Savings mechanism, where heuristic information is defined as $\eta_{i j}=1 / s_{i j}$ and the savings as $s_{i j}=s_{0 i}+s_{j 0}-s_{i j}$. During the execution of iterations there is obtained value of pheromone concentration $\tau_{i j}$, which specifies quality of the combination of two customers $i$ and $j$.

There are many possibilities, how the ACO can be parallelized, [7],[8],[10],[11]. An illustrative example of the parallel ACO, that is applicable for the VRP, is the following:

```
1: Initialization;
2: Foreach parallel Colony do:
    Foreach Iteration do:
```

```
Foreach Ant do:
    Create Savings based solution;
    Apply local search;
    Select elitists Ants;
If Synchonize step then:
    Exchange Solution(s);
    Update Pheromone matrix;
```

3: Finalization;

The application of Local search algorithm is optional, if it is applied usually one of improvement heuristics, or their combinations are used. Synchronization step and amount of exchanged information is defined by the parallelization algorithm. In the following section we discuss several parallelization strategies.

## 2 Parallel Metaheuristics

In general, there are three dimensions of classification used for the description of parallel metaheuristic strategies [7], [8], where the first of them specifies Control Cardinality. This classification examines how the global search is controlled. In the case of a single process, it is denoted by $1 C$. Another case, when more collaborative or even independent processes control search, it is $p C$. The second classification reflects the art of information exchange. In parallel computing, this dimension is specifies as Communications, where four classes are defined: whether communication mechanism exchanges more knowledge about solutions (e.g. statistical information) or not and also whether synchronous or asynchronous communication is used. Combining of those alternatives is noted as follows: $R S$ as rigid synchronization, $K S$ as knowledge synchronization, $C$ as collegial synchronization and $K C$ as knowledge collegial synchronization. The third dimension of classification indicates the search Differentiation of parallel processes. There are specified four cases: SPSS, each process uses Same initial Population and Same search Strategy; $S P D S$, each process uses Same initial Population but Different search Strategies; MPSS, Multiple initial Populations are used by processes but Same search Strategies; $M P D S$, Multiple initial Populations are used by processes and Different search Strategies.

Historically the first parallel implementations are independent and synchronous. Typically this $1 C / R S$ model is implemented with one master and several slaves, where each process is initiated, stopped at synchronization points and continued by controlling process. During synchronization, which occurs in specified situations e.g. number of iterations, master evaluates received solutions and re-distributes results to slave processes. Distribution of good or global best solutions, is performed by different communication infrastructure [7]. However, knowledge exchange was first introduced with synchronous communication, currently research is focused more on asynchronous communication models $p C / K C$, which now more-less define the "state of the art" in parallel metaheuristics.

There are more reasons for this. The first is the fact, that it is hard to predict how much time each population needs to finish its iteration when $M P S S, M P D S$ or $S P D S$ with different settings are used, therefore wasted time during synchronization is more probably. Another reason can be found in lower flexibility of the algorithm setting possibilities. This problem is

Table 1: Available ACO frameworks

| Name | Language | Object-oriented | Parallel | License |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| AntLib [13] | C++ | Yes | No | $?$ |
| ACOTSP [19] | C | No | No | GPL |
| libaco [17] | C++ | Yes | No | LGPL |
| ACOptim | C++ | Yes | Yes | GPL |
| PARMETAOPT [2] | C++ | Yes | Yes | $?$ |
| HeuristicLab [20] | .NET | Yes | Yes | GPL |

reduced when asynchronous communication is used and population doesn't have to wait for other processes.

The significant goal of current research of parallel metaheuristics is to gain advance from parallel execution of algorithm not only to speedup execution or only to obtain better quality solution but to increase both of them together. Therefore algorithms are not only executed in parallel but sophisticated cooperation is introduced. This combines found solutions to obtain new characteristics of good solutions and tries to create or prefer solutions with similar characteristics during next search. Such cooperative strategies belong to the $p C / K C$ class of the taxonomy. Although asynchronous procedures follow similar principles of starting from the same or different solution and use the same or different search strategy, the knowledge creating mechanism is fundamental principle of parallel metaheuristics. For example, one can store more parts of good solutions and combine them to create new solutions or to store less but whole solutions possibly with additional information (e.g. neighborhood information). First approach is widely used in genetic metaheuristics [7].

### 2.1 Existing software

In the literature there are several libraries and frameworks which apply metaheuristics for solving the problem. For instance, there are projects like GAlib [21], OpenBEAGLE [15], ParMetaOpt [2] and pALS [1] in early development phase and projects in stable development phase like ParadisEO [5] and HeuristicLab [20]. Only few of them are suitable for parallel environment (e.g. HeuristicLab, ParMetaOpt, pALS and ParadisEO) and those primary implement metaheuristics based on genetic algorithms. To the best knowledge of the authors libraries or frameworks which contain implementations of the ACO are listed in Table 1. Although several currently available software packages contain mechanisms of creating knowledge-based parallel algorithms, so far, there is no one known which is suitable for the ACO and stays simple, customizable, easy to use and free of charge.

## 3 ACOptim: Ant Colony based Optimizer

This chapter presents basic architecture and design principles used by creation of the framework. Basic motivation to create ACOptim framework was to examine several knowledge-based parallel ACO algorithms for the VRP. Therefore ACOptim should be an flexible modularized tool providing several communication strategies with simple possibility to extend new strategies. To achieve modularization, the object oriented paradigms has been adopted. Each module loadable into the framework provides basic customization of execution in dependence on input parameters. ACOptim has been developed in Linux operating system, with keeping in mind to stay portable console application, so it can run in batch mode.

The ACOptim framework consists of four basic groups of components: generally used support layer, generic Ant Colony Problem solver, communication and synchronization layer and problem specific layer. Simplified architecture overview is displayed in Figure 1.

The support layer consists of objects which are used in the whole framework, like: $\boldsymbol{S o}$ lution, PheromoneMatrix, DistanceMatrix, TimeDiff. There is defined object Operator, which specifies element operation over two matrixes. Inherited subclass can be used for manipulating matrixes in several situations, for example when a new pheromone matrix is received by ant colony.

Module interface LoadModule defines a set of operations to be implemented by any module which should be loaded into system. First, it provides module identification, specification of module type and functions for interaction with user as providing of help and parsing of input arguments. Second, it is used by ModuleManager which according to the module type and its configuration, affects behaviour of ant colony solver.

There exist following types of modules: terminate condition, synchronizer, communicator, algorithm calculation, solution constructor and local search optimizer. In the case of need, it is possible to extend this list by introducing new module types. Any module type can be re-implemented by subclass to achieve customized functionality.

Generic Ant Colony Problem Solver represented by Problem class is responsible to run several Ant Colonies (AntColony) according to the used configuration and infrastructure. It is also responsible to specify parameters of executed colonies in MPSS and MPDS parallel diversification strategies.

Transportation and synchronization layer is responsible for: low level communication within or between nodes, higher level interaction and communication strategy between colonies. It defines also Ant Colony stopping criteria. Objects which should be send trough network or should be saved into files has to implement serializing and de-serializing functionality.

When ACOptim is applied to a custom problem, specific layer has to be implemented by the user. An example of applying ACOPtim to the VRP can be found in the next chapter.

### 3.1 Application of ACOptim to the VRP

Before starting application of ACOptim to the VRP, it is important to understand that VRP is actually a combination of two problems. The first subproblem is to divide customers into groups which will be served within one route. The second aspect is to solve Traveling Salesman

Problem over each group of customers. According to the ACO definition it is necessary to define complete graph, which will be traversed by artificial ants. Afterwards it is required to define mechanism of pheromone update and ensure that all VRP constrains will be satisfied by ants. Optionally it is possible to define improvement heuristics. There exists approach mentioned in the previous chapter which defines Savings based complete graph to be traversed by ants. This approach has been used in our paper, where class SolutionConstructor has been inherited and solution creating in Saving based environment has been implemented. The same class has been used for capacity constrain checking, where ant prior to selection of next customer, checks if it is not overloaded. If no such customer can be found, the depot is selected. We have used general pheromone evaporating mechanism.

For improving the solution quality, two local search procedures has been implemented. The first procedure calls swap heuristic exchanges between groups of customers, while all constrains stay satisfied. The second one used heuristic 2-opt, which exchanges order of served customers within one route. This functionality has been achieved by inheriting of the class LocalSearch.

There are proposed three communication strategies in this paper. First communication strategy called "Best Solution Only" (bso), applies sending best founded solution, so far including elite, to other colonies. When such solution is received, local pheromone matrix is updated by this solution. Second tested strategy, denoted $n c$, uses no communication at all. This strategy is usually used as reference strategy. This approach is similar to independent run of several experiments, where the best run is used as result. The last presented communication strategy, denoted $b p m$, uses combination of pheromone matrices, where better pheromone matrix is selected. When a better solution is found, corresponding pheromone matrix is spread between colonies and each colony combines the local pheromone with the received pheromone. In comparison to bso, this approach seems not to lead so quickly into local optima, because the pheromone matrix is kept with more alternative promising solution areas.

### 3.2 Computational results

We have used two larger Christofides instances [6] and three larger Golden instances [16] in our experiments. Tested instances contained from 150 to 420 customers. The experiments were executed on the Luna cluster of University of Vienna, consisting of 72 SUN X4100 nodes with two 64 -bit dual core processors, each. We have assigned one colony to one core, therefore we could use 4 colonies working over the common shared memory of one node. Each colony has been executed in separate POSIX thread. Inter-node communication has been implemented by sending UDP (User Datagram Protocol) packets with scaling factor of 32 kB . Each group of four colonies uses one thread for receiving of UDP packets. Synchronization of threads within one node has been achieved by POSIX mutual exclusion locks. Pheromone matrix has been transferred compressed using zlib library. Except experiments, without communication, asynchronous communication strategy has been used between colonies.
During each iteration two local search heuristics has been applied. First heuristic uses vehicle swap algorithm between routes, second is well known 2-opt heuristic applied within one route. During experiments we have generally used iteration based stopping criteria, where number of iterations iter $=2 n$. Reported results are average values gained over 10 runs for each

instance. Total number of artificial ants equals number of customers $n$. Calculation parameters $\alpha=\beta=5, \rho=0.95$, elite $\sigma=6$ and neighborhood size $k=\lfloor n / 2\rfloor$. Used parameters has been proposed by [18]. Results from Table 2 show that communication strategy has significant influence on the solution quality. We can see that bso strategy has produced worst solutions. This is caused by fact that the algorithm relatively quickly reaches local optima. Surrounding of this solution is afterwards ineffectively explored by all cooperative colonies. Interesting results are reported by $n c$ strategy without communication between colonies. This strategy provides best measured speedup. Proposed strategy bpm where better values of received and local pheromone matrix is combined, provides best quality solution with similar speedup of previous strategies. None of examined strategies increases solution quality with increasing number of cores.

## 4 Conclusions

We have presented a new framework for computation of the ACO with application to the VRP. In this paper we have focused on synchronization between ant colonies and examined three proposed strategies from solution quality and speedup point of view. We have shown that communication strategy has impact to calculated solution quality. Multithreaded framework ACOptim has been developed as expandable, portable, built on object oriented foundations and written in C++ language under general public license. Therefore it provides good basics and functionality for further examination of complex communication strategies. In our further work we would like to design and examine more complex communication strategies, with keep in mind the solution quality and speedup. We would like to improve the proposed framework to automatically generate statistical reports and introduce parameterized instance generator.

## Acknowledgement

Table 2: Calculated average results of each measured instance according to the number of threads and used strategy. Th denotes number of used threads, $V$ denotes calculated quality solution and $S$ denotes speedup.

| Instance | Th | $V_{\text {bso }}$ | $S_{\text {bso }}$ | $V_{n c}$ | $S_{n c}$ | $V_{b p m}$ | $S_{\text {bpm }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| E151 - <br> Christofides 150 customers | 1 | $1054.74 \pm 1.24$ | 1.00 | $1049.80 \pm 1.22$ | 1.00 | $1051.08 \pm 1.41$ | 1.00 |
|  | 2 | $1052.49 \pm 0.97$ | 1.95 | $1051.92 \pm 1.24$ | 1.94 | $1052.32 \pm 1.41$ | 1.88 |
|  | 4 | $1056.18 \pm 1.76$ | 3.81 | $1051.73 \pm 0.96$ | 3.76 | $1055.19 \pm 1.16$ | 3.80 |
|  | 8 | $1063.65 \pm 1.78$ | 7.44 | $1058.55 \pm 0.78$ | 7.39 | $1058.09 \pm 0.57$ | 7.30 |
|  | 16 | 1075.26 $\pm 3.32$ | 13.41 | $1060.69 \pm 1.39$ | 13.38 | $1060.28 \pm 1.37$ | 13.50 |
|  | 32 | $1071.90 \pm 4.36$ | 23.75 | $1061.74 \pm 1.49$ | 25.54 | $1058.86 \pm 0.97$ | 24.96 |
| E200 Christofides 199 customers | 1 | $1339.09 \pm 2.25$ | 1.00 | $1337.06 \pm 2.65$ | 1.00 | $1334.73 \pm 2.47$ | 1.00 |
|  | 2 | $1340.13 \pm 1.59$ | 1.94 | $1336.13 \pm 1.96$ | 1.96 | $1336.64 \pm 2.28$ | 1.90 |
|  | 4 | $1351.44 \pm 2.50$ | 3.77 | $1340.77 \pm 1.39$ | 3.80 | $1338.60 \pm 1.42$ | 3.76 |
|  | 8 | $1350.89 \pm 3.33$ | 7.18 | $1348.95 \pm 0.86$ | 7.26 | $1343.71 \pm 1.34$ | 7.24 |
|  | 16 | $1360.30 \pm 2.35$ | 13.74 | $1355.82 \pm 1.93$ | 14.16 | $1348.96 \pm 1.62$ | 13.90 |
|  | 32 | $1371.65 \pm 2.45$ | 23.27 | $1360.12 \pm 1.58$ | 24.52 | $1352.26 \pm 2.54$ | 24.83 |
| E301 - Golden 300 customers | 1 | $1037.60 \pm 1.44$ | 1.00 | $1040.23 \pm 1.79$ | 1.00 | $1036.98 \pm 1.61$ | 1.00 |
|  | 2 | $1039.39 \pm 2.36$ | 1.92 | $1038.49 \pm 1.82$ | 1.96 | $1034.02 \pm 1.54$ | 1.85 |
|  | 4 | $1039.52 \pm 1.70$ | 3.72 | $1041.66 \pm 1.65$ | 3.79 | $1035.72 \pm 1.85$ | 3.62 |
|  | 8 | $1045.71 \pm 3.54$ | 7.24 | $1047.79 \pm 2.09$ | 7.49 | $1041.76 \pm 2.20$ | 7.25 |
|  | 16 | $1055.55 \pm 2.24$ | 14.25 | $1052.83 \pm 1.44$ | 14.52 | $1050.65 \pm 1.53$ | 14.24 |
|  | 32 | $1066.49 \pm 1.23$ | 23.63 | $1059.95 \pm 1.59$ | 26.58 | $1054.98 \pm 1.27$ | 25.74 |
| E361 - Golden 360 customers | 1 | $1426.33 \pm 2.32$ | 1.00 | $1426.14 \pm 3.91$ | 1.00 | $1422.88 \pm 1.99$ | 1.00 |
|  | 2 | $1418.93 \pm 2.64$ | 1.88 | $1423.10 \pm 1.47$ | 2.00 | $1414.30 \pm 1.50$ | 1.82 |
|  | 4 | $1425.25 \pm 2.93$ | 3.68 | $1427.04 \pm 1.96$ | 3.86 | $1428.36 \pm 1.64$ | 3.61 |
|  | 8 | $1427.81 \pm 2.03$ | 7.15 | $1435.05 \pm 1.67$ | 7.50 | $1430.71 \pm 2.29$ | 7.07 |
|  | 16 | $1445.17 \pm 3.18$ | 14.02 | $1435.91 \pm 2.85$ | 14.78 | $1434.55 \pm 1.56$ | 13.76 |
|  | 32 | $1454.54 \pm 2.28$ | 25.38 | $1449.48 \pm 1.74$ | 26.22 | $1440.77 \pm 2.56$ | 25.60 |
| E421 - Golden 420 customers | 1 | $1905.25 \pm 2.09$ | 1.00 | $1902.99 \pm 3.38$ | 1.00 | $1901.11 \pm 2.85$ | 1.00 |
|  | 2 | $1895.27 \pm 3.21$ | 1.90 | $1905.57 \pm 2.29$ | 1.98 | $1890.96 \pm 2.43$ | 1.80 |
|  | 4 | $1895.88 \pm 2.09$ | 3.56 | $1904.21 \pm 2.48$ | 3.89 | $1903.52 \pm 3.42$ | 3.60 |
|  | 8 | $1906.68 \pm 2.50$ | 7.23 | $1910.55 \pm 2.66$ | 7.55 | $1905.78 \pm 1.93$ | 7.21 |
|  | 16 | $1932.97 \pm 4.43$ | 12.57 | $1919.88 \pm 2.05$ | 14.62 | $1916.99 \pm 2.77$ | 13.68 |
|  | 32 | $1937.72 \pm 3.78$ | 25.06 | $1931.72 \pm 2.76$ | 26.62 | $1917.79 \pm 3.71$ | 25.20 |

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# ANALYSIS OF TOOL GEOMETRY <br> FOR SCREW EXTRUSION MACHINES 

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#### Abstract

The contribution deals with the technology of biomass briquetting into the solid highgrade biofuel by screw extrusion machines. It is focused mainly on the theory of compacting tools for screw briquetting presses, their analysis, stress conditions and geometry. The main aim is analyzing of pressing screw geometry and determination process of its design. Analysis of force conditions on the screw is necessary for designed geometry verification and for stress analysis. The determination process of the frictional power is instrumental to main power drive design. Knowledge of these processes is the base of the new tools research for screw presses, the increase of tools lifetime and the competitiveness of whole technology.


Key words. biomass, briquetting, tool, screw, screw machine, screw profile
Mathematics Subject Classification: Primary 28A75; Secondary 49Q10.

## 1 Introduction

The world trend of the fossil fuel substitution by renewable energy sources affects the research direction of production machines in field of the material agglomeration. Biomass represents the most perspective renewable energy source with the most effective possibilities of the energy storage. It gives rise to a necessity of the research in field of the biomass treatment and transformation into solid high-grade biofuel. The briquetting technology is successfully used for production of solid biofuel destined for industrial sector as well as household. The biomass briquetting is based on the pre-treatment material pressing and its extruding through the pressing die. It is a complicated production process where the mount of conditions affecting the binding mechanism within the briquette has to be fulfilled. The production process and the production quality are influenced by many technological and structural parameters. The briquetting technology of biomass uses three types of machines - hydraulic presses, mechanical crank presses and screw extrusion machines. Ever principle of machine with own advantages and disadvantages has its justification in the production of solid high-grade biofuel. In light of the achieved production quality, as a determining factor on the biofuel market, the most suitable is the principle of screw

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extrusion machine. This principle allows production of different shapes of briquettes (cylindrical, nangular; with or without hole, etc.). The main disadvantage of this type of press is high operating costs following the low lifetime of very expensive tools - screws. Nowadays, manufactured pressing screws reach lifetime only a few decades of working hours. The biomass briquetting by the screw presses is progressive technology of solid biofuel production. It is high actual and perspective to deal with the research in this area. Increase of the tools lifetime and decrease of the operating costs will be reflected on the lower price of production and may contribute for wider utilization of renewable energy sources.

## 2 Tools for screw briquetting presses

The briquetting process by screw presses is continual without beats. The high compacting pressure is characteristic what causes high density and high strength of briquettes. The surface of briquette is high-class. By screw presses is possible to produce different shapes of briquettes with or without hole.
There are put material and geometric requirements on the tools. Material requirements include high wear resistance, toughness and thermal stability. Specification of geometrical requirements in not simple and it is dependent on the pressed material. The main geometrical request is creating the rapid increase of pressure into the pressed material. The tool geometry has also to insure the material axial movement and the fluency of the pressing process. Tools consist of the feeding screw, the pressing screw and the pressing chamber included particulate dies (Fig. 1).


Figure 1. Tools of screw extrusion machines
(1-feeding screw, 2 - pressing screw, 3-pressing chamber, 4-dies)
The feeding screw is not stressed by high working load. Its main task is fluent axial movement of material toward the pressing screw and the homogenous filling the whole screw profile crosssection. The pressing screw as the tool is exposing to high workload, abrasion and temperature. Its geometry provides a high degree of compaction of material in the pressing chamber and the material extrusion through a particular dies, thereby achieving a compact pellet of high density, strength and surface quality. This contributes to compact briquette of high density, strength and surface quality. Pressing screw is the most loaded machine part with the highest rate of wear. The most worn part of the pressing screw is the spike (Fig. 2) and the first 1.5 thread, which shows the distribution of the workload on the tool. Material movement, compression, wear rate, stress

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distribution depends primarily on the chosen screw geometry. Therefore it is extremely important in the design of the screw to pay great attention to its geometry.


Figure 2. Pressing screw (1-working screw part, 2 - calibration spike)
Pressing chamber must be sufficiently strong to withstand the internal pressure of compression. Each die in the pressing chamber have to copy the pressing screw, its geometry has to prevent the rotation of the material and provide its axial displacement. Their geometry is chosen so that on the one side of the pressing chamber, they copy the screw and on the other side, they pass fluently to the required shape of briquettes. They are highly stressed by compression pressure, temperature and abrasion especially. Therefore, their material has to be hard and abrasion resistant on the surface, tough in the core.

## 3 Analysis of pressing screw

### 3.1 Volume of screw profile

Assuming that the thread space is filled with material to $100 \%$, the volume of the screw profile represents the amount of material transported by the screw. Feeding screw is designed to achieve a fill volume of the screw profile. It is set coaxially with the pressing screw and form one unit. The main task of the feeding screw is the homogenization of entering material throughout cross-section of the screw profile.


$$
\begin{aligned}
& \alpha 1 \text { - face angle } \\
& \alpha 2 \text { - backoff angle } \\
& s \text { - pitch } \\
& \text { e - sliding surface } \\
& h \text { - profile depth } \\
& D \text { - outside diameter of screw }
\end{aligned}
$$

Figure 3. Screw profile (meridian cross-section)
Screw profile volume of i-course screw can be expressed ( $i$ - number of courses):

$$
\begin{align*}
\frac{V}{\pi \cdot D^{3}}= & \left(1-\frac{h}{D}\right) \cdot \frac{h}{D} \cdot\left[\frac{s}{D} \cdot\left(1-\frac{i \cdot e}{s}\right)+\frac{1}{2} \cdot \frac{\sin \left(\alpha_{1}+\alpha_{2}\right)}{\sin \alpha_{1} \cdot \sin \alpha_{2}}\right]-\left[\frac{1}{2} \cdot \frac{h}{D}-\left(\frac{h}{D}\right)^{2}+\frac{2}{3}\left(\frac{h}{D}\right)^{3}\right] \cdot \frac{\sin \left(\alpha_{1}+\alpha_{2}\right)}{\sin \alpha_{1} \cdot \sin \alpha_{2}}- \\
& -\left(1-\frac{2 h}{D}\right) \cdot\left(\frac{r_{1}}{D}\right)^{2} \cdot\left[\frac{1-\cos \alpha_{1}}{\sin \alpha_{1}}-\frac{\pi}{360} \cdot\left(180-\alpha_{1}\right)\right]-\left(1-\frac{2 h}{D}\right) \cdot\left(\frac{r_{2}}{D}\right)^{2} \cdot\left[\frac{1-\cos \alpha_{2}}{\sin \alpha_{2}}-\frac{\pi}{360} \cdot\left(180-\alpha_{2}\right)\right]-  \tag{1}\\
& -\frac{1}{6} \cdot\left(\frac{r_{1}}{D}\right)^{3} \cdot\left[\frac{10 \cdot \sin \alpha_{1}-\sin 2 \alpha_{1}}{1-\cos \alpha_{1}}-\frac{\pi}{30} \cdot\left(180-\alpha_{1}\right)\right]-\frac{1}{6} \cdot\left(\frac{r_{2}}{D}\right)^{3} \cdot\left[\frac{10 \cdot \sin \alpha_{2}-\sin 2 \alpha_{2}}{1-\cos \alpha_{2}}-\frac{\pi}{30} \cdot\left(180-\alpha_{2}\right)\right]
\end{align*}
$$

The expression can be considerably simplified if we implement some assumptions valid for special types of structures, such as $\alpha_{1}=\alpha_{2}=\frac{\pi}{2}$ or $\frac{r_{1}}{D}=\frac{r_{2}}{D} \rightarrow 0$.

$$
\begin{equation*}
\frac{V}{\pi D^{3}}=\left(1-\frac{h}{D}\right) \cdot \frac{h}{D} \cdot\left[\frac{s}{D} \cdot\left(1-\frac{i \cdot e}{s}\right)\right] \Rightarrow V=\pi \cdot h \cdot(D-h)(s-i . e) \tag{2}
\end{equation*}
$$

The volume profile of one thread can be changed following the screw length. Such changes will be achieved by:

- changing the outer diameter (conical screw),
- changing the profile depth (conical screw core), changing the pitch angle (screw with progressive pitch).Several parameters can be changed also at the same time. Then the ratio of volumes in the monitored areas on the screw is known as the compression ratio, thus: $k=V / V_{i}$


### 3.2 Speed and force relationships in the screw

Material, in our case biomass, is delivered to the screw profile in loose form. The feeding screw profile under the hopper is permanently flooding, thus the filling of the screw profile is kept at the interface constant value during the compression process. Fluent and constant supply of material from the hopper is provided by the filling screw. The advantage of using the filling screw is the possibility of partial pre-compacting of material before entry into the feeding respectively pressing screw area.

For describing the movement of material in the screw is introduced following assumptions:

- thread profile is completely filled with solid,material moves in the direction of the screw pitch, neglecting the influence of curvature on the bottom of screw profile ,friction coefficient between material and monitored surfaces is constant,processed material does not transmit shear stresses.
If the assumptions are valid and in steady conditions, we can monitor the movement of defined particles in Fig 4.

$\alpha$-spiral angle
$\varphi$ - feeding angle
$u_{o}$ - circumferential speed
$v_{r}$ - relative circumferential speed component
$v_{a}$-absolute circumferential speed component

Figure 4. Speed conditions in screw profile
The material is moving at a speed $v_{r}$ relative to screw (in the moving coordinates). The material is moving at a speed $v_{a}$ relative to the die (in the fixed coordinates). From Fig 4 we can derive:

$$
\begin{align*}
& v_{a}=\frac{u_{0} \cdot \sin \alpha}{\sin (\alpha+\varphi)}=\frac{u_{0} \cdot \sin \alpha}{\sin \alpha \cdot \cos \varphi+\cos \alpha \cdot \sin \varphi}=\frac{u_{0} \cdot \operatorname{tg} \alpha}{\operatorname{tg} \alpha \cdot \cos \varphi+\sin \varphi}  \tag{3}\\
& v_{r}=\frac{u_{0} \cdot \sin \varphi}{\sin (\alpha+\varphi)}=\frac{u_{0} \cdot \sin \varphi}{\sin \alpha \cdot \cos \varphi+\cos \alpha \cdot \sin \varphi}=\frac{u_{0} \cdot \operatorname{tg} \varphi}{\sin \alpha+\cos \alpha \cdot \operatorname{tg} \varphi} \tag{4}
\end{align*}
$$

For the material transport is applied only axial velocity component $v_{a x}$, which is given by:

$$
\begin{equation*}
v_{a x}=v_{a} \cdot \sin \alpha=u_{0} \frac{\operatorname{tg} \alpha \cdot \sin \varphi}{\operatorname{tg} \alpha \cdot \cos \varphi+\sin \varphi}=u_{0} \cdot \frac{\operatorname{tg} \alpha \cdot \operatorname{tg} \varphi}{\operatorname{tg} \alpha+\operatorname{tg} \varphi} \tag{5}
\end{equation*}
$$

If is valid, that the $v_{a x}$ has to be greater than 0 , then the numerator in the fraction expression has to be greater than zero. Therefore has to be that the feeding angle $\varphi$ belongs to the interval $0<\varphi<\pi / 2$. If $\varphi=0$ then $\operatorname{tg} \varphi=0$, and so $v_{a x}=0$. The first boundary state occurs. In other words, we can say that the material will not be transported in the direction of the axis of the screw, but will rotate with it. The second boundary state occurs, if $\varphi=\pi / 2$. This value can not be appointed directly to the above relation, because we get an indefinite term. Therefore the original relationship is necessary to modify:

$$
\begin{equation*}
v_{a x}=v_{a} \cdot \sin \alpha=u_{0} \frac{\operatorname{tg} \alpha \cdot \sin \varphi}{\operatorname{tg} \alpha \cdot \cos \varphi+\sin \varphi} \tag{6}
\end{equation*}
$$

Substituting for $\varphi=\pi / 2$ we get:

$$
\begin{equation*}
v_{a x}=u_{0} \frac{\operatorname{tg} \alpha \cdot \sin 90^{\circ}}{\operatorname{tg} \alpha \cdot \cos 90^{\circ}+\sin 90^{\circ}}=u_{0} \cdot \operatorname{tg} \alpha \Rightarrow u_{0}=v_{a} \tag{7}
\end{equation*}
$$

In this case, the material is moving in the direction of the screw axis without circumferential velocity component (such as nut and bolt). It follows that the real value of the angle $\varphi$ will lie somewhere between these extremes. Using those relationships and knowledge can be expressed extruded amount. For simplicity, here is mentioned relationship for angular profile with the neglecting radius on the bottom of the screw profile:

$$
\begin{equation*}
\dot{V}_{S}=v_{a x} \cdot \pi \cdot(D-h) \cdot h \cdot\left(1-\frac{i e}{s}\right) \cdot \rho_{S H} \tag{8}
\end{equation*}
$$

Substituting:

$$
\begin{equation*}
\dot{V}_{s}=u_{0} \cdot \frac{\operatorname{tg} \alpha \cdot \operatorname{tg} \varphi}{\operatorname{tg} \alpha+\operatorname{tg} \varphi} v_{a x} \cdot \pi \cdot(D-h) \cdot h \cdot\left(1-\frac{i e}{s}\right) \cdot \rho_{S H} \tag{9}
\end{equation*}
$$

( $\dot{V}_{S}$ - volume flow of material, $\rho_{S H}$ - bulk density of material).
For circumferential speed $u_{o}$ is valid:

$$
\begin{equation*}
u_{0}=\pi . D . n \tag{10}
\end{equation*}
$$

( $D$ - screw diameter, $n$ - number of screw revolutions). Then we can write:

$$
\begin{equation*}
V s=\pi^{2} . D .(D-h) \cdot h .\left(1-\frac{i e}{s}\right) \cdot n \cdot \frac{\operatorname{tg} \alpha \cdot \operatorname{tg} \varphi}{\operatorname{tg} \alpha+\operatorname{tg} \varphi} \cdot \rho_{S H} \tag{11}
\end{equation*}
$$

As conditions will vary in length $l$, the feeding angle will vary too. When the bulk material is pressed, the feeding angle $\varphi$ shrink lengthwise.
On the defined particles when the screw profile is completely filling, the forces affect by the following Fig. 5.


Figure 5. Force conditions in screw profile
Three-dimensional system of forces is converted to the planar so that all forces are reduced to the outer diameter of the screw. We respect the same pitch by introduction the pitch angles, see Fig. 6.

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Figure 6. Pitch angles on screw profile
From Fig. 6 for forces balance in the direction of relative speed is valid:

$$
\begin{align*}
& F \cdot \frac{D-h}{D}=p \cdot f_{p} \cdot s \cdot\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{p} \cdot \frac{d l}{\sin \alpha_{p}} \cdot \sin \left(\alpha_{p}+\varphi\right) \\
& \text { 2.p.f. } f_{z} \cdot h \cdot \frac{d l}{\sin \alpha_{0}} \cdot \frac{D-h}{D}+p \cdot f_{z} \cdot s \cdot\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{s} \cdot \frac{d l}{\sin \alpha_{s}} \cdot \frac{D-2 h}{D}+d p . h \cdot s \cdot\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{0} \cdot \frac{D-h}{D}+F \cdot \frac{D-h}{D} \cdot f_{s}=  \tag{12}\\
& =p . f_{p} \cdot s .\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{p}}{\sin \alpha_{p}} \cdot d l \cdot \cos \left(\alpha_{p}+\varphi\right)
\end{align*}
$$

Where: $f_{z}$ - friction coefficient between the material and the screw, $f_{p}$ - friction coefficient between the material and the die.
Exclusion of F we get:

$$
\begin{align*}
& \text { 2.p.f.f.h. } \frac{d l}{\sin \alpha_{0}} \cdot \frac{D-h}{D}+\text { p.f.f.s. }\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{s} \cdot \frac{d l}{\sin \alpha_{s}} \cdot \frac{D-2 h}{D}+d p . h . s .\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{0} \cdot \frac{D-h}{D}= \\
& =\text { p.f.f.s. }\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{p}}{\sin \alpha_{p}} \cdot d l .\left[\cos \left(\alpha_{p}+\varphi\right)-f_{s} \cdot \sin \left(\alpha_{p}+\varphi\right)\right] \tag{13}
\end{align*}
$$

It is also valid:

$$
\begin{align*}
& \text { h.s. }\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{0} \cdot \frac{D-h}{D} \cdot \frac{d p}{p}= \\
& =\left\{f_{p} \cdot s .\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{p}}{\sin \alpha_{p}} \cdot\left[\cos \left(\alpha_{p}+\varphi\right)-f_{s} \cdot \sin \left(\alpha_{p}+\varphi\right)\right]-2 f_{z} \cdot \frac{h}{\sin \alpha_{0}} \cdot \frac{D-h}{D}-f_{z} \cdot s \cdot\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{s}}{\sin \alpha_{s}} \cdot \frac{D-2 h}{D}\right\} d l \tag{14}
\end{align*}
$$

After treatment:

$$
\begin{align*}
\frac{d p}{p} \cdot \frac{D}{d l} & =\frac{1}{\cos \alpha_{0}} \cdot \frac{f_{p}}{\frac{h}{D}} \cdot \frac{1}{1-\frac{h}{D}} \cdot \frac{\cos \alpha_{p}}{\operatorname{tg} \alpha_{p}} \cdot\left[\cos \varphi-f_{z} \cdot \sin \varphi-\operatorname{tg} \alpha_{p} \cdot\left(f_{z} \cdot \cos \varphi+\sin \varphi\right)\right]-  \tag{15}\\
& -\frac{2 \cdot f_{z}}{\frac{s}{D} \cdot\left(1-\frac{i e}{s}\right) \cdot \sin \alpha_{0} \cdot \cos \alpha_{0}}-\frac{f_{z} \cdot \frac{D-2 h}{D-h}}{\frac{h}{D} \cdot \cos \alpha_{0} \cdot \operatorname{tg} \alpha_{s}}
\end{align*}
$$

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Assuming that the right-side expression the variables will not depend on the length $l$ nor the pressure $p$, then integrating and using the boundary condition $l=0 \rightarrow p=p_{0}$, where $l$ is the working length of the screw with filled profile, we get:

$$
\begin{align*}
\frac{D}{l} \cdot \ln \frac{p}{p_{0}}= & \frac{f_{p}}{\frac{h}{D} \cdot \operatorname{tg} \alpha_{p} \cdot \cos \alpha_{0} \cdot \frac{D-h}{D}}\left[\cos \varphi-f_{z} \cdot \sin \varphi-\operatorname{tg} \alpha_{p} \cdot\left(\sin \varphi-f_{z} \cos \varphi\right)\right]- \\
& -\frac{2 \cdot f_{z}}{\frac{s}{D} \cdot\left(1-\frac{i e}{s}\right) \cdot \sin \alpha_{0} \cdot \cos \alpha_{0}}-\frac{f_{z} \cdot \frac{D-2 h}{D-h}}{\frac{h}{D} \cdot \cos \alpha_{0} \cdot \operatorname{tg} \alpha_{z}} \tag{16}
\end{align*}
$$

The expression on the right side will denote as the constant of proportionality A.

$$
\begin{equation*}
A=\frac{f_{p}}{\frac{h}{D} \operatorname{tg} \alpha_{p} \cos \alpha_{0} \frac{D-h}{D}}\left[\cos \varphi-f_{z} \sin \varphi-\operatorname{tg} \alpha_{p}\left(\sin \varphi-f_{z} \cos \varphi\right)\right]-\frac{2 f_{z}}{\frac{s}{D}\left(1-\frac{i e}{s}\right) \sin \alpha_{0} \cos \alpha_{0}}-\frac{f_{z} \frac{D-2 h}{D-h}}{\frac{h}{D} \cos \alpha_{0} \operatorname{tg} \alpha_{z}} \tag{17}
\end{equation*}
$$

Then it is a valid:

$$
\begin{equation*}
p=p_{0} \cdot e^{A \cdot \frac{l}{D}} \tag{18}
\end{equation*}
$$

The above relations show that the pressure in the screw profile is exponentially dependent on the length of the screw. Proportionality constant $A$ depends on the geometry of screw profile and the friction coefficient between material and die $f p$, and the material and screw $f z$. The condition of steep growth pressure requires that the coefficient of friction $f p$ is the largest and coefficient $f z$ the smallest. Coefficient $f z$ can be greatly affected by the quality of screws surface. The aim is to achieve the least surface roughness. Enlargement of the coefficient $f p$ can be achieved by rougher surface treatment of the die or grooves in the screw axis. Using the grooves will not only increase friction, but form closure prevents rotation of material and causes the so-called block axial flow.

### 3.3 Frictional power of screw

Movement of material in the filled screw profile has connection with friction power. The friction power is transformed into heat.

Elementary friction power on the die is given by expression:

$$
\begin{equation*}
d P_{p}=k_{a} \cdot p . f p . s \cdot\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{p} \cdot \frac{d l}{\sin \alpha_{p}} \cdot v_{a} \tag{19}
\end{equation*}
$$

( $k_{a}$ is coefficient of pressure anisotropy).
Elementary friction power on the screw is given by expression:

$$
\begin{equation*}
d P_{Z}=k_{a} \cdot\left[p . f s . s .\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{s} \cdot v_{r s}+2 . p 2 f s . h \cdot \frac{d l}{\sin \alpha_{0}} \cdot v_{r 0}+p . f p . f s . s .\left(1-\frac{i e}{s}\right) \cdot \cos \alpha_{p} \cdot \frac{d l}{\sin \alpha_{p}} \cdot v_{r 0} \cdot \sin (\alpha+\varphi)\right] \tag{20}
\end{equation*}
$$

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Using those relationships and integration within the limits from 0 to 1 under the above assumptions we get:

$$
\begin{gather*}
P p=k_{a} \cdot f p . s .\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{p}}{\sin \left(\alpha_{p}+\varphi\right)} \cdot u_{0} \cdot \frac{p_{l}-p_{0}}{\ln \frac{p_{l}}{p_{0}}} \cdot l  \tag{21}\\
P z=u_{0} \cdot \sin \varphi \cdot l \cdot \frac{f s}{\sin \left(\alpha_{0}+\varphi\right)} \cdot \frac{p_{l}-p_{0}}{\ln \frac{p_{l}}{p_{0}}} \cdot\left[k p . s .\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{s}}{\sin \alpha_{s}} \cdot \frac{\sin \left(\alpha_{0}+\varphi\right)}{\sin \left(\alpha_{s}+\varphi\right)}+2 \cdot h \cdot \frac{1}{\sin \alpha_{0}} \cdot \frac{D}{D-h}-k a . f p . s .\left(1-\frac{i e}{s}\right) \cdot \frac{\cos \alpha_{p}}{\sin \alpha_{p}}\right] \tag{22}
\end{gather*}
$$

The resulting friction power with filled profile:

$$
\begin{equation*}
P v=P p+P z \tag{23}
\end{equation*}
$$

Torque on the screw:

$$
\begin{equation*}
M k=\frac{P v}{2 \cdot \pi \cdot n} \tag{24}
\end{equation*}
$$

Friction power $P v$ and torque on the screw $M k$ are necessary for drive of machines design.

## 4 Application of the theory of geometry pressing screw design

The usefulness of that theory we demonstrate on three variants of the pressing screw geometry (Fig. 7, 8, 9). For comparison simplicity of variants, the same input parameters are chosen for all variants: outer screw diameter ( $\mathrm{D}=70 \mathrm{~mm}$ ), inner screw diameter $(\mathrm{d}=40 \mathrm{~mm})$, cross-sectional area of thread $\left(S=325 \mathrm{~mm}^{2}\right)$, width of the screw guide surface $(\mathrm{e}=5 \mathrm{~mm})$, number of threads $(\mathrm{z}=3)$.


Figure 7. Screw cross-section - variant A

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Figure 8. Screw cross-section - variant B


Figure 9. Screw cross-section - variant C
Figure 10 shows that the work pressure is growing slowest in the screw geometry of variant A . Conversely, in the variant C the pressure grows steepest with the number of threads. The thread area is constant in all three variants. Just screw pitch varies depending on changes in the profile. In terms of condition of steep pressure growth is the most advantageous the geometry of variant C . It can not be definitively held that the mentioned thread profile is optimal in all aspects. Certainly it is necessary to verify that assumption also from other aspects - strength, technological and economic.


Figure 10. Dependence of pressure available by screw and number of screw-threads at constant initial pressure ( $p_{0}=1.05 \mathrm{MPa}$ )

## 5 Conclusion

Compaction of biomass is relatively complicated process. Therefore good design of the pressing tool geometry is a fundamental for the success of the technology. The mentioned analysis of the pressing screw geometry can help in the design of new progressive tools that would eliminate the deficiencies of current instruments and allow the increase in biomass briquetting technology and application in a larger share of utilization of solid high-grade biofuel within the renewable energy sources.

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# MATHEMATICS OF INSTRUMENTED INDENTATION 

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#### Abstract

Indentation testing is a common method to investigate mechanical properties of solids near their surface. The elastoplastic properties such as microhardness and Young modulus may be determined from the curves load - depth of penetration (Depth Sensing Indentation - DSI). In our work basic mathematical elements of DSI theory used for explanation and interpretation of experimental results for some pure metals are presented.


Key words: microhardness, instrumented indentation, pure metals,
Mathematics Subject Classification: 74Fxx - Mechanics of deformable solids

## 1 Introduction

Material science is a discipline nearly as old as mankind. Over a long period (up to $30^{\text {th }}$ last century) it had been developed only empirically but from that time many new physically based methods have been introduced and intensively developed. For example the hardness methods were established by the end of $19^{\text {th }}$ century (Mohs scale (1898), Brinell hardness test (1900), Mayer test (1908), Rockwell test (1914), Vickers test (1924) and Knoop test (1939)). In the second half of the $20^{\text {th }}$ century a new approach to mechanical properties of material investigation has began. The founders of such approach were H. Hertz, D. Tabor, Sneddon and especially Bulychev and Alekin who published an analysis of continuous conical and spherical indentation, later called "Instrumented indentation", or "Depth Sensing Indentation"(DSI). Instrumental indentation in micro- and nanoscale is now very popular technique for measuring the mechanical properties of surface of bulk materials and thin films [1-3]. This method provides to record the load penetration curves during the loading and unloading process. According to Doerner and Nix [4] and Oliver and Pharr [5] many important parameters such as hardness, and Young modulus can be determined especially from unloading curves.

## 2 Instrumented indentation

It has been observed that sharp indenters (Vickers, Berkovich, Knoop) deform material both elastically and plastically and that the indenter is shelded by a hydrostatic core and plastic zone. The final model is based on the similarity to the processes in radial expansion of a spherical cavity under internal pressure [3] (Fig.1)


Fig. 1. Expanding cavity model [6]


Fig.2. Contact mechanics [7]

The rudiments of micro- and nanoindentation begins with study of contact mechanics between solid bodies, which were done by H.Hertz at the end of $19^{\text {th }}$ century. Hertz [8] determined that the radius of the circle of contact $a$ cold be calculated in terms of load $P$, the radius $R$ and elastic modulus $E^{*}$ by the relation (Fig.2)

$$
\begin{equation*}
a^{3}=\frac{3}{4} \frac{P \cdot R}{E^{*}} \tag{1}
\end{equation*}
$$

where $E^{*}$ is the combined elastic modulus of the indenter and specimen:

$$
\begin{equation*}
\frac{1}{E^{*}}=\frac{\left(1-v_{1}^{2}\right)}{E_{1}}+\frac{\left(1-v_{2}^{2}\right)}{E_{2}} \tag{2}
\end{equation*}
$$

( $v_{1}$ resp. $v_{2}$ is Poisson's ration of material resp. indenter and $E_{1}$, resp. $E_{2}$ is Young modulus of material, resp. indenter). For radius $R$ we have (see Fig.2):

$$
\begin{equation*}
\frac{1}{R}=\frac{1}{R_{1}}+\frac{1}{R_{2}} \tag{3}
\end{equation*}
$$

(for flat specimen $\mathrm{R}_{1} \rightarrow \infty$ ).
For an rigid spherical indenter the profile of deformed surface can be expressed ([7], Fig.3) as

$$
\begin{equation*}
\delta^{3}=\left(\frac{3}{4 E^{*}}\right)^{2} \frac{P^{2}}{R} \tag{4}
\end{equation*}
$$

And finally with using (1):

$$
\begin{equation*}
P=\frac{4}{3} E^{*} R^{\frac{1}{2}} h^{\frac{3}{2}} \tag{5}
\end{equation*}
$$

And contact area $A=2 \pi R h_{c}$

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( $h_{c}$ is the depth of the circle of contact).

a)

b)

Fig 3. To evaluation of $P$ for round a) and conical b) indenter [7]
Analogous considerations give for conical indenter [7]:

$$
\begin{equation*}
P=\frac{2}{\pi} E^{*} h^{2} \operatorname{tg} \alpha \tag{6}
\end{equation*}
$$

and

$$
A=\pi \cdot h_{c}^{2} \operatorname{tg}^{2} \alpha
$$

(where $\alpha$ is the indenter cone half-angle)

## For Berkovich indenter:

$$
\begin{equation*}
P=2 a \cdot E^{*} h \tag{7}
\end{equation*}
$$

and

$$
\begin{aligned}
A & =3 \sqrt{3} h_{c}^{2} \operatorname{tg}^{2} \Theta=24.49 h_{c}^{2} \\
\text { (where } \theta & =65,27^{0} \text { and } \alpha=70,296^{0} \text { ) }
\end{aligned}
$$

## For Vickers indenter:

$$
\begin{equation*}
P=2 a \cdot E^{*} h \tag{8}
\end{equation*}
$$

and

$$
\begin{aligned}
& A=4 h_{c}^{2} \operatorname{tg}^{2} \Theta=24,504 h_{c}^{2} \\
& \text { (where } \theta=68^{\circ} \text { and } \alpha=70,3^{\circ} \text { ) }
\end{aligned}
$$

## For Knoop indenter:

$$
\begin{equation*}
P=2 a \cdot E^{*} h \tag{9}
\end{equation*}
$$

and

$$
\begin{gathered}
A=2 h_{c}^{2} \operatorname{tg} \Theta_{1} \operatorname{tg} \Theta_{2}=108,24 . h_{c}^{2} \\
\left(\text { where } \theta_{l}=86,25^{0}, \theta_{2}=65^{0}, \alpha=77,64^{0}\right. \text { ) }
\end{gathered}
$$

And for Cube corner indenter:

$$
\begin{equation*}
P=2 a \cdot E^{*} h \tag{10}
\end{equation*}
$$

And

$$
A=3 \sqrt{3} h_{c}^{2} \operatorname{tg}^{2} \Theta=2,60 h_{c}^{2}
$$

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$$
\text { (where } \theta=35,26^{0}, \alpha=42,278^{\circ} \text { ) }
$$

### 2.1. Contact Stiffness

The elastic contact between cone and flat specimen is done by the equation (6):

$$
\begin{equation*}
P=\frac{2}{\pi} E^{*} h^{2} \operatorname{tg} \alpha \tag{6a}
\end{equation*}
$$

Taking the derivation of $P$ with respect to $h$ we obtain:

$$
\begin{equation*}
\frac{d P}{d h}=2\left[\frac{2}{\pi} E^{*} \operatorname{tg} \alpha\right] h \tag{11}
\end{equation*}
$$

And after substitution to the (6a) we have:

$$
\begin{equation*}
P=\frac{1}{2} \frac{d P}{d h} h \tag{12}
\end{equation*}
$$

which means that the slope of the load - displacement curve for an elastic loading at any particular point on the curve is twice that given by the ration $P / h$.
For $\mathrm{r}=0$ is the relation between the displacement of the indenter and the radius of the circle of contact done by the equation

$$
\begin{equation*}
h=\frac{\pi}{2} a \cdot \cot g \alpha \tag{13}
\end{equation*}
$$

Since $A=\pi \cdot \mathrm{a}^{2}$, we have

$$
\begin{equation*}
\frac{d P}{d h}=2 E^{*} a \tag{14}
\end{equation*}
$$

## 3. Experimental procedure.

The equipment. The Fischerscope H100 Xyp is a computer controlled microindentation system by Fischer Technology. Its main characteristics are:

- Load range $0,4 \sim 1000 \mathrm{mN}$,
- Depth sensitivity $\pm 1 \mu \mathrm{~m}$,
- Maximal indentation depth $700 \mu \mathrm{~m}$.

The equipment is located in Laboratory of Mechanical Properties of Thin Solid Films, Institute of Physical Electronics, Masaryk University Brno. The calibration of equipment was performed on certified standard samples with well known mechanical properties and without any work hardening. These samples are usually carried out by massive planparallel blocks of boron silicate glass with small surface roughness. Typical schematical load - indentation depth curve with parameters commonly measured is shown on Fig.4.



Fig.4. Typical load - indentation depth curve and Vickers indent.
Measured materials. The samples of polycrystalline tin and copper (purity of 99,99 \%) were mechanically grained and chemically polished to remove Beilby's layer. In order to find the possible influence of the layer on microhardness both measurements were provided on samples with and without Beilby's layer [9]. The microindentation experiments were performed by using Vickers indenter tips. All experimental results are shown on Figs 5,6.

## 4. Results and discussion.

According to our measurements, the Beilby's layer does not influence significantly microhardness of pure tin and copper (Figs. 5, 6). It can be explained either as a consequence of a small thickness of the layer ( $\sim 100 \mathrm{~nm}$ ) or as a result of similar mechanical properties of Beiby's layer and bulk material under it. The answer can be confirmed by nanoindentation measurements because nanoindents penetrate only this layer unlike microindents which are much more greater (micrometers).


Fig. 5. Microhardness vs. Load for pure tin. Results for polished (red dots) and only grained samples (blue dots) show the occurence of ISE.


Fig.6. Microhardness vs. Load for pure copper. Results for polished (red dots) and only grained samples (blue dots) do not show occurence of ISE.

Second result of our measurements confirm existence of the Indentation Size Effect (ISE), i.e. dependence of the microhardness on the load (or on the indentation depth) [10,11]. According to many authors the explanation of ISE can be done by

- abroaded surface layers and oxides,
- chemical contamination,
- inadequate measurement capability of small areas of indents,
- elastic recovery of indents,
- indenter-specimen friction.
- occurrence of the geometrically necessary dislocations.

As we can see from the Figs. 5 and 6, the ISE for pure tin is considerable unlike for pure copper where was not observed. The explanation of that difference can be done by existence of geometrically necessary dislocations. The structure of tin is tetragonal, unlike copper, which is faced centered cubic (f.c.c.). It means that in copper there is at the beginning of indentation more working slip systems than in tin, i.e. for small indents is mechanical resistence (hardness) of tin comparatively higher than that for copper one.

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# TO THE IMPACT OF THE OCCUPATION ON THE HOUSEHOLDS INCOME 

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#### Abstract

The paper is focused on the analysis of household income. We try to describe and analyze how the occupation of the head of household influence the household income per capita and per EU scale equivalency and what is the development of this influence in time (1992 2008). We have at a disposal micro-data on household incomes from Micro-censuses (1992, 1996, 2002) and from the Statistics of Income and Living Conditions (annual data from 2005 2008). We also use the aggregated and weighted data from both these surveys. Two different approaches are used. Firstly, the regression models are built up. These models for every year explain the household income by many factors including the occupation of the head of household. Secondly, the deciles distributions of income divided by occupations are analyzed. At both approaches, the incomes per capita and per EU scale equivalency are considered.


Key words. Household income, micro-census, SILC, head of household, occupation
Mathematics Subject Classification: Primary 62P20; Secondary 91B82.

## 1 Introduction

Surveys of household income have a very long tradition and are an important part of the social statistics. Till 2002, the surveys (micro-censuses) had been realized in intervals of $4-6$ years, from 2005 the income and living condition are surveyed using harmonized surveys called Statistics of Income and Living Conditions (EU-SILC). The aim of the survey is to gather representative data on income distribution and on housing, labour, financial and health conditions not for the whole population, but also for various household types and adults living in the households.

Our attention is paid to the impact of occupation on the income distribution. How does the occupation of head of household influence the income per capita and per EU scale equivalency? What is the impact on deciles distribution? Is there any trend in time (from 1992 to 2008)?

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## 2 Data and Methodology

For our analysis the data from the Czech Statistical Office (CZSO) are used. Two types of surveys were realized in period 1992 - 2008. We have micro-data sets from micro-censuses 1992, 1996 and 2002. The number of surveyed household is $4,353,7,485$ and 28,150 units respectively. Also the micro-data sets are at a disposal from the EU-SILC surveys for each year from period 2005 - 2008 (number of units varies between approx. 8,000 and 16,000 ). The aggregated data are also used for our analysis, because these data, published by the CZSO are weighted and grossed-up ${ }^{1}$. It implies advantages and disadvantages of using micro-data vs. aggregated data. While the micro-data allow us to use advanced statistical techniques for the multivariate analysis, they are not good for direct aggregation due to the biases caused by non-responses. For our analysis we use micro-data and the aggregated data as well. For the purpose of assessment the impact of occupation, we use Classification of Occupation (CZ-ISCO).

### 2.1 Regression Analysis

As the first approach, we use the regression analysis for describing the impact of occupation. For every year we try to find regression function

$$
\begin{equation*}
Y=f\left(x_{1}, x_{2}, \ldots, x_{n}\right), \tag{2.1}
\end{equation*}
$$

Where $Y$...income of individual household, $x_{1}, x_{2}, \ldots, x_{n} . \ldots$ characteristics of the head of household (occupation, age, edcuation level, gender...) or of the whole household (number of children etc.).

Some of the factors (incl. the occupation, sex, education) are represented by the dummy variables (see table 1). As the reference category there was chosen a woman who has a university education and works as a manager, senior official, law-giver. Regression parameters give us the information on the influence of the related factor. The regression model is built-up for every year and for household income per capita and per EU scale equivalency.

Table 1

| Occupation |  | Education |  |
| :--- | :--- | :--- | :--- |
| non-working | dummy_0 | non or not finished primary | dummy_ed0 |
| managers, senior officials, law-givers* | dummy_1 | Primary | dummy_ed1 |
| professionals, research workers | dummy_2 | lower secondary, apprenticeship | dummy_ed2 |
| teachers, technicians, medical staff | dummy_3 | secondary, post-secondary | dummy_ed3 |
| lower administrative staff | dummy_4 | baccalaureate, graduate, postgraduate* | dummy_ed4 |
| service sector workers <br> skilled workers in agriculture and <br> forestry | dummy_5 | ph.d. studies | dummy_ed5 |
|  |  |  |  |
|  | dummy_6 |  |  |

[^16]| craftsmen, artisans, skilled producers | dummy_7 |  |  |
| :--- | :--- | :--- | :--- |
| machinery operators | dummy_8 | Sex |  |
| unskilled workers, element.occupations | dummy_9 | Female* | dummy_sex0 |
| armed forces | dummy_10 | Male | dummy_sex1 |

*Reference group

The analysis and mainly the comparison in time are complicated by the fact that the selection of variables surveyed in micro-censuses has been slightly changed between surveys. Also the methodology of micro-censuses differs from the methodology of EU-SILC.

### 2.2 Decile Distribution

For the second approach, we use aggregated data from Micro-Census 2002 and from the EU-SILC 2005 - 2008. We analyze the deciles distribution of household income per capita and per EU scale equivalency, divided by occupations (using 1-digit CZ-ISCO).

There are two possible points of view on the distributions. On the one hand, we can see the structure of households by occupation for each individual decile. On the other hand, we can see the structure of households by the income-level (by income decile) for each individual occupation category. From the comparison between years we can see the development of the impact of occupancy on the household income, both per capita and per EU scale equivalency.

## 3 Results

### 3.1 Regression Analysis

We have used regression analysis to data from years 1992, 1996 and 2002. Data for additional years, which was obtained from the EU-SILC was unusable for the regression analysis in this research.

### 3.1.1 Household income 1992

For this research we have a reference household, where we consider as a head of household a single woman without children, with university level of education and working as manager, senior official or law-giver. The average income of a reference household per year per head was 60642 CZK.
In this case we describe changes for income per head. Incomes per economic and consumer units behave similarly.
Difference in income between profession in the group managers, senior officials, law-givers and the group unskilled workers, elementary occupations is 17856 CZK. If we compare income in group unskilled workers, elementary occupations and non-working, the difference is 4140 CZK. Every child in household reduces income per head of about 8567 CZK per year. Every non-working adult reduces income per head of about 5304 CZK per year. An average per year per head income
increases of about 4170 CZK when a man serves as a head of household. With a lower level of education, average income per head falls down, maximum difference is 9240 CZK per year. This situation is illustrated in Figure 1.


Figure 1

### 3.1.2 Household income 1996

Situation in 1996 is quite similar like in 1992. For an income level of households per head, economic unit and consumer unit there is still important education, profession, sex and age of a head, number of children, number of non-working persons in household and number of nonworking pensioners. There is only one important difference between year 1992 and 1996 because of increasing the price level. For this reason the average income was higher than in year 1992. And for the reference household there is a difference in average income per head of about 70744 CZK per year.

### 3.1.3 Household income 2002

In 2002, the price level was still increasing. The impact of education, profession, sex and age of the head, number of children, number of non-working persons in household and number of nonworking pensioners on average income per head, per consumer unit and per economic unit was not changed too much. The income per head for households with a head working as a lower administrative staff the same was the same as for households with a head working as a professional

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or research worker. But the income was higher in comparison to households in which the head worked as a teacher, technician or a medical staff.
The most important change in this year was a fact, that households with a non-working head had not the lowest income per head among households. Households, where a head worked as a craftsman, artisan or a skilled producer, had a lower income per head than household, where the head was non-working.


Figure 2

### 3.2 Decile Distribution

In a decile distribution the households are sorted by income per capita or per consumer unit to 10 deciles groups. In the lowest decile there are households with the lowest income (per head, per consumer unit) while the highest one comprises households with the highest income.

### 3.2.1 Decile distribution in 2005

In 2005 the head of households included to the lowest decile (sorted by income per capita) was most often non-working ( $49.5 \%$ ). If he was working, he was most often a craftsman, artisan or skilled producer ( $22.9 \%$ ). In the highest decile, the head worked as a professional, research worker ( $20.6 \%$ ) or as a manager, senior officials or law-giver ( $18.3 \%$ ). They are only $3.9 \%$ non-working heads in the highest decile.

If we sort households to deciles by income per consumer unit, $67.7 \%$ of heads in the lowest decile are non-working. Working heads worked mainly in a group craftsman, artisan, skilled producer ( $11.5 \%$ ). In the highest decile, heads of households worked most often in groups teacher,
technician, medical staff (26.4 \%), manager, senior official, law-giver (19.9 \%) and professional or research worker (19.8 \%). (See also figure 3).


Figure 3 - Decile distribution in 2005


Figure 4 - Decile distribution in 2006

### 3.2.2 Decile distribution in 2006

In 2006 head of households in the lowest decile (sorted by income per capita) was most often non-working ( $45.6 \%$ - it is less than in 2005). If he was working, he was most often a craftsman, artisan, skilled producer ( $19.7 \%$ ). On the other side, head of household in the highest decile worked most often as a teacher, technician or a medical staff ( $25.2 \%$ ), then professional, research worker $(22.6 \%)$ or manager, senior official and law-giver ( $13.6 \%$ ). There is only $6.3 \%$ non-working heads of households in the highest decile, it is about $3 \%$ higher than in 2005.

Sorting households to deciles by income per consumer unit, $68.9 \%$ of heads in the lowest decile are non-working. Working heads of households worked mainly in group craftsman, artisan, skilled producer ( $9.6 \%$ ). In the highest decile, heads of households worked most often as a teacher, technician, medical staff ( $24.4 \%$ ), then as professional, research worker ( $21.9 \%$ ) and manager, senior official and law-giver (16.9 \%). See also figure 4.

### 3.2.3 Decile distribution in 2007

In 2007, the share of non-working heads of households included to the lowest deciles increased to $47.5 \%$. Working head of households worked mainly as a craftsman, artisan, skilled producer ( $17.1 \%$ ). In the highest decile, heads of households worked most often still in groups teacher, technician, medical staff ( $27.3 \%$ ), professional, research worker ( $22.9 \%$ ) and manager, senior official, law-giver ( $13.6 \%$ ). Number of heads of households included to the highest decile, which worked as artisan and skilled producer appreciably increased (13.1 \%).

Also in 2007 there was an increase of non-working heads of households in the lowest decile (sorted by income per consumer unit). Almost $71 \%$ of heads of households included to the lowest decile was non-working. Working heads worked mainly in group craftsman, artisan and skilled producer ( $8.1 \%$ ). In the highest decile, heads of households worked most often in groups teacher, technician, medical staff ( $25.7 \%$ ), professional or research worker ( $23.1 \%$ ) and manager, senior official, law-giver ( $15.7 \%$ ). See also figure 5 .

### 3.2.4 Decile distribution in 2008

In 2008 head of households included to the lowest decile (sorted by income per capita) was most often non-working $(45.3 \%)$. The most often group of professions remains unchanged like in 2007 - craftsman, artisan, skilled producer ( $18.2 \%$ ). In the highest decile, the head worked mainly as a teacher, technician, medical staff (25.3 \%), professional or research worker (19.0 \%) and manager, senior official and law-giver ( $15.2 \%$ ).

As well as in 2007, there is an increase of non-working heads of households in the lowest decile (by income per consumer unit). Almost $72 \%$ heads of households included to the lowest decile were non-working. Number of working heads of households decreased in most represented group craftsman, artisan and skilled producer to $7.0 \%$. In the highest decile, heads of households worked most often as a teacher, technician, medical staff ( $24.5 \%$ ), professional or research worker ( $18.7 \%$ ) and manager, senior official and law-giver (17.1 \%). See also figure 6.


Figure 5 - Decile distribution in 2007


Figure 6 - Decile distribution in 2008

### 3.2.5 Decile distribution for non-working head of households

While the decile distribution of households for all the groups of working heads of households by ISCO classification corresponds to our expectations (e. g. higher ratio of households with more qualified heads in higher deciles), for non-working heads the situation was surprising. In our reference period (2005-2008), most households with a non-working head were not included to the lowest decile (as we expected), but to the deciles 4 and 5 (i. e. middle-class), see also table 2 . This could be caused by the system of social benefits.

Table 2 - Non-working head of household in deciles (in \%)

| Year | lowest <br> $10 \%$ | second <br> $10 \%$ | third <br> (0\% | fourth <br> $10 \%$ | fifith <br> $10 \%$ | sixth <br> $10 \%$ | seventh <br> $10 \%$ | eighth <br> $10 \%$ | ninth <br> $10 \%$ | highest <br> $10 \%$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2005 | 13,30 | 6,61 | 11,49 | 16,40 | 16,89 | 14,63 | 10,22 | 6,19 | 3,23 | 1,05 |
| 2006 | 11,95 | 7,99 | 12,36 | 14,91 | 15,51 | 14,98 | 11,55 | 6,22 | 2,88 | 1,65 |
| 2007 | 13,30 | 6,61 | 11,49 | 16,40 | 16,89 | 14,63 | 10,22 | 6,19 | 3,23 | 1,05 |
| 2008 | 11,61 | 8,17 | 13,17 | 15,52 | 15,34 | 13,98 | 10,20 | 6,88 | 3,20 | 1,92 |

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## 4 Discussion

Two different data sources and two different analytical approaches have been used. While the data from Micro-Censuses are at a disposal once per several years (1992, 1996, 2002), the EU-SILC survey is realized every year from 2005. Due to the methodological changes between Micro-Census and EU-SILC, it is not possible to make a direct comparison between the first and the second periods. Since the Micro-Censuses were also changed from 1992 - 2002, we preferred the comparison of the data using regression models for each of the survey. The decile distribution used for EU-SILC data gives us the other point of view for the same issue.

## 5 Conclusion

The issue of the impact of occupation on the households' income has been solved. Using two data sources and two methodological approaches we can conclude that the income is strongly influenced by the occupation of the head of household both for the income per capita and per EU scale equivalency. As a side-effect we analyzed the other income factors such as sex, level of education, number of children and so on.
But, contrary to our expectation, the income situation of household with a non-working head is not so. Using the decile distribution we can see that these households are mainly included to the middle class.

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# FINITE ELEMENTS METHOD APPLICATION AT THE ENGINEERING OF BRIQUETTING MACHINE FOR BIOMASS 

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#### Abstract

Finite element method is a progressive tool for developed engineering work. Softwares which used the finite element method for simulating physical phenomenon are more and more sophisticated. Engineers have to find a way how to use these software tools correctly. For these types software simulators exist verifications methods which compare Analytical solution with Computed results. These particular master tasks are useful for real usage and calibrate the software for application in real life. The biomass compacting is a very complicated process, because the biomass is a "live" material and there are many factors influencing this process. We used finite element method for our development work and optimizing the machines construction.


Key words: biomass, finite element method, pressing, compacting, briquetting, Catia V5R20, solver verification.

Mathematics Subject Classification: Primary 97M50, 62K15.

## 1 Background

The finite element method is a numerical technique which gives approximate solutions to differential equations that model problems arise in physics and engineering. As in simple finite difference schemes the finite element method requires a problem defined in geometrical space (or domain) to be subdivided into a finite number of smaller regions (a mesh). In finite differences, the mesh consists of rows and columns of orthogonal lines; in finite elements, each subdivision is unique and does not need to be orthogonal. For example, triangles or quadrilaterals can be used in two dimensions, and tetra hexagons or hexahedrons in three dimensions. Over each finite element, the unknown variables (e.g., temperature, velocity, etc.) are approximated using known functions; these functions can be linear or higher-order polynomial expansions that depend on the geometrical locations (nodes) used to define the finite element shape. In contrast to finite difference procedures (conventional finite difference discretizations, as opposed to the finite volume method, which is

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integrated), the governing equations in the finite element method are integrated over each finite element and the solution summed ("assembled") over the entire problem domain. As a consequence of these operations, a set of finite linear equations is obtained in terms of a set of unknown parameters over each element. Solution of these equations is achieved using linear algebra techniques.

## 2 History

The history of the finite element method is particularly interesting, especially since the method has only been existing since the mid 1950s. Earlier work on numerical solution of boundary value problems can be traced to the use of finite difference schemes; Southwell discusses the use of such methods on his book published in 1946. The beginnings of the finite element method actually stem from these earlier numerical methods and the frustration associated which attempting to use finite difference methods on more difficult, geometrically irregular problems (Roache 1972). Beginning in the min 1950s, efforts to solve continuum problems in elasticity using small, discrete „elements" to describe the overall behavior of simple elastic bars began to appear. Argyris (1954) and Turner at all (1956) were the first to publish on the use of such techniques for the aircraft industry. The actual coining of the term "finite element" appeared in a paper by Clough (1960). The earlier use of finite elements was restricted to the application of such techniques for structural related problems. However, the versatility of the method and its underlying rich mathematical basis were soon recognized by others for application in nonstructural areas. Zienkiewicz and Cheung (1965) were among the first to apply the finite element method to field problems (e.g., heat conduction, irrotational fluid flow, etc.) involving solution of Laplace and Poisson equations. Much of the earlier work on nonlinear problems can be found in Oden (1975). Efforts to model heat transfer problems which complex boundaries are discussed in Hueber (1975); a comprehensive threedimensional finite element model for heat conduction is described by Heuser (1972). An earlier application of the finite element technique to viscous fluid flow is given by Martin (1968). A virtual explosion in usage of the method has occurred since the mid 1970s. Numerous articles and texts have been published, and new applications appear routinely in the literature. Excellent reviews and descriptions of the method can be found in the texts by Finlayson (1972), Desai (1979), Becker at al. (1981), Fletcher (1984), Reddy (1984), Segerlind (1984), Hugnes (1987), Bickford (1990), and Zinkewiwcz and Taylor (1989). A rigorous mathematical discussion is given in the text by Johnson (1987), and programming the finite element method is described by Smith (1982). A short monograph on development of finite element method is given by Owen and Hinton (1980). The underlying mathematical basic of finite element method first lines with the classical Rayleigh-Ritz method and variation calculus procedures introduced by Rayleigh (1977) and Ritz (1909). These theories provided the reasons why the finite element method worked well for the class of problems in which variation statements could be obtained (e. g. linear diffusion-type problems). However, as interest, expanded theory could no longer be used to describe such problems. This is particularly accident in fluid-related problems involving convection. Extension of the mathematical basic to nonlinear and nonstructural problems was achieved through the method of weighted residuals, originally conceived by Galerkin (1915) in the earlier 20th century. The method of weighted residuals was found to provide Rayleigh-Ritz method can be applied. Basically, the method requires the governing product to be integrated over space; this integral is then required to vanish. Technically Galerkin's method is a particular case of the general weighted residuals procedure where various typed of weights can be utilized; in the case of Galerkin's method, the weights are chosen to be the same as the functions used to define the unknown variables. Galerkin's method

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yields results identical to the Rayleigh of functions; the weighted residual method yields the finite volume technique. A rigorous description of the method of weighted residuals can be found in Finlayson (1972). Most practitioners of the finite element method now employ Galerkin's method to establish the approximations to the governing equations.

## 3 Mathematical models and engineering decisions

Controversial safety problems of dams, bridges, offshore drilling platforms, aircraft, automobiles, etc., pose many difficult questions (
Figure 1). As a simple illustration, consider the safety of a bridge. The bridge has to resist loads caused by traffic, high wind, and earthquake. Does this mean, however, that the bridge has to resist maximum traffic load, high wind, and earthquake imposed simultaneously? What does "resist" mean? What kind of damages is acceptable under various conditions? What intensity of earthquake and wind should one assume in designing the structure? The problem of safety and durability are formulated differently in one time than in another. For example, in the 1970s the U. S. Air Force changed its design requirements from the maximum stress criterion to criteria based on linear elastic fracture mechanics.


Figure 1 Schematic representation of the engineering decision-making process

### 3.1 The General Mathematical Model

The general mathematical model, also called the mathematical model, is an idealized representation of reality. In constructing mathematical models we take into consideration our purposes in wanting to model a physical system or process, what we know about the physical system or process, and what we consider important about it. The mathematical model will represent reality only if we succeed in taking into consideration all factors that will affect the conclusions drawn from the mathematical solution. Among the factors we need to consider is how well we know the data needed for the mathematical model, such as the material properties, geometric properties, and loading. Uncertainties about the data can be taking into consideration by stochastic models or by using bounding assumptions which are conservative with respect to the appropriate safety and durability requirements. Because of their complexity, general mathematical models do not permit exact solutions. Of necessity, the solutions are approximate. Correct interpretation of these approximate solutions is possible only if one is aware of the assumptions incorporated in the mathematical model and the implied limitations.
Finite element analysis has now become an integral part of Computer Aided Engineering (CAE) and is being extensively used in the analysis and design of many complex real-life systems. While it

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started off as an extension of matrix methods of structural analysis and was initially perceived as a tool for structural analysis alone, its applications now range for structures to bio-mechanics to electromagnetic field problems. Simple linear static problems as well as highly complex nonlinear transient dynamic problems are effectively solved using the finite element method. The filed of finite element analysis has natured and now rests of rigorous mathematical foundation. Many powerful commercial software packages are now available, enabling its widespread use in several industries. Classical analytical methods consider a differential element ant develop the governing equations, usually in the prom of partial differential equations. When applied to real-life problem situations, it is often difficult to obtain an exact solution to these equations is view of complex geometry and boundary conditions. The finite element method (FEM) can be viewed simply as a method of finding approximate solutions for partial differential equations or as a tool to transform partial differential equations into algebraic equations, which are then easily solved. Some of key ideas used in finite element formulation are now summarized: - Since the solution for field variable satisfying both the boundary conditions and the differential equation is unknown, we begin with an assumed trial solution. The trial solution is chosen such that the boundary conditions are satisfied. The trial solution assumed, in general, does not satisfy the differential equation exactly and teases a domain residual defined as the error in satisfying the differential equation. - In general, the domain residual varies from point to point within the domain and cannot be exactly reduced to zero everywhere. We can choose to make in vanish at select points within the entire domain. Thus, the weighted sum of the domain residual computed over the entire domain is rendered zero. - The accuracy of the assumed trial solution can be improved by taking additional, higher order terms, but the computations become tedious and do not readily render themselves for automation. Also, for complex real-life problems, choosing a single continuous trial function valid over the entire domain satisfying the boundary conditions is not a trivial function valid over the entire domain satisfying the boundary conditions is not a trivial task. We therefore prefer to discretise the domain into several segments (called finite elements) and use several piece-wise continuous trial functions, each valid within a segment (finite element). - Trial functions used in each segment (finite element) are known as element level shape functions. These are defined in the form of interpolation functions used to interpolate the value of the field variable at an interior point within the element from its value at certain key points (called the nodes) in the element. Typical elements commonly used in finite element analysis are shown in Figure 2.


Figure 2: Typical finite elements (frame element, $2 d$ e., plate and shell e., 3-d e.)

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With these shape functions, the weighted sum of the domain residual is computed for each element and then summed up over all the elements to obtain the weighted sum for the entire domain. - For all elements using the same shape functions, the computations will be identical and, thus, for each type of element we have element level characteristic matrices. These characteristic matrices for several types of elements are derived a priori and programmed into finite element software such as ANSYS, NASTRAN, IDEAS, etc. And other categories of software called CAx systems which are more complex and oriented to complex engineering design solutions as CATIA, SOLID WORKS, SOLID EDGE, INVENTOR, PROENGINEER, etc. These categories of software solution contain lot of tools necessary for development worked during whole life cycles of product. These solutions are every year more complex and improvement. The user can choose to discretise (model) his domain with a variety of different finite elements. The computer program sets up the characteristic matrices for each element and then sums them all up for entire finite element mesh to set up and solve the system level equations. The basic steps of finite element analysis, as outlined above, are quite generic and can be applied to any problem-be it from the field of structural mechanics or heat transfer, or fluid flow or electromagnetic fields, given the appropriate differential equation and boundary conditions. In view of the similarity in the form of governing differential equations, the finite element formulation for a particular type of differential equation can be used to solve a class of problems. For example, a differential equation of the type.

$$
\begin{equation*}
A C \frac{d^{2} f}{d x^{2}}+q=0 \tag{1}
\end{equation*}
$$

Described of a rod when we use the connotation that $f$ represents the axial deformation, $q$ represents the load, and $A, C$ stand for cross-sectional area and Young's modules, respectively. The same equation, when interpreted with the connotation that $f$ stands for temperature, $q$ represents internal heat source and $A, C$ stand for cross-sectional area and coefficient of thermal conductivity, respectively will be the governing equation for one-dimensional heat conduction. Thus, finite element formulation developed for the above differential equation can be readily used to solve either of physical problems. Sometimes, the governing equations are more readily available in the form of minimization of a functional. For example, in problems of structural mechanics, the equilibrium configuration is the one that minimizes the total potential of the system. Finite element formulation can be developed readily for a problem described by a functional, rather than a differential equation. When both the forms are available for a given for each other. The finite element method essentially grew up as a tool for structural mechanics problems, as an extension of the matrix methods of structural analysis. While such an approach towards the study of finite element formulation enables easy visualization in the form of lumped springs, masse, etc., the approach outlined above highlights the generic nature of the method, applicable for a variety of problems belonging to widely varying physical domains. It is felt that this approach gives a proper perspective on the entire field of finite element analysis. In the chapters that follow, we elaborate on the various basic steps outlined above for one- ant two- dimensional, static and dynamic problems. We now present several examples of application of finite element analysis to real-life problems, to give an overview of the capabilities of the method. Our application examples are drawn from the fields of structural mechanics, aerospace, manufacturing, processes, electromagnetic, etc.

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### 3.2 Equation Solvers

After the computational model has been created, it is then fed to solver to solve the discretised system, simultaneous equations for the field variables at the nodes of the mesh. This is the most computer hardware demanding process. Different software packages use different algorithms depending upon the physical phenomenon to be simulated. There are two very important considerations when choosing algorithms for solving a system of equations: one is the storage required, and another is the CPU (Central Processing Unit) time needed. There are two main types of method tor solving simultaneous equations: direct methods and iterative methods. Commonly used direct methods include the Gauss elimination method and the LU decomposition method. Those methods work well for relatively small time of duration.

## 4 CATIA Elfini Solver Verification

For our application finite element method at the construction process of compacting machines FOR biomass we are using very sophistic software solution CATIA Structural Analysis. This software module is part of complex software solutions CAx categories developed by Dassault Systemes. Actual used version of product is CATIA V5R20. This module used Elfini Solver. For verification functionality any solvers you can use compare principle. This principle is base on compares analytical or reference solution with solution of solver represent by result in CATIA. Exempli gratia we presented some examples part of verification process. A Structural Analysis contains the Static Analysis, Modal Analysis, Buckling Analysis, Dynamic Response Analysis, Thermo Mechanical Analysis, Analysis Assembly and Composite. For actual task of engineering work we need mainly Static and Assembly Analysis. Description of several linear static analysis problems, several elements assembled with a fastened connection and comparison of results. In literature are introduced problem as Thick Beam, Twisted Beam, Bending of a Beam, Thick Cylinder, etc.
Static Analysis contains static linear analysis problems which illustrate some of the features and capabilities of CATIA-ELFINI. Static linear analysis consists in finding the deformed shape and the internal strains and stresses of an elastic structure subject to prescribed boundary conditions (displacement and traction types). In literatures you can find following tests: Space Structure on Elastic Supports, Cylindrical Roof Under its Own Weight, Hemispherical Shell under Concentrated Load, Morley's Problem, Pinched Cylinder, Simply-supported Square Plate, Thick Beam, Twisted Beam, Bending of a Beam, Thick Cylinder etc.
Bending of a Beam test lets you check analysis results for the bending of a beam, in the context of a static case. You will use 3D meshes. Reference: SFM, Afnor Technique, Guide de validation des prologiciels de calcul de structures, pp124-125. Specifications are in the Table3, analytical solution is represent by educational (2) and the table below presents the analysis results. The results correspond to the means of the stress and the displacements in the section A .

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Table 1: Analysis Specifications and pictures of "TE10"


Table 2: Geometry Specifications


Table 3: Analysis Specifications (HR20) And compare equations of analytical solution

|  | $\sigma_{z z}=\frac{E}{R} x$ <br> $\sigma_{x x}=\sigma_{y y}=\sigma_{x y}=\sigma_{y z}=\sigma_{x z}=0$ <br> $u_{A}=\frac{l^{2} a}{2 R}$ <br>  <br> Figure 3 Misses stress by Parabolic <br> Hexahedron mesh (HR20) |
| :--- | :--- |
| $v_{B}=\frac{l^{\prime} a}{R}$ |  |
| $v_{F}=v_{G}=v \frac{a^{2}}{R}$ |  |

Table 4: Normalized results (computed results divided by analytical solution) are listed.

| Localizati on | Type of Values | Analy-ticalSolution | Values |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Linear Tetrahedron (TE4) |  | Parabolic Tetrahedron (TE10) |  | LinearHexahedron(HE8) |  | Parabolic Hexahedron (HE20) |  |
|  |  |  |  |  |  |  |  |  |  | - 1 |
|  |  |  | Computed Results | $\begin{gathered} \text { Norma- } \\ \text { lized } \\ \text { Results } \end{gathered}$ | $\begin{gathered} \text { Computed } \\ \text { Results } \end{gathered}$ | $\begin{array}{\|l} \text { Norma- } \\ \text { lized } \\ \text { Results } \end{array}$ | Computed <br> Results | Normalized <br> Results | Computed Results | Normalized Results |
| Section | $\sigma_{\mathrm{zz}}[\mathrm{Mpa}]$ | 10 | 7.6 | 0.76 | 10 | 1 | 9.97 | 0.997 | 9.99 | 0.99 |
| A | $\mathrm{U}_{\mathrm{A}}$ [mm] | -0.4 | -0.329 | 0.823 | -0.388 | 0.97 | -0.389 | 0.973 | -0.39 | 0.975 |
| B | $\mathrm{W}_{\mathrm{B}}$ [mm] | 0.2 | 0.17 | 0.85 | 0.197 | 0.985 | 0.197 | 0.985 | 0.197 | 0.985 |
| $\mathrm{F}_{\text {or }} \mathrm{G}$ | $\mathrm{V}_{\mathrm{F}}=-\mathrm{V}_{\mathrm{G}}[\mathrm{mm}]$ | 0.015 | 0.012 | 0.8 | 0.01485 | 0.99 | 0.0149 | 0.993 | 0.0149 | 0.993 |
| $\mathrm{D}_{\text {or }} \mathrm{E}$ | $\mathrm{V}_{\mathrm{D}}=-\mathrm{V}_{\mathrm{E}}[\mathrm{mm}]$ | -0.015 | -0.012 | 0.8 | -0.01485 | 0.99 | -0.0149 | 0.993 | -0.0149 | 0.993 |

This was one of compare verification test which certify functionality of this finite element simulating software.

## 5 Briquetting - Pressing spiral worm

Biomass briquetting by spiral worm achieves the best quality of briquettes. This principle can create different shapes of briquettes (cylinder, $n$-angle parallelepiped, with hole or without) (Figure 4). Material is compacted continually and structural defects are not created. Material is compacted to the cylindrical endless briquette, which is cut to suitable length. Continual compacting of material insures high grade of mechanical pressing quality indexes.


Figure 4: Briquettes made by spiral worm
Every principle of briquetting using other type of pressing tool, has advantages and disadvantages. But the pressing quality is not always determining for briquetting principle selection. Producers also have to consider the production costs and capital costs on the unit of compacted material to be competitive on the biofuel market. Therefore we do research work in this field to increase the pressing quality. But the quality cannot raise at the expense of costs if biofuel has to be successfully using and gradually gets competitive to fossil fuel.

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### 5.1 Finite element real life applications

## Within the development project construction process of compacting machines for biomass we

 have to solve a lot of question about durability, stability, tribology and acceptable load of part of construction or whole assemblies of machines. One of examples is partial task which hasbeen solved. Task was oriented of Analysis of an compressed spiral worm Assembly (
Figure 5). This analyses solved critical construction nodal that is a joint between two part of screw assembly. The joint is realized by parallel key. . It must be very compact because there is limited space in the main construction. One of other problem is Hub Key Groove made for key in blind hole. For this reason is drilled gap advisory hole in to the spiral worm. This hole whither stability of the spiral worm. For that geometrical and dimensional limitations is necessary verify this nodal. In Catia Structural Analysis was created simulation of this operative state. Because the complex process of continues compacting Biomass as wood is very hard to describe. Compacting is a technology which influences many factors. The most variables is coming to the compacting process with the compacted material, kind, structure, chemical composition, mechanical conditions, humidity. For this reasons analysis was simplified.


Figure 5: Assembly of compacting spirall worm

## High pressure worm spiral (green

Figure 5) was replaced by virtual part as well as other contact parts. It was defined load only of gear which we suppose from the previous calculations. There were defined relevant constrains and contacts. Results were represented graphically and numerically. As a result is that spiral worm is manufactorable. Very hard steal must be used in manufacture which satisfies solved technical parameters.


Figure 6: Simplified simulation, compare Von Misses stress, constrains, virtual part and load

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Figure 7: Detail view, compare Von Misses stress.

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# SOFTWARE MATHEMATICA WITH GUI FOR APPLICATIONS WITH ARTIFICIAL NEURAL NETWORKS 

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#### Abstract

This article is focused on Mathematica software (www.wolfram.com) GUI and its usage in the study course Methods of artificial intelligence at Tomas Bata University in Zlin, Faculty of Applied Informatics. Methods of artificial intelligence in this case is applied mathematics in the fields of neural networks and optimization. The article will describe advantages and disadvantages of GUI environement of Mathematica software.


Key words. Mathematica, GUI, webMathematica, neural networks.
Mathematics Subject Classification: Primary 68T01, 68T20.

## 1 Introduction and motivation

This paper will be focused on presentation how Mathematica GUI can be used during lessons, tutorials and lectures in the field of artificial intelligence. Our university - Tomas Bata University in Zlin has bought a Unlimited Site License for Mathematica software and since that time not only computer laboratories have been equipped by this software but also students and employees have been allowed to ask the home use license of Mathematica. This caused spreading of usage of Mathematica over all study programmes and study years. Students learn the basics of the software in the first year and then they use it in different subjects like Mathematics, Differential Equations, Cryptology, classical Optimization and also in Methods and Applications of Artificial Intelligence and others. The possibility of visualization and very short programming code is the main thing which allows to show the explained problem very quickly without time demanding programming to see first results.
Mathematica has two possibilities of nice functional programs for presentations: command Manipulate which can be translated on demonstrations.wolfram.com into a self running program only with Mathematica Player and GUI (graphics user interface) environment where Mathematica or Mathematica Player Pro is necessary for its run. Mathematica Player compared to Mathematica Player Pro can be downloaded free of charge and it is possible to see the code or run Manipulate translated by demonstrations.wolfram.com. In this case no parts where editing is necessary are

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allowed. Mathematica Player cannot edit anything, in Mathematica environment it is possible to have a Manipulate with editable fields. In the case of Mathematica Player Pro the core of Mathematica is available but no edit is possible too. Manipulate case is perfect if no external data is required. It is the case all parameters and data have to be inside the Manipulate and a user only chooses the combination of his/her interests.
The case with GUI is suitable for offering many choices and even uploading of external data or opening of more windows, storing results inside the Mathematica window or export into text of graphical files. There is also two big disadvantages from the programmer point of view and these are: a small support in documentation and that the code has two be written in one big function and it seems that also definitions of user created functions has to be inside the one big function otherwise it is not working properly.
As advantages of GUI were more important for us we have decided to use GUI in our case. The application, which is under development now, will be defended as a diploma thesis during June 2011. Some parts have been already finished as simple neural network with one neuron, Hopfield net and correction of corrupted letters, BAM, multilayer Perceptron with Backpropagation algorithm and multilayer Perceptron with evolutionary techniques as a training algorithm are in the process of testing. The application counts with adding of further blocks of other neural networks types in the future.
The aim of this application is to show possibilities of neural networks to students on tutorials and lectures without writing a code into Mathematica software.
The article will describe introduction into neural networks, secondly figures of the application itself will be shown. In the third part evolutionary algorithms will be mentioned and example of usage with approximation of data.

## 2 Neural networks

Artificial neural networks [1] - [3] are inspired in the biological neural nets and are used for complex and difficult tasks. The most often usage is classification of objects. ANN are capable of generalization and hence the classification is natural for them. Some other possibilities are in pattern recognition, control, filtering of signals and also data approximation and others.
Neural networks are of two kinds with and without supervision. Perceptron and feedforward nets are with supervision. They need a training set of known solutions to be learned on them. Supervised ANN has to have input and also required output. ANN with unsupervised learning is capable of self-organization.
The neural network works so that suitable inputs in numbers have to be given on the input vector. These inputs are multiplied by weights which are adjusted during the training. In the neuron the sum of inputs multiplied by weights are transferred through mathematical function like sigmoid, linear, hyperbolic tangent etc. Therefore ANN can be used for data approximation [1] - [3].
These single neuron units (0) are connected to different structures to obtain ANN (e.g. Figure 2). These networks were design for different tasks. Feedforward nets have different training algorithms - the most known Backpropagation, Pruning algorithm, gradient methods, Levenberg-Marquardt [4] but in last years also evolutionary optimization techniques are used to train ANN, mainly genetic algorithms or Differential evolution [5], [7] In the current phase evolutionary algorithms like SOMA [6], [7] or Differential evolution are implemented and tested because it is supposed that classification or approximation of some tasks could be a very hard optimization problem to find suitable weights in neural networks.


Figure 1. Neuron model, where TF (transfer function like sigmoid), $x_{1}-x_{n}$ (inputs to neural network), $b$ - bias (usualy equal to 1 ), $w_{1}-w_{n}, w_{b}-$ weights, $y$-output.


Figure 2. ANN models with a) one hidden layer and b) with two hidden layer net and more outputs where $\sum \delta=T F\left[\sum\left(w_{i} x_{i}+b w_{b}\right)\right]$ and in this case $\sum=T F\left[\sum\left(w_{i} x_{i}+b w_{b}\right)\right]$, where TF is sigmoid. These pictures are taken from Neural Networks Toolbox for Mathematica environment (www.wolfram.com) since this tool was used during the simulations. Also names are taken from this tool to avoid other speculations what it means..

The previous nets are most often used neural nets. But in the application also Hopfield net and BAM net can be found for demonstration of their capabilities. Mainly Hopfield net which is able to correct corrupted text. It is possible to put either text file with letters or signs for training or GUI offers to open a second window (
Figure 3) where a user can click into the grid according the choice to create a sign. Such signs are added into training sets.

## 3 Printscreens of the application

The following figures (
Figure 3-Figure 6) show example printscreens of the application created in GUI of Mathematica software. Currently, the application works with Perceptron, Adaline - one neuron net and the their types of training. BAM and Hopfield net work with typing of letters or uploading of external data from text file. The training history is also available in the last fold as well as weights set in each epoch.

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Figure 3. Example of Hopfield net part of the application and a window with possibility of new letters clicking

## 4 Approximation of $\operatorname{Sin}(x)$ with usage of evolutionary techniques

Besides above described modules, there is a module of one neuron and multilayer perceptron structure with evolutionary technique training algorithm which is in the testing phase now and adding of this module is in the near future plans.
Evolutionary algorithms are optimization algorithms and are used for different task, usually very hard or impossible for solving by means of classical mathematical tools [7]. These techniques have been tried to use it as a training algorithm in the case of neural networks or even for synthesis of the whole structure and connections [8]. These techniques allow of usage of different transfer functions or different kinds of connection, even not fully connected nets. Usual technique is to optimize of all weights to minimize of the global error - to go through the training set, to count the global error and to change of weight values.
As a demonstration this technique is applied on the approximation of $\sin (x)$ function in the interval
$<0,2 \mathrm{Pi}>$. In this case SOMA algorithm was used [7]. It serves as a demonstration for students.
Following figure (
Figure 4) shows one of the results of this approximation. It is not completely with a global error equal to zero but as it can be seen such approach is possible. More examples are planned as well as user can choose different connections or different transfer functions inside the neurons in the net. The same applications will be implemented also with classical approach to see the difference and comparison between these methods.

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Figure 4. Example of approximation of $\sin$ (x), red - approximation, blue - original data

## 5 Conclusion

This article deals with an application in GUI of Mathematica environment for the subject Methods of artificial intelligence. The main aim is to show to students during lectures and tutorials how different types of neural networks work. There will be implemented also a experimental part with evolutionary technique training algorithm. The application could be used also for scientific computation as uploading of external data is possible.


Figure 5. Example of Perceptron training


Figure 6. Example of Perceptron training history

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# UTILIZATION OF LINEAR TEMPORAL LOGIC FOR GENERATED C PROGRAM CODE 

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#### Abstract

This paper presents a novel approach to software development, mainly useful for embedded devices. Embedded software is described in a programming language with very high level of abstraction. We first generate a special verifiable code from the description and prove that it has certain properties defined by LTL formulae. Then we generate final C code with the same properties.


Key words and phrases. linear temporal logic, formal verification, generative programming.
Mathematics Subject Classification. Primary 68N30, 68M15.

## 1 Introduction

Embedded computer systems are a very large domain. Nowadays, large portion of industrial products are controlled by a computer within, e.g., cars, washing machines, cameras, etc. An embedded software is the ultimate source of flexibility and controllability of the embedded system [1].

There are special requirements for the software that runs on embedded computers. The embedded system should be usually cheap, and thus an embedded software should use as little resources, i.e., CPU and memory, as possible. Power consumption is also important in many cases.

These constraints make embedded software development extremely conservative, see [2], [3]. Whereas programmers of desktop applications use new high level object oriented languages, e.g., Java, C\#, Python, embedded software development relies mostly on legacy tools such as plain C or assembly; despite the fact that the performance of embeddable microprocessors grows for decades.

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Another requirement is dependability. Embedded systems usually work in so called reactive mode, that is, they react on events in the real world. Therefore, a malfunction of an embedded system may directly cause real world damages. It is also typically too expensive to fix an error in a deployed embedded piece of software.

Formal methods can contribute substantially to the reliability of embedded systems. Unfortunately, the state of the practice is often far behind the state of the art of formal methods. This fact has several reasons: formal methods are considered hard and require special languages and tools. Furthermore, formal methods can be applied more easily to well defined high-level languages such as Java or, even better, Haskell, than to low-level languages such as C or assembly.

In this paper, we advocate a novel approach to the embedded software development based on generative programming. It is based on a description in a high-level programming language. We generate two types of codes from the description. The first is suitable for formal verification, we use explicit model checker and linear temporal logic (LTL) formulae. The second one is C code suitable for production use.

Our approach also fits well with advanced software engineering methods such as aspect oriented programming or design by contract [10] because we use flexible dynamically typed language for the description.

The paper is organized as follows: the second section describes the philosophy of our approach and tools we are using; the third section is devoted to the formal verification, literally model checking. The fourth section describes our experiments with LTL verification on an example. The fifth section concludes our achievements.

## 2 Generative Programming

### 2.1 Programming Languages Abstraction

Generative programming is a process of creating a program code that is done by an automated tool, i.e., the code is not directly written by the human.

Every compiler of a programming language such as C or Ada can be viewed as an automated code generator; a programmer writes a human-readable code (the actual source code), and the compiler generates a code runnable by a computer-a low level native code. Without compilers and automated low-level code generation, creating large applications would be unfeasible.

Level of abstraction is one of the most important attributes of a programming language. If the language abstracts from implementation details, the code can be usually more easily understood and verified.

Although the higher level languages and their environments have indisputable benefits for development process, their performance overhead is not always bearable for embedded applications. For example, standard Sun Java Virtual Machine (JVM) needs at least several megabytes of memory just to start its execution. Usually one has to employ special approaches such as customizable JVMs with small memory-footprint [4] [5], or ahead-of-time compilation [6]. This special approaches, however, demand additional effort.


Figure 1: Code Generation Scheme

In the world of embedded software, the traditional pattern is to translate C language to the machine code. However, much more advanced approaches can be taken. The translation process can utilize more sophisticated methods than traditional compilers do, for example, abstract interpretation. It is possible to take a language that offers higher level of abstraction than C (or even Java) and to produce a machine code that does not require any special environment such as JVM.

### 2.2 Our Toolchain

Our proposed approach relies on Python. Python is object oriented dynamically typed programming language with very high level of abstraction. To execute a Python program, one needs the Python interpreter, i.e., a virtual machine. Because of the overhead, it is usually not feasible to run interpreted Python directly on an embedded device with limited resources.

We also rely on tools from the PyPy project. PyPy is an experimental implementation of the Python language developed at ETH Zurich [11]. PyPy provides a tool that translates fairly rich subset of Python (called RPython) to a more efficient code. The translation is based on abstract interpretation, utilizes type inference, and offers great amount of flexibility. With PyPy, we are able to generate both efficient C code suitable for running on an embedded device and Java bytecode suitable for formal verification from one RPython source code.

The PyPy powered translation process consists of two steps:

1. Dynamic program source code is run in an interpreted manner. Classes and objects are constructed dynamically. The result of this step is an initialized object system in the memory of the interpreter. In this step, it is possible to take advantage of dynamic object oriented environment and utilize advanced methods such as aspect oriented programming or meta-programming.
2. In the second step, an abstract interpretation itself is performed. The abstract interpretation starts from a selected entry point, i.e., a procedure/function, and follows the control structure of the program reachable from the point. Static data types are inferred. Output code is emitted during the interpretation.

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A significant advantage of the generative approach is that the generated output code can be easily customized for a particular application, see [12]. This is especially useful when generating C code. For example, memory management can be seen as a translation aspect. Garbage collection algorithm can be selected during the translation and appropriate low-level operations, e.g., reference counting, are emitted to the output C code. There are also possibilities how to customize thread scheduling [13].

As mentioned above, an embedded application is implemented in RPython (a subset of Python), and we translate it to Java bytecode and C code. So the application runs in three modes.

- It runs on top of the Python interpreter. In this mode, a developer can take advantage of very comfortable development environment.
- It is translated to Java bytecode and runs on top of the JVM. We use this mode mainly for verification with Java Pathfinder.
- It is translated to C and compiled to native code, runs directly in the operating system. This mode is intended for production use.

In order to run the application in three different environments, we developed a library called parlib that lies between the application and the underlying platform. We have a special version of parlib for each environment. Each version appropriately handles platform specific issues such as I/O and threading. The translation process is depicted in figure 1.

For verification purposes, it is crucial that the code in the form of Java bytecode and the C code behave identically. This is guaranteed by two properties of the translation.

First, we use the same thread mutual exclusion semantics in both codes. Whereas standard Python programs use locks from POSIX threads, Java uses its own monitors. In our parlib library, we implemented monitors for Python programs; see also our previous work [7]. Significant advantage of this method is that Java Pathfinder needs Java native monitors in order to perform state-space optimizations.

Second, both codes are generated from the same intermediate code defined by PyPy translator. The intermediate code is platform independent. There are, however, two features that are available for Java bytecode and not for C code: automatic memory management and exception handling. These features has to be explicitly implemented by generated C code. There are two well defined transformations that are applied to the intermediate code in order to implement garbage collector and exception handlers in C code.

## 3 Formal Verification

### 3.1 Model-checking

Program correctness is really important in the field of embedded applications. There are plenty of techniques that help detecting program defects, both design- and run-time. The most common is type checking that is usually provided by compilers of statically typed programming
languages. Modern software engineering practices such as aspect oriented programming or design by contract may also help. Traditional error discovering technique is testing, i.e., comparing the actual program output with the expected results.

One of the strongest technique is formal verification. One of the most popular formal tool for verifying software functionality is model-checking.

Model-checking is a process of checking whether a given system, i.e., a piece of software, is a model of a given logic formula. The process is done by enumeration (explicit or implicit) of all the states reachable by the system and the behaviors that traverse through them [14].

The major drawback of model-checking is that it scales badly. When the system size grows linearly, the state space of the system tends to grow exponentially. The nature of the growth is given by the fact, that every component added to the system may cause multiplication of the number of system states. That is why, the model-checking is often not performed on the original piece of software, but rather on an abstracted model of the program. The abstracted model is usually constructed by hand and utilizes some kind of modeling language, for instance Promela. This simplification implies that the model does not necessarily have the same behavior as the original program has.

### 3.2 Linear Temporal Logic

Linear Temporal Logic (LTL) [8] is popular way for describing formal properties of computer program. LTL is a modal logic with modalities referring to time [8]. It is a subset of richer Generalized Computational Tree Logic (CTL*). Its atoms are atomic propositions reflecting the current state of a system.

A model for a temporal formula $\varphi$ is an infinite sequence of states (i.e., a word)

$$
\begin{equation*}
\pi=\pi_{0} \pi_{1} \pi_{2} \ldots \tag{1}
\end{equation*}
$$

where each state $\pi_{i}$ provides an interpretation for the atomic propositions mentioned in $\varphi$.
The set of LTL formulae is defined inductively starting from countable set of atomic propositions, Boolean operators, and the temporal operators $\mathbf{X}$ (Next) and $\mathbf{U}$ (Until):

$$
\begin{equation*}
\varphi:=a|\neg \varphi| \varphi \wedge \varphi|\boldsymbol{X} \varphi| \varphi \boldsymbol{U} \varphi \tag{2}
\end{equation*}
$$

Given a model $\pi$, as above, we present an inductive definition for the notion of a temporal formula $\varphi$ holding at a position $j \geq 0$ in $\pi$, denoted by $(\pi, j) \models \varphi$. For a state formula $\varphi$,

$$
(\pi, j) \models \varphi \Longleftrightarrow \pi_{j} \models \varphi
$$

That is, we evaluate $\varphi$ locally, using the interpretation given by $\pi_{j}$.

```
\((\pi, j) \vDash \neg \varphi \Longleftrightarrow(\pi, j) \not \models \varphi\)
\((\pi, j) \models \varphi \wedge \psi \Longleftrightarrow(\pi, j) \models \varphi\) and \((\pi, j) \models \psi\)
\((\pi, j) \models \boldsymbol{X} \varphi \Longleftrightarrow(\pi, j+1) \models \varphi\)
\((\pi, j) \models \varphi \boldsymbol{U} \psi \Longleftrightarrow\) for some \(k \geq j,(\pi, k) \models \psi\), and for every \(i\) such that \(j \leq i<k,(\pi, i) \models \varphi\)
```

There are also two additional useful operators: $\mathbf{F}$ (in the future, eventually) and $\mathbf{G}$ (globally) defined as follows:

$$
\begin{aligned}
& (\pi, j) \models \boldsymbol{F} \varphi \Longleftrightarrow(\pi, j) \models \text { true } \boldsymbol{U} \varphi \\
& (\pi, j) \models \boldsymbol{G} \varphi \Longleftrightarrow(\pi, j) \models \neg \boldsymbol{F} \neg \varphi
\end{aligned}
$$

According to [9], temporal properties can be partitioned into two classes: safety and liveness. The classes can be informally characterized as:

- A safety property states that some bad thing never happens.
- A liveness property states that some good thing eventually happens.

Safety properties typically represent requirements that have to be continuously maintained by the system. For example, safety property should specify mutual exclusion: a lock is acquired at most by one thread. Liveness properties, on the other hand, represent requirements that need not hold continuously, but have to be eventually of repeatedly fulfilled. For example, it is guaranteed that one of the threads requiring a lock eventually acquires it.

### 3.3 Java Pathfinder

For verification, we use Java Pathfinder (JPF), which is an explicit model-checker developed at NASA [15]. JPF is able to check programs in the form of Java bytecode. It can be seen as a special implementation of Java Virtual Machine. Whereas standard JVM executes program sequentially, JPF investigates all possible states of the program by traversing all possible execution paths. JPF verifies Java bytecode of a real Java program, i.e., not an abstracted model.

In our approach, the Java bytecode generated from our RPython program can be seen as an inabstracted model of the final C code that is intended for production use.

JPF as is does not provide any LTL-based verification. Fortunately, JPF has an open architecture and one can create modules for custom verification procedures. We use LTL verifier developed by Nguyen and Khoo [16]. This verifier uses method calls as atomic propositions.

## 4 Case Study

### 4.1 Program for Logging Events

Let us apply our approach to an example that is taken from real-world application. Having a network video recorder (NVR) which is an embedded computer system that manages IP cameras over network. Its main goal is to store records from the associated cameras together with metadata such as whether a camera detected a motion, how many frames-per-second are in the video record, temperature, etc.

Metadata are stored in a database that acts as a data warehouse. The metadata are not stored in the raw form but summarized into time intervals. There is a predefined hierarchy of lengths of intervals: 10 seconds, 3 minutes, 1 hour and 1 day. In the case of motion detects and similar events, value of every interval is the total number of the events, that happened within the interval. For physical quantities such as temperature, the value of the longer interval is computed as average from subintervals that are lower in the hierarchy.

The data warehouse is built on-the-fly by NVR component called Logger. It handles incoming metadata from camera drivers. The camera driver runs in a separate thread and reads video stream through network socket; raw video is saved to disk and extracted metadata are pushed to Logger.

The key class of Logger is called Summarizer. It maintains the state of the current intervals in memory, i.e., updates the values according to the events incoming from drivers, and writes the final value into the database when a particular interval elapses. After the interval is successfully written, its value in memory is cleared, i.e., new interval is started. The process of flushing interval values from the memory to the database is performed by a thread called IntervalWriter. It sleeps most of the time, wakes up only when interval ends to perform the write \& clear operation.

It is important to note that camera driver runs in soft real-time mode, i.e., it has to be guaranteed that it is never blocked when interacting with Summarizer. This proposition holds, because the possibly slow operation of inserting interval value into the database is performed in IntervalWriter thread.

Apart from standard events that are summarized, there are special events called alarms with completely different policy. Whenever an alarm occurs it has to be written into a persistent alarm $\log$ as soon as possible. For this task, there is a thread called AlarmWriter that waits for an alarm; when one arrives, thread is waken up and writes the alarm into the log. Unlike intervals, alarm data are stored in operating memory as shortly as possible.

### 4.2 Experiments

In order to check the program by Java Pathfinder, we created a simplified version; simplification is done by replacing interaction with real environment by interaction with some model of the environment. We have a virtual camera driver that does not communicate with real device but only semi-randomly generated events. The database is replaced by a mockup that simply does nothing. It is important to emphasize that Summarizer itself and the thread interaction stays unsimplified. With this setup, the model checker can enumerate all possible states of Logger.

There are plenty of properties that we can possibly specify by LTL formulae. For our purpose, we investigate the reaction of the system when interval elapses. The following formula denotes that whenever an interval elapses, it has to be written to the database ${ }^{1}$.

```
G((method:Summarizer.intervalElapsed)
    -> (X(F(method:Database.writeInterval)))
)
```

${ }^{1}$ We use syntax from used LTL verifier, " $->"$ denotes implication, $" \sim$ negation.

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Java Pathfinder can check that our implementation has this property. If we inject a bug into Summarizer, i.e., remove the writeInterval method call from the program, JPF finds this bug and provides a program trace as a proof.

There is also a requirement that when an interval elapses, the intermediate value for the interval has to be cleared in order to start a new interval. This can be expressed by simple modification of the formula mentioned above, that is:

```
G((method:Summarizer.intervalElapsed)
    ->(X(F(method:Database.clearInterval)))
)
```

The order of writeInterval and clearInterval operations has to be always valid, i.e., the interval is cleared only after it is written. The following formula denotes that whenever interval elapses, it is not cleared until it is written to the database.

```
G((method:Summarizer.intervalElapsed)
    -> (X(
            (~(method:Summarizer.clearInterval))
            U(method:Database.writeInterval)
        )
        )
)
```

We can ensure the correctness of the formula by injecting a bug into Summarizer. If we swap the clearInterval and writeInterval method calls in the program, model checker will discover this misbehavior. Note that all these three methods are executed by one thread, the IntervalWriter thread.

To demonstrate the ability to find bugs in multi-threaded programs, we will investigate the part of the program dealing with alarms. As mentioned above, alarms are detected by camera driver (in driver thread) and always have to be written to the alarm log (by Alarm Writer thread). The following LTL formula holds if every occurrence of alarm is eventually followed by write operation.

G((method:Driver.alarm0ccurred)
-> (X (F (method:AlarmLog.writeAlarm))))
After we run the verification procedure, we are sure that Logger has this property. Again, if we inject a bug, i.e., remove a thread notification that wakes up Alarm Writer from alarmOccurred method, JPF discovers the bug.

Apart from the LTL formulae, JPF checks that the code has the following properties:

- no deadlock occurs,
- none of the threads ends by an uncaught exception,
- there is no unsafe data access among threads, i.e., there is no race condition.

In practice, it is impossible to prove that a real program is ultimately correct. However, mere proving some isolated and almost trivial properties is excellent testing method.

### 4.3 Generated C Code

After the program is verified, the final production C code can be generated. Proper investigation of the C code is beyond the scope of this paper. In our previous work [7], we show that the memory footprint is not heavily affected by the fact that the program was generated from abstract high level description. Of course, hand-written C code may be more efficient, mainly because of manual memory management.

## 5 Conclusion

In this article, we have presented a novel approach to embedded software development. In our approach, we implement an intended embedded application in an object oriented programming language with very high level of abstraction, that is, in our case, a subset of Python.

From this implementation, we generate code that is suitable for formal verification, literally model-checking. We employ linear temporal logic (LTL) as a tool for specification of correct behavior of the program. The LTL-based properties are investigated by explicit model checker called Java Pathfinder.

From the high level implementation, we also generate very efficient native machine code that is suitable for production use on an embedded device with limited resources. Thanks to the nature of our tool-chain, the generated code has the same properties (defined by LTL formulae) as were checked by the model checker.

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# IMPROVEMENTS LEADING TO BETTER PERFORMANCE OF ALGORITHMS USED IN THE GENETICALLY PROGRAMMING PROCESS IN ASSEMBLY LANGUAGE 

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#### Abstract

Genetically programming processes are based on program loops which are repeating in many cycles, because many generations are required for the evolution of a genetic code. The time consumed for execution of such loop cycle has enormous impact on the whole performance of every genetically programming process of evolution. Author of the text decided to try alternative design methods of a genetically programming process and pure x86 assembly language was chosen as a programming language. The target goal of such research is a CPU code running as fast as possible without unnecessary time loses when implementing a genetically programming process. The paper describes improvements of algorithms designed in assembly language which are leading to a better performance of the whole genetically programming process according to author's experiences.


Key words: genetically programming, assembly language, minimalist implementation, efficient computing, cellular automaton

Mathematics Subject Classification: Computer science, Artificial intelligence 68T04.

## 1 Short history of the research

On the beginning of my research at autumn 2008, I was thinking about genetically programming processes and how to increase their performance. As first improvement and increase of evolution speed I tried to define a new, universal language for coding of wide range of algorithms which could be evaluated fast and with a high efficiency.
Theoretical base (supported by a mathematical background) of the new "language" for universal coding of algorithms, was introduced on the beginning of 2009. Inspiration by cellular automata is evident and processing of the algorithm coded with the "language" can be compared to the processing of a cellular automata.
An algorithm described by this "language" is represented by cells, logic functions coded inside them and connections with other cells at the same time. Each of the cells contains binary information which can be propagated to other cells and it is equivalent to a flow and propagation of

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binary information through a simulation of a logic circuit in discrete steps, with every possible complexity.

The cellular representation of the algorithm, called "Cellular processor of logical functions" is then simply converted to a sequence of bits - a binary vector, which is further evaluated by the genetic algorithm.

Most powerful option to handle large amount of bits and bytes and effective manipulation of them are algorithms coded in assembler language. At the summer 2009 started coding of the assembler program for genetically evolution of algorithms which are represented by Cellular processor of logical functions and the application is called "GenAlg".

At the summer 2010, after one year of development during evenings at home (Development of this application is not part of my profession, only at free time), the first draft of the application has been running. Since then I am improving all program's subparts to gain faster convergence during the evolution, to increase the effectiveness, to implement new useful features and to develop good examples of problem solving based on the usage of the application.
n following parts of the paper, detailed information is given about the core of algorithm representation - the "Cellular processor of logical functions", short outline of the whole genetically programming loop is described and as the main part of the paper, improvements and experiences coming from the application operation are introduced.

## 2 Cellular processor of logical functions

Basic construction unit of the cellular processor of logical functions is one cell $\mathrm{B}_{\mathrm{n}, \mathrm{k}}$, which in cooperation with other cells forms one-dimensional cellular automaton with the absolute cell address range $<0, \mathrm{n}_{\max }>$. Each of the cell $\mathrm{B}_{\mathrm{n}, \mathrm{k}}$ is coded in 32 bits and carries the binary information:

- 1 bit: the last valid value calculated during the current step " k " in

$$
\begin{equation*}
y_{n, k} \in\left\{(0)_{2},(1)_{2}\right\}, \tag{2.1}
\end{equation*}
$$

- 1 bit: the previous value valid in previous step " $k$ - 1 " in

$$
\begin{equation*}
y_{n, k-1} \in\left\{(0)_{2},(1)_{2}\right\}, \tag{2.2}
\end{equation*}
$$

- 11 bits: the relative link from the first other cell

$$
\begin{equation*}
\mathrm{a}_{\mathrm{n}}=<-\left(\mathrm{n}_{\max }+1\right) / 2,+\left(\mathrm{n}_{\max }+1\right) / 2-1> \tag{2.3}
\end{equation*}
$$

- 11 bits: the relative link from the second other cell $\mathrm{b}_{\mathrm{n}}=<-\left(\mathrm{n}_{\max }+1\right) / 2,+\left(\mathrm{n}_{\max }+1\right) / 2-1>$,
- 8 bits: the logical function $\mathrm{F}_{\mathrm{n}}$ coded by the 8 -bit table (described by one byte)

$$
\begin{equation*}
\mathrm{F}_{\mathrm{n}}=\left[\mathrm{f}_{\mathrm{n}, 0}, \mathrm{f}_{\mathrm{n}, 1}, \mathrm{f}_{\mathrm{n}, 2}, \mathrm{f}_{\mathrm{n}, 3}, \mathrm{f}_{\mathrm{n}, 4}, \mathrm{f}_{\mathrm{n}, 5}, \mathrm{f}_{\mathrm{n}, 6}, \mathrm{f}_{\mathrm{n}, 7}\right]=<(0)_{10},(255)_{10}>. \tag{2.4}
\end{equation*}
$$

During each of the iteration step „ $\mathrm{k}^{\text {" }}$, all new output values of all cells $\mathrm{B}_{\mathrm{n}, \mathrm{k}}$ are calculated from previous output value of cell $B_{n+\text { an,k }}$ (address of the cell $A$ is calculated from $n+a_{n}$ ), previous output value of the cell $B_{n+b n, k}$ (address of the cell $B$ is calculated from $n+b_{n}$ ) and the previous output value of the cell $\mathrm{B}_{\mathrm{n}, \mathrm{k}}$ itself. All these three binary values are used as an address $(0 \ldots 7)$ to get a new output value $\left(f_{n, 0} \ldots f_{n, 7}\right)$ of the cell $B_{n, k}$ from the 8-bit table $F_{n}$.

## 3 Program algorithm

My implementation of genetically programming algorithm consists of several programming blocks (inspired by [1], p.27) - assembler subroutines:

- 1: Definition of genetic algorithm's parameters, control and termination parameters
- 2: Generation of the first random population
- 3: Calculation of fitness for all members of the population
- 4: Sorting of members of the population according to their fitness
- 5: Testing of termination conditions
- 6: Selection of population members, removal of unsuccessful members
- 7: Definition of pairs of survived population members and number of offspring
- 8: Selection of crossing operators to generate children from defined pairs as the combination of genetic information
- 9: Random mutation operators to modify genetic information of new members of the population


## 4 Calculation of fitness

Fitness of a member of the generation is calculated in such a way, that a target function (a function algorithm which is to be found during the evolution) is represented with test patterns.
Each of the test pattern consist of one input binary vector, number of steps (of cellular processor) to be performed and one output binary vector which is expected, after all steps of the cellular processor finished.

Input vector is mapped to first cells of the tested binary vector, output vector is mapped to last cells of the binary vector. Cells in-between which are not used for inputs and outputs are variables and they contain information about the processing. During each of the step of calculation of fitness, all test patterns one by one are applied to the tested member and the number of binary outputs which are equal to expected binary outputs is calculated. The goal of the evolution is to achieve maximum number of binary outputs equal to outputs defined in all test patterns.
If the maximum of fitness is achieved, the equivalent of the target function is coded inside configuration of cells of the cellular processor.

To achieve better accuracy, more test patterns are necessary, but the evolution is very slow then.

It is recommended to perform the evolution in smaller steps. For example: To use a small group of patterns on the beginning of the evolution and to extend it in later phases. Another possibility is to have several groups of patterns which are changed during the evolution after several generations. This can help to escape from potential local extremes in the evolution process.

## 5 Selection of population members

During first versions of my application I used for selection of population members exponential function, which can be used as a probability function. On the figure Nr.1, there is on left side the probability function with $\mathrm{k}=2$ and on the right side the same function normalized to 1000 population members and $\mathrm{p}(\mathrm{x})$ normalized to 400 as the maximum probability, stored in RAM as integer numbers.

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$$
\begin{equation*}
p(x)=e^{-\frac{x}{k}} \tag{5.1}
\end{equation*}
$$

$x$ : ordinal number of the algorithm candidate starting with zero (max. fitness)
$k$ : constant which must be selected according to previous experiences


Figure 1: Left -probability function, Right -normalized to 1000 candidates and $p_{\max }=400$
After several time I have experimented with my application, I have seen that the effectivity of the selection algorithm has problems with the genetically diversity of members inside the population. Very soon after small amount of generations, there were many copies of one, two or three members inside the population with the different genetic information and the progress of the evolution was too slow.

I experimented after with different other types of selection functions and with one, the genetically diversity was much better than before as only simple exponential function was used. As a starting point I used the previous exponential function and then I added effect of the sinus function. This type of selection leads to :


Figure 2: The selection function corresponds (normalized to 1000 candidates and $p_{\max }=400$ ) to

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$$
\mathrm{p}(\mathrm{x})=400-0,73 *\left(400 *(1-(\operatorname{EXP}(-\mathrm{x} / 200)))+150 * \operatorname{SIN}\left(2 * \mathrm{PI}^{*}(1000-\mathrm{x}+50) / 300\right)\right)
$$

This function leads to 3 local maximum: On places 215, 523 and 825 . Around these points the probability function gives to members of a generation higher chance that they will pass to the next generation and this type of behaviour improves the genetically diversity.

## $6 \quad$ Parallelism of several evolutions

When experimenting with the application, the progress of the evolution was not fast enough, the genetically diversity has been still not so big and it was easy to be stopped in some local extremes. The number of parallel running evolutions has been increased. The main strategy is, to mix members between parallel running evolutions after some number of evolution steps.
Currently I use 8 parallel evolutions and their members are mixed in this way (moved from one parallel evolution to other) while $\mathrm{s} 3>\mathrm{s} 2>\mathrm{s} 1$ :

- after s1 evolution steps
member1(ev1)->member1(ev2); member1(ev2)->member1(ev3); member1(ev3)->member1(ev4); member1(ev4)->member1(ev1); member1(ev5)->member1(ev6); member1(ev6)->member1(ev7); member1(ev7)->member1(ev8); member1(ev8)->member1(ev1);
member2(ev2)->member2(ev1); member2(ev3)->member2(ev2); member2(ev4)->member2(ev3); member2(ev1)->member2(ev4); member2(ev6)->member2(ev5); member2(ev7)->member2(ev6); member2(ev8)->member2(ev7); member2(ev1)->member2(ev8).
- after s2 evolution steps
member1(ev2)->member1(ev1); member1(ev3)->member1(ev2); member1(ev4)->member1(ev3); member1(ev1)->member1(ev4); member1(ev6)->member1(ev5); member1(ev7)->member1(ev6); member1(ev8)->member1(ev7); member1(ev1)->member1(ev8);
member2(ev1)->member2(ev2); member2(ev2)->member2(ev3); member2(ev3)->member2(ev4); member2(ev4)->member2(ev1); member2(ev5)->member2(ev6); member2(ev6)->member2(ev7); member2(ev7)->member2(ev8); member2(ev8)->member2(ev1).
- after s3 evolution steps
member1(ev1)->member1(ev7); member1(ev2)->member1(ev8); member1(ev3)->member1(ev5); member1(ev4)->member1(ev6); member1(ev5)->member1(ev3); member1(ev6)->member1(ev4); member1(ev7)->member1(ev1); member1(ev8)->member1(ev2);
member2(ev1)->member2(ev5); member2(ev2)->member2(ev6); member2(ev3)->member2(ev7); member2(ev4)->member2(ev8); member2(ev5)->member2(ev1); member2(ev6)->member2(ev2);


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member2(ev7)->member2(ev3); member2(ev8)->member2(ev4).

## $7 \quad$ Future development

It is necessary to find more improvements in all of steps of the evolution cycle, as it is believed that there is still a possibility to gain better performance in the future. As next, some good examples for testing the application should be found in the future to be able to compare the performance of the application to other already existing programs which are based on more traditional concepts.

## References

[1] ZELINKA I., Oplatková Z., Šeda M., Ošmera P., Včelař F.: Evoluční výpočetní techniky, Principy a aplikace. BEN - technická literatura, Praha, 2009.

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[^0]:    ${ }^{1}$ A concept similar to the "Ends lemma" - albeit in different context - is discussed by Vougiouklis in [16]. As far as I am aware, this is the only "Ends lemma"-like idea by a non-Czech author.

[^1]:    ${ }^{2}$ In fact, they were part of an article Identities and inverses of "Ends lemma" based hyperstructures submitted to Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium, which was after rather a long time withdrawn from publication. Meanwhile, new results built upon the article have been obtained and published. Some findings of the unpublished article are presented here in the form of this contribution.
    ${ }^{3}$ For the sake of this contribution the term reversible need not be defined.
    ${ }^{4}$ Cf. Definition 3.7.
    ${ }^{5}$ In some earlier articles the term "Ends lemma"-based hyperstructure is used.

[^2]:    ${ }^{6}$ In [15] the same idea is used to construct another hyperstrure based on matrix multiplication. The results used in the following example would be analogous.

[^3]:    ${ }^{1} \varphi \neq \mathrm{id}_{Q}$.
    ${ }^{2}$ That is, $\operatorname{card}(Q) \geq 1$.
    ${ }^{3}$ The assumption $L_{e}=\mathrm{id}_{Q}$ would imply ex $=\mathrm{x}=\mathrm{xx}$, hence $x=e$ by cancellation, and $Q=\{e\}$, i.e. the trivial case.

[^4]:    ${ }^{4}$ Remark that the identity (7) has been discussed also in [2].

[^5]:    ${ }^{1}$ one of them is found e.g. in [8]

[^6]:    ${ }^{2}$ this is a criterion of transversality of two $A$-subspaces; Two $A$-subspaces are called transversal $A$-subspaces in the sense of Veldkamp if both their intersection and their sum are $A$-subspaces, too (see e.g. [18])
    ${ }^{3}$ generally, it is not true over an arbitrary local ring $A$, see e.g. [18]

[^7]:    ${ }^{4}$ Let us remark that this proposition is not true for modules over arbitrary ring (for example, let us consider a (free) module $\mathbb{Z}$. The submodule of even numbers is kernel of no endomorphism of $\mathbb{Z}$ )

[^8]:    ${ }^{1}$ the generator of the algebra $A$ is usually denoted by $\varepsilon$. In this article it will be denoted by $\eta$ because in electromagnetism there is by $\varepsilon$ the permitivity denoted

[^9]:    ${ }^{2}$ it follows from this that for example orthogonal product on $E$ induced such product on $M$

[^10]:    ${ }^{3}$ the paper of P. A. Kotelnikov, published in Annals of Imperial University (Kazan, 1895), has been destroyed during Russian revolution

[^11]:    ${ }^{4}$ it is a vector of density of flow of electromagnetic power, see [4]
    ${ }^{5}$ An operator $\frac{\mathrm{d}}{\mathrm{d} \eta}$ on $M$ is in [16] defined by $\frac{\mathrm{d}}{\mathrm{d} \eta}\left(\boldsymbol{x}_{0}+\eta \boldsymbol{x}_{1}\right)=\boldsymbol{x}_{1}$.

[^12]:    ${ }^{1}$ Dating from 1984 and last revised in 2004.

[^13]:    ${ }^{1} V y \bar{a} m a$ is the measure of a man from the soles to the root of the hair on the forehead. It is a standard of measure for the altars of the daily sacrifices. Vyāma has 96 angulas and angula is the width of a finger, añgula is approximately equal to $\frac{3}{4}$ inch or 1.8 cm .
    ${ }^{2}$ Purusa is the mesure of a man with arms lifted up. The length of a purusa is standardised as 120 arigulas.

[^14]:    ${ }^{3}$ One pada is equal to 6 arigulas.

[^15]:    ${ }^{1}$ Let us note that the genus is a birational invariant of a curve which gives us a criterion of the rationality of the curve. Namely $\mathcal{X}$ is rational if and only if $\mathrm{g}(\mathcal{X})=0$.

[^16]:    ${ }^{1}$ For technical details, see also Methodological Remarks of Micro-census 2002 (CZSO 2004) and of EU-SILC (CZSO 2006-2008).

